NOTES ON QUANTIFIED MODAL LOGIC

1. Preliminaries

1.1. The Language of Quantificational Logic.

1.1.1. *Syntax for QL.* The vocabulary for the language *QL* consists of the following:

(1) An infinite number of *constants*—lowercase letters from the start of the alphabet, potentially with subscripts:

$$a, b, c, d, e, a_1, b_1, c_1, d_1, e_1, a_2, b_2, \ldots$$

(2) An infinite number of *variables*—lowercase letters from the end of the alphabet, potentially with subscripts:

$$w, x, y, z, w_1, x_1, y_1, z_1, w_2, \ldots$$

(3) For every natural number $N \geq 1$, an infinite number of N-place predicates—capital letters, potentially with subscripts:

$$\begin{array}{cccccccc} A^1, & B^1, & \cdots, & Y^1, & Z^1, & A^1_1 & B^1_1, & \cdots \\ A^2, & B^2, & \cdots, & Y^2, & Z^2, & A^2_1 & B^2_1, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^N, & B^N, & \cdots, & Y^N, & Z^N, & A^N_1 & B^N_1, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots \end{array}$$

(4) The identity relation:

$$=$$

(5) Logical operators:

$$\forall, \ \sim, \ \rightarrow$$

(6) Parentheses:

$$(, \ )$$

Nothing else is included in the vocabulary of *QL*.

Terminology: we call both constants and variables *terms* of *QL*.

These notes are heavily indebted to G. E. Hughes and M. J. Cresswell (1996), *A New Introduction to Modal Logic*, Routledge, London; Greg W. Fitch, *Naive Modal Logic*, unpublished lecture notes; and Theodore Sider (2010) *Logic for Philosophy*, Oxford University Press, Oxford.

1.1.2. *Rules for Wffs.* We specify what it is for a string of symbols from the vocabulary of *QL* to constitute a *well-formed formula*, of *wff* of *QL* recursively with the following.

- Π) If $\ulcorner \Pi^N \urcorner$ is an N-place predicate and $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$ are N terms, then $\ulcorner \Pi^N \tau_1 \tau_2 \ldots \tau_N \urcorner$ is a wff—known as an *atomic* wff.
- $=$) If $\ulcorner \tau_1 \urcorner$ and $\ulcorner \tau_2 \urcorner$ are terms, then $\ulcorner \tau_1 = \tau_2 \urcorner$ is a wff—also known as an *atomic* wff.
- $\sim$) If $\ulcorner \phi \urcorner$ is a wff, then $\ulcorner {\sim}\phi \urcorner$ is a wff.
- $\rightarrow$) If $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$ are wffs, then $\ulcorner (\phi \rightarrow \psi) \urcorner$ is a wff.
- $\forall$) If $\ulcorner \phi \urcorner$ is a wff and $\ulcorner \alpha \urcorner$ is a variable, then $\ulcorner (\forall \alpha)\phi \urcorner$ is a wff.
- – Nothing else is a wff.

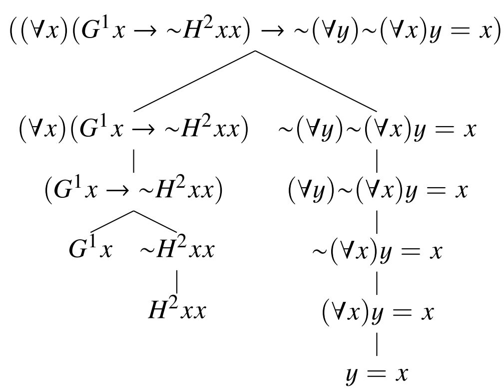
For instance, the following is a wff of *QL*:

$$((\forall x)(G^1x \rightarrow {\sim}H^2xx) \rightarrow {\sim}(\forall y){\sim}(\forall x)y = x)$$

We could show this by providing the following proof, which appeals to the rules for wffs above:

1.	'G^1x' is a wff.	(Π)
2.	'H^2xx' is a wff.	(Π)
3.	'$y = x$' is a wff.	($=$)
4.	So, '${\sim}H^2xx$' is a wff.	2, ($\sim$)
5.	So, '$(G^1x \rightarrow {\sim}H^2xx)$' is a wff.	1, 4 ($\rightarrow$)
6.	So, '$(\forall x)(G^1x \rightarrow {\sim}H^2xx)$' is a wff.	5, ($\forall$)
7.	So, '$(\forall x)y = x$' is a wff.	3, ($\forall$)
8.	So '${\sim}(\forall x)y = x$' is a wff.	7, ($\sim$)
9.	So '$(\forall y){\sim}(\forall x)y = x$' is a wff.	8, ($\forall$)
10.	So '${\sim}(\forall y){\sim}(\forall x)y = x$' is a wff.	9, ($\sim$)
11.	So '$((\forall x)(G^1x \rightarrow {\sim}H^2xx) \rightarrow {\sim}(\forall y){\sim}(\forall x)y = x)$' is a wff.	6, 10 ($\rightarrow$)

Another way of notating a proof like this is with a syntax tree like the following:



1.1.3. *Definitions.* We introduce the following stipulative definitions, for any wffs $\ulcorner\phi\urcorner$, $\ulcorner\psi\urcorner$ of QL, and any variable $\ulcorner\alpha\urcorner$ of QL:

(1) $\ulcorner(\phi \vee \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner((\phi \rightarrow \psi) \rightarrow \psi)\urcorner$
(2) $\ulcorner(\phi \wedge \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner\sim(\phi \rightarrow \sim\psi)\urcorner$
(3) $\ulcorner(\phi \leftrightarrow \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner\sim((\phi \rightarrow \psi) \rightarrow \sim(\psi \rightarrow \phi))\urcorner$
(4) $\ulcorner(\exists\alpha)\,\phi\urcorner \stackrel{\text{def}}{=} \ulcorner\sim(\forall\alpha)\sim\phi\urcorner$
(5) $\ulcorner\tau_1 \neq \tau_2\urcorner \stackrel{\text{def}}{=} \ulcorner\sim\tau_1 = \tau_2\urcorner$

The penultimate definition tells us that, for instance, if we write '$(\forall x)(\exists y)R^2xy$', this is simply shorthand for the wff '$(\forall x)\sim(\forall y)\sim R^2xy$'.

1.1.4. *Conventions.* As a matter of convention, we will omit the outermost parentheses, and suppress the superscripts on the predicates of QL. Thus, rather than writing

$$((\forall x)(G^1x \rightarrow \sim H^2xx) \rightarrow (\exists y)(\forall x)y = x)$$

we could instead simply write

$$(\forall x)(Gx \rightarrow \sim Hxx) \rightarrow (\exists y)(\forall x)y = x$$

1.1.5. *Syntactic Defintions.* We introduce the following syntactic defintions.

(1) The MAIN OPERATOR of a wff of QL is the logical operator whose associated rule would be *last* appealed to when building that wff up according to the rules for wffs. It is the operator which is added at the top of the wff's syntax tree.
(2) $\ulcorner\phi\urcorner$ is a SUBFORMULA of $\ulcorner\psi\urcorner$ if and only if we would have to show that $\ulcorner\phi\urcorner$ is a wff in order to show that $\ulcorner\psi\urcorner$ is a wff. Alternatively, $\ulcorner\phi\urcorner$ is a subformula of $\ulcorner\psi\urcorner$ iff $\ulcorner\phi\urcorner$ appears by itself on a leaf in the syntax tree for $\ulcorner\psi\urcorner$.
(3) The SCOPE of a quantifier—$\ulcorner(\forall\alpha)\urcorner$ or $\ulcorner(\exists\alpha)\urcorner$—is the subformula for which that quantifier is the main operator.
(4) A variable $\ulcorner\alpha\urcorner$ in a wff of QL is BOUND if and only if it occurs within the scope of a quantifier, $\ulcorner(\forall\alpha)\urcorner$ or $\ulcorner(\exists\alpha)\urcorner$, whose associated variable is $\ulcorner\alpha\urcorner$.
(5) A variable $\ulcorner\alpha\urcorner$ in a wff of QL is FREE if and only if it does not occur within the scope of a quantifier, $\ulcorner(\forall\alpha)\urcorner$ or $\ulcorner(\exists\alpha)\urcorner$, whose associated variable is $\ulcorner\alpha\urcorner$.

For instance, in the wff

$$(\forall x)Fxy \rightarrow (\exists y)Gyx$$

the 'x' in 'Fxy' is bound by the universal quantifier '$(\forall x)$'. The 'y' in 'Fxy' is free, since it is not the scope of any y-quantifier. The 'y' in 'Gyx' is bound by the existential quantifier '$(\exists y)$'. The 'x' in 'Gyx' is free, since it is not the scope of any x-quantifier.

1.1.6. *Semantics for Quantificational Logic.* In propositional logic, we defined our semantics by way of (bivalent) *interpretations*. Similarly, for some of the non-classical propositional logics we saw, we defined our semantics with *trivalent interpretations*. With propositional modal logic, we defined the semantics with the notion of a *model* (a set of worlds, a binary relation, and a bivalent propositional interpretation). For Quantificational Logic, our semantics will appeal to the notion of a *QL-model*, which is just a set of some things—called the *domain*, $\mathscr{D}$, of the model—and an interpretation $\mathscr{I}$—which is just a function from terms of *QL* to the things in the domain $\mathscr{D}$, and from the N-place predicates of *QL* to N-tuples of the things in $\mathscr{D}$.

> QL-Model:
> A QL-MODEL $\mathcal{M}$ is a pair $< \mathscr{D}, \mathscr{I} >$ of a (non-empty) *domain* $\mathscr{D}$ and an interpretation function $\mathscr{I}$. $\mathscr{I}$ maps terms of *QL* to entities in $\mathscr{D}$, and N-place predicates of *QL* to sets of N-tuples of entities in $\mathscr{D}$. Thus, for every term $\ulcorner \tau \urcorner$ of *QL*,
> $$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$
> And for every N-place predicate $\ulcorner \Pi^N \urcorner$ of *QL*,[1]
> $$\mathscr{I}(\Pi^N) = \{\dots, < \mathfrak{u}_1, \mathfrak{u}_2, \dots, \mathfrak{u}_N >, \dots\} \subseteq \underbrace{\mathscr{D} \times \mathscr{D} \times \cdots \times \mathscr{D}}_{N \text{ times}} = \mathscr{D}^N$$

For this definition, we'll take a set of 1-tuples of entities in $\mathscr{D}$ to be just a set of entities in $\mathscr{D}$. So $\mathscr{I}$ maps 1-place predicates to sets of entities in $\mathscr{D}$. To get an intuitive idea for what's going on here, think about the set of things that $\mathscr{I}$ maps a 1-place predicate to as those things in the domain which have the property denoted by the predicate. So: $\mathfrak{u} \in \mathscr{I}(F)$ if and only if $\mathfrak{u}$ has the property denoted by '*F*'. Similarly, think of the set of pairs of entities that $\mathscr{I}$ maps a 2-place predicate to as the set of pairs of entities in $\mathscr{D}$ such that the first bears the relation denoted by the predicate to the second. So: $< \mathfrak{u}, \mathfrak{v} > \in \mathscr{I}(R)$ if and only if $\mathfrak{u}$ bears the relation denoted by '*R*' to $\mathfrak{v}$.

Recall, in propositional logic, our bivalent interpretation could be used to construct a *valuation function*, which mapped us from any arbitrary wff of propositional logic to $\{0, 1\}$. Similarly, in quantificational logic, we will use a *QL*-model to construct a valuation function which maps us from arbitrary wffs of *QL* to $\{0, 1\}$.

Before getting to that, however, we must define the notion of a *variant QL*-model. Given a *QL*-model $\mathcal{M} =< \mathscr{D}, \mathscr{I} >$, a variable $\ulcorner \alpha \urcorner$, and an entity $\mathfrak{u} \in \mathscr{D}$, we may define the variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$ as follows: the domain of $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is identical to the domain of $\mathcal{M}$, and the interpretation function for $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is exactly like the interpretation function for $\mathcal{M}$, *except* that $\mathscr{I}_{\alpha\to\mathfrak{u}}(\alpha) = \mathfrak{u}$. That is: a variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is exactly like the model $\mathcal{M}$, except that, in the variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$, the variable $\ulcorner \alpha \urcorner$ refers to the entity $\mathfrak{u}$.

> Variant *QL*-Model:
> Given a *QL*-model $\mathcal{M} =< \mathscr{D}, \mathscr{I} >$, a variable of *QL* $\ulcorner \alpha \urcorner$, and some $\mathfrak{u} \in \mathscr{D}$, the VARIANT *QL*-MODEL $\mathcal{M}_{\alpha\to\mathfrak{u}} \stackrel{\text{def}}{=}< \mathscr{D}, \mathscr{I}_{\alpha\to\mathfrak{u}} >$, with $\mathscr{I}_{\alpha\to\mathfrak{u}}$ defined as

[1] A point of clarification: $\mathscr{I}(\Pi^N)$ is allowed to be empty.

follows:

$$\mathscr{I}_{\alpha \to \text{u}} \stackrel{\text{def}}{=} (\mathscr{I} - < \alpha, \mathscr{I}(\alpha) >) \cup < \alpha, \text{u} >$$

An alternative, but equivalent, definition of $\mathscr{I}_{\alpha \to \text{u}}$ is given by the following: for any N-place predicate $\ulcorner \Pi^N \urcorner$,

$$\mathscr{I}_{\alpha \to \text{u}}(\Pi^N) = \mathscr{I}(\Pi^N)$$

and, for any term $\ulcorner \tau \urcorner$,

$$\mathscr{I}_{\alpha \to \text{u}}(\tau) = \begin{cases} \mathscr{I}(\tau) & \text{if } \tau \neq \alpha \\ \text{u} & \text{if } \tau = \alpha \end{cases}$$

With this definition in hand, we may provide a definition of a QL-valuation:

QL-Valuation:
Given a QL-model $\mathcal{M} =< \mathscr{D}, \mathscr{I} >$, we define a QL-valuation function, $V_{\mathcal{M}}$, in the following way: for any N-place predicate $\ulcorner \Pi^N \urcorner$, any N terms $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$, any variable $\ulcorner \alpha \urcorner$, and any wffs of QL $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\Pi^N \tau_1 \tau_2 \ldots \tau_N) = 1$ iff $< \mathscr{I}(\tau_1), \mathscr{I}(\tau_2), \ldots, \mathscr{I}(\tau_N) > \in \mathscr{I}(\Pi^N)$.

(2) $V_{\mathcal{M}}(\tau_1 = \tau_2) = 1$ iff $\mathscr{I}(\tau_1) = \mathscr{I}(\tau_2)$.

(3) $V_{\mathcal{M}}(\sim\phi) = 1$ iff $V_{\mathcal{M}}(\phi) = 0$.

(4) $V_{\mathcal{M}}(\phi \to \psi) = 1$ iff $V_{\mathcal{M}}(\phi) = 0$ or $V_{\mathcal{M}}(\psi) = 1$.

(5) $V_{\mathcal{M}}((\forall\alpha)\phi) = 1$ iff, for all $\text{u} \in \mathscr{D}$, $V_{\mathcal{M}_{\alpha \to \text{u}}}(\phi) = 1$.

It's important, in understanding the second clause above, that we clearly distinguish between object- and meta-language uses of the identity sign. In (2) above, the first use of '=' occurs in the object language. There, we are *mentioning* the wff $\ulcorner \tau_1 = \tau_2 \urcorner$. The second use of '=' occurs in the metalanguage. There, we are *using* the identity sign to say that the thing $\mathscr{I}$ maps τ_1 to *is the very same thing as* the thing $\mathscr{I}$ maps τ_2 to.

Given this semantics for QL, we may show that our stipulative definition of $\ulcorner (\exists\alpha)\phi \urcorner$ as $\ulcorner \sim(\forall\alpha)\sim\phi \urcorner$ gives us the familiar truth-conditions for '∃'. First we'll show that, if any arbitrary QL-model $\mathcal{M}$ makes $\ulcorner \sim(\forall\alpha)\sim\phi \urcorner$ true, then there is some $\text{u} \in \mathscr{D}$ such that the variant model $\mathcal{M}_{\alpha \to \text{u}}$ makes $\ulcorner \phi \urcorner$ true.

1.	Suppose that there is an arbitrary QL-model $< \mathscr{D}, \mathscr{I} >$ such that $V_{\mathcal{M}}(\sim(\forall\alpha)\sim\phi) = 1$.	*Assumption*
2.	Then, $V_{\mathcal{M}}((\forall\alpha)\sim\phi) = 0$.	1, *def.* ~
3.	So it is not the case that $V_{\mathcal{M}}((\forall\alpha)\sim\phi) = 1$.	2, *bivalence*
4.	So, it is not the case that, for all $\text{u} \in \mathscr{D}$, $V_{\mathcal{M}_{\alpha \to \text{u}}}(\sim\phi) = 1$.	3, *def.* ∀
5.	So, there is some $\text{u} \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha \to \text{u}}}(\sim\phi) \neq 1$.	4, *QL*
6.	So there is some $\text{u} \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha \to \text{u}}}(\sim\phi) = 0$.	5, *bivalence*

7. So there is some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\phi) = 1$. 6, *def.* ~
8. So, for any QL-model $< \mathscr{D}, \mathscr{I} >$, if $V_{\mathcal{M}}(\sim(\forall\alpha)\sim\phi) = 1$, then there is some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\phi) = 1$. 1–7, → *I*.

Next, we may show that given any arbitrary QL-model $\mathcal{M}$, if there is some $u \in \mathscr{D}$ such that the variant model $\mathcal{M}_{\alpha\to u}$ makes $\ulcorner\phi\urcorner$ true, then $\mathcal{M}$ makes $\ulcorner\sim(\forall\alpha)\sim\phi\urcorner$ true.

1. Suppose that there is an arbitrary QL-model $< \mathscr{D}, \mathscr{I} >$, with some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\phi) = 1$. *Assumption*
2. So there is some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\sim\phi) = 0$. 1, *def.* ~
3. So, there is some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\sim\phi) \neq 1$. 2, *bivalence*
4. So, it is not the case that, for all $u \in \mathscr{D}$, $V_{\mathcal{M}_{\alpha\to u}}(\sim\phi) = 1$. 3, *QL*
5. So it is not the case that $V_{\mathcal{M}}((\forall\alpha)\sim\phi) = 1$. 4, *def.* ∀
6. So $V_{\mathcal{M}}((\forall\alpha)\sim\phi) = 0$. 5, *bivalence*
7. So $V_{\mathcal{M}}(\sim(\forall\alpha)\sim\phi) = 1$. 6, *def.* ~
8. So, for any QL-model $< \mathscr{D}, \mathscr{I} >$, if there is some $u \in \mathscr{D}$ such that $V_{\mathcal{M}_{\alpha\to u}}(\phi) = 1$, then $V_{\mathcal{M}}(\sim(\forall\alpha)\sim\phi) = 1$. 1–7, → *I*, ∀*G*

Putting these together with our stipulative definition above, we have shown that, for any QL-model $\mathcal{M}$,

$$V_{\mathcal{M}}((\exists\alpha)\phi) = 1 \quad \Longleftrightarrow \quad \text{there is some } u \in \mathscr{D} \text{ such that } V_{\mathcal{M}_{\alpha\to u}}(\phi) = 1$$

Now that we have proven this semantic fact, we should feel free to use it in our semantic proofs in the future.

1.1.7. *Consequence for Quantificational Logic.* A wff of QL, $\ulcorner\phi\urcorner$, is a QL-consequence of a set of wffs Γ—or, to say the same thing another way, the argument from Γ to $\ulcorner\phi\urcorner$ is *QL-valid*—written

$$\Gamma \models_{QL} \phi$$

if and only if there is no QL-model $\mathcal{M}$ such that $V_{\mathcal{M}}(\gamma) = 1$, for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi) = 0$. Or, equivalently, iff every QL-model which makes every member of Γ true makes $\ulcorner\phi\urcorner$ true as well.

And a wff of QL, $\ulcorner\phi\urcorner$, is a QL-tautology—or, to say the same thing in another way, $\ulcorner\phi\urcorner$ is *QL-valid*—written

$$\models_{QL} \phi$$

if and only if there is no QL-model $\mathcal{M}$ such that $V_{\mathcal{M}}(\phi) = 0$. Or, equivalently, iff every QL-model makes $\ulcorner\phi\urcorner$ true.

1.1.8. *Establishing Validity in QL.* If we wish to show that an argument or wff of QL is QL-valid, we may provide a semantic proof. For instance, suppose that we wish to show that

$$\{(\forall x)(Fx \to Gx)\} \models_{QL} (\forall x)Fx \to (\forall x)Gx$$

Then, the following semantic proof will suffice.

1.	Suppose that there is some QL-model $< \mathscr{D}, \mathscr{I} >$ such that $V_{\mathcal{M}}((\forall x)(Fx \to Gx)) = 1$ and $V_{\mathcal{M}}((\forall x)Fx \to (\forall x)Gx) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}((\forall x)(Fx \to Gx)) = 1$	1
3.	So, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Fx \to Gx) = 1$.	2, *def.* ∀
4.	So, for all $\mathfrak{u} \in \mathscr{D}$, either $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Fx) = 0$ or $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Gx) = 1$.	3, *def.* →
5.	And $V_{\mathcal{M}}((\forall x)Fx \to (\forall x)Gx) = 0$.	1
6.	So $V_{\mathcal{M}}((\forall x)Fx) = 1$ and $V_{\mathcal{M}}((\forall x)Gx) = 0$.	5, *def.* →
7.	So $V_{\mathcal{M}}((\forall x)Fx) = 1$.	6
8.	So, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Fx) = 1$.	7, *def* ∀
9.	So, for all $\mathfrak{u} \in \mathscr{D}$, it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Fx) = 0$.	8, *bivalence*
10.	So, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Gx) = 1$	4, 9, *DS*
11.	And $V_{\mathcal{M}}((\forall x)Gx) = 0$.	6, ∧*E*
12.	So, it is not the case that $V_{\mathcal{M}}((\forall x)Gx) = 1$.	11, *bivalence*
13.	So, it is not the case that, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(Gx) = 1$.	12, *def.* ∀
14.	Our assumption has led to a contradiction.	10, 13
15.	So it is not the case that there is any QL-model $< \mathscr{D}, \mathscr{I} >$ such that $V_{\mathcal{M}}((\forall x)(Fx \to Gx)) = 1$ and $V_{\mathcal{M}}((\forall x)Fx \to (\forall x)Gx) = 0$.	1–14, ~*I*

Similarly, we may show that

$$\models_{QL} (\exists x)x = x$$

This is a QL-tautology because we require that our domains, $\mathscr{D}$, be non-empty. To show this more rigorously, we may provide a semantic proof like the following.

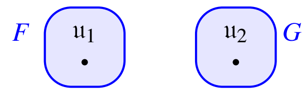
1.	Suppose that there is a QL-model $\mathcal{M} =< \mathscr{D}, \mathscr{I} >$ such that $V_{\mathcal{M}}((\exists x)x = x) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}((\exists x)x = x) = 0$.	1, ∧*E*
3.	So, it is not the case that $V_{\mathcal{M}}((\exists x)x = x) = 1$.	2, *bivalence*
4.	So, it is not the case that, for some $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(x = x) = 1$.	3, *def.* ∃
5.	So, for all $\mathfrak{u} \in \mathscr{D}$, it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(x = x) = 1$.	4, *QL*
6.	So, for all $\mathfrak{u} \in \mathscr{D}$, $\mathscr{I}_{x\to\mathfrak{u}}(x) \neq \mathscr{I}_{x\to\mathfrak{u}}(x)$.	5, *def.* =
7.	There is some $\mathfrak{u} \in \mathscr{D}$—call it '$\mathfrak{u}_1$'	*def. QL-model*
8.	So, $\mathfrak{u}_1 \in \mathscr{D}$.	7
9.	If $\mathfrak{u}_1 \in \mathscr{D}$, then $\mathscr{I}_{x\to\mathfrak{u}_1}(x) \neq \mathscr{I}_{x\to\mathfrak{u}_1}(x)$.	6, ∀*E*
10.	So $\mathscr{I}_{x\to\mathfrak{u}_1}(x) \neq \mathscr{I}_{x\to\mathfrak{u}_1}(x)$.	8, 9, → *E*
11.	So $\mathfrak{u}_1 \neq \mathfrak{u}_1$.	10, *def.* $\mathscr{I}_{x\to\mathfrak{u}_1}$
12.	Our assumption has led to a contradiction.	11
13.	So there is no QL-model $\mathcal{M} =< \mathscr{D}, \mathscr{I} >$ such that $V_{\mathcal{M}}((\exists x)x = x) = 0$.	12, ~*I*

1.1.9. *Establishing Invalidity in QL.* If we wish to show that an argument of QL is QL-invalid, it is enough to provide a single QL-model in which the premises of the argument are true, yet the conclusion is false. For instance, suppose that we wish to show that

$$\{(\forall x)(Fx \vee Gx)\} \not\models_{QL} (\forall x)Fx \vee (\forall x)Gx$$

Then, we may provide the following QL-model:

$$\begin{aligned} \mathscr{D} &= \{\mathfrak{u}_1, \mathfrak{u}_2\} \\ \mathscr{I}(F) &= \{\mathfrak{u}_1\} \\ \mathscr{I}(G) &= \{\mathfrak{u}_2\} \end{aligned}$$



In this model, $V_{\mathcal{M}}((\forall x)(Fx \vee Gx)) = 1$. To see this, note that $\mathfrak{u}_1 \in \mathscr{I}(F)$. Therefore, $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx) = 1$; and, by the definition of '$\vee$', $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx \vee Gx) = 1$. Additionally, $\mathfrak{u}_2 \in \mathscr{I}(G)$. Therefore, $V_{\mathcal{M}_{x \to \mathfrak{u}_2}}(Gx) = 1$; and, by the definition of '$\vee$', $V_{\mathcal{M}_{x \to \mathfrak{u}_2}}(Fx \vee Gx) = 1$. So, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x \to \mathfrak{u}}}(Fx \vee Gx) = 1$. So, by the definition of '$\forall$', $V_{\mathcal{M}}((\forall x)(Fx \vee Gx)) = 1$.

However, in this model, $V_{\mathcal{M}}((\forall x)Fx \vee (\forall x)Gx) = 0$. This is so because both $V_{\mathcal{M}}((\forall x)Fx) = 0$ and $V_{\mathcal{M}}((\forall x)Gx) = 0$. To see that $V_{\mathcal{M}}((\forall x)Fx) = 0$, note that $V_{\mathcal{M}_{x \to \mathfrak{u}_2}}(Fx) = 0$. To see that $V_{\mathcal{M}}((\forall x)Gx) = 0$, note that $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Gx) = 0$. Then, by the definition of '$\vee$', $V_{\mathcal{M}}((\forall x)Fx \vee (\forall x)Gx) = 0$.

So this model demonstrates that $\{(\forall x)(Fx \vee Gx)\} \not\models_{QL} (\forall x)Fx \vee (\forall x)Gx$.

1.2. **Axiomatization of Quantificational Logic.** To provide an axiomatic system for quantificational logic, we will not, as we did with propositional logic, introduce a set of *axioms*; rather, we will introduce a set of *axiom schemata*. An axiom schema contains metavariables like 'ϕ' and 'α' which range over the vocabulary of QL. To accept the following schema ($\forall$1) as an axiom schema means that, at any point in an axiomatic proof, we may write down the result of going through ($\forall$1) and replacing each occurrence of 'ϕ' with a wff of QL, each occurrence of 'τ' with a term of QL, and each occurrence of 'α' with a variable of QL:

$$\vdash_{QL} \phi, \quad \text{for all } PL\text{-valid schemata } \ulcorner \phi \urcorner \tag{PL}$$

$$\vdash_{QL} (\forall \alpha)\phi \to \phi[\tau/\alpha] \tag{$\forall$1}$$

provided that $\ulcorner \tau \urcorner$ *is free in* $\ulcorner \phi[\tau/\alpha] \urcorner$

$$\vdash_{QL} (\forall \alpha)(\phi \to \psi) \to (\phi \to (\forall \alpha)\psi) \tag{$\forall$2}$$

provided that $\ulcorner \alpha \urcorner$ *is not free in* $\ulcorner \phi \urcorner$

$$\vdash_{QL} \tau = \tau \tag{=1}$$

$$\vdash_{QL} \tau_1 = \tau_2 \to (\phi \to \phi[\tau_2 // \tau_1]) \tag{=2}$$

provided that $\ulcorner \tau_2 \urcorner$ *is free in* $\ulcorner \phi[\tau_2 // \tau_1] \urcorner$

In ($\forall$1), $\ulcorner \phi[\tau/\alpha] \urcorner$ refers to the result of going through the wff $\ulcorner \phi \urcorner$ and uniformly replacing every occurrence of $\ulcorner \alpha \urcorner$ with $\ulcorner \tau \urcorner$. For instance, ($\forall$1) allows us to write down '$(\forall y)Py \to Pc$', '$(\forall x)(\forall y)Rxy \to (\forall y)Rey$' and '$(\forall x)(Fx \vee Gx) \to (Fx \vee Gx)$' at any point in a QL axiomatic proof.

The proviso requiring that the substituted term $\ulcorner \tau \urcorner$ be free in $\ulcorner \phi[\tau/\alpha] \urcorner$ is important. Without this proviso, it would be an axiom that

$$\vdash_{QL} (\forall x)(\exists y)Rxy \to (\exists y)Ryy$$

But this is not, and ought not be, a tautology of *QL*; just because everything bears a relation to something, this doesn't mean that something bears that relation to itself.

(*PL*) allows us to write down any substitution instance of a *PL*-valid schema. For instance, *PL* tells us that, for any $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$, $\ulcorner \phi \to (\psi \to \phi) \urcorner$ is a theorem; so (*PL*) tells us that, *e.g.*,

$$\vdash_{QL} (\forall x)(\forall y)Rxy \to (\forall zHzz \to (\forall x)(\forall y)Rxy)$$

($\forall$2) allows us to write down, *e.g.*,

$$\vdash_{QL} (\forall x)(Fa \to Gx) \to (Fa \to (\forall x)Gx)$$

It also allows us to write down, *e.g.*,

$$\vdash_{QL} (\forall x)((\forall x)Fx \to Gx) \to ((\forall x)Fx \to (\forall x)Gx)$$

The restriction that $\ulcorner \alpha \urcorner$ not appear free in $\ulcorner \phi \urcorner$ is important. For instance, ($\forall$2) does *not* allow us to write down

$$(\forall x)(Fx \to Gx) \to (Fx \to (\forall x)Gx)$$

And this is good, since the above wff is not a tautology, given our semantics for *QL*. (Can you think of a counter-model?)

(=1) allows us to write down, at any point, as theorems, things like '$x = x$' or '$a = a$'.

In (=2), $\ulcorner \phi[\tau_2 // \tau_1] \urcorner$ refers to the result of going through $\ulcorner \phi \urcorner$ and replacing *some*—though not necessarily *all*—of the occurrences of $\ulcorner \tau_1 \urcorner$ with $\ulcorner \tau_2 \urcorner$. For instance, (=2) allows us to write down '$a = b \to (Fa \to Fb)$' and '$c = d \to ((Dc \wedge Ec) \to (Dd \wedge Ec))$' at any point in a *QL* axiomatic proof.

The proviso requiring that the substituted term $\ulcorner \tau_2 \urcorner$ be free in $\ulcorner \phi[\tau_2 // \tau_1] \urcorner$ is important. Without this proviso, it would be an axiom that

$$\vdash_{QL} x = y \to ((\forall y)y = y \to (\forall y)y = x)$$

But then, since $\vdash_{QL} (\forall y)y = y$, it would turn out that $\{x = y\} \vdash_{QL} (\forall y)y = x$. But this is not a valid argument in *QL*, since the thing denoted by 'x' could be identical to the thing denoted by 'y' without *everything* being identical to the thing denoted by 'x'.

In addition to these axiom schemata, we introduce the following rules of inference:

> *Propositional Logic Rules* (*PLR*): If $\ulcorner \psi \urcorner$ follows from $\ulcorner \phi \urcorner$ according to propositional logic, then, from $\ulcorner \phi \urcorner$, infer $\ulcorner \psi \urcorner$.
>
> *Generalization* (*G*): If a wff $\ulcorner \phi \urcorner$ is a theorem of *QL*, then you may infer $\ulcorner (\forall \alpha)\phi \urcorner$ as a theorem of *QL*, where $\ulcorner \alpha \urcorner$ is a variable of *QL*.
>
> $$\text{from } \vdash_{QL} \phi, \text{ infer } \vdash_{QL} (\forall \alpha)\phi$$

Then, here is a QL axiomatic proof establishing that '$(\forall x)Fx$' is equivalent to '$\sim(\exists x)\sim Fx$':

1.	$\vdash_{QL} (\forall x)Fx \to Fx$	$(\forall 1)$
2.	$\vdash_{QL} (\forall x)Fx \to \sim\sim Fx$	1 (PLR)
3.	$\vdash_{QL} (\forall x)((\forall x)Fx \to \sim\sim Fx)$	2, (G)
4.	$\vdash_{QL} (\forall x)((\forall x)Fx \to \sim\sim Fx) \to ((\forall x)Fx \to (\forall x)\sim\sim Fx)$	$(\forall 2)$
5.	$\vdash_{QL} (\forall x)Fx \to (\forall x)\sim\sim Fx$	3, 4, (PLR)
6.	$\vdash_{QL} (\forall x)Fx \to \sim\sim(\forall x)\sim\sim Fx$	5, (PLR)
7.	$\vdash_{QL} (\forall x)\sim\sim Fx \to \sim\sim Fx$	$(\forall 1)$
8.	$\vdash_{QL} (\forall x)\sim\sim Fx \to Fx$	7, (PLR)
9.	$\vdash_{QL} \sim\sim(\forall x)\sim\sim Fx \to Fx$	8, (PLR)
10.	$\vdash_{QL} (\forall x)Fx \leftrightarrow \sim\sim(\forall x)\sim\sim Fx$	6, 9, (PLR)

And here is one demonstrating that '$(\forall y)(Fy \to Fy)$' is a theorem:

1.	$\vdash_{QL} Fy \to Fy$	(PL)
2.	$\vdash_{QL} (\forall y)(Fy \to Fy)$	1, (G)

Here's one that shows that '$(\forall x)Rxx \leftrightarrow (\forall z)Rzz$' is a theorem:

1.	$\vdash_{QL} (\forall x)Rxx \to Rzz$	$(\forall 1)$
2.	$\vdash_{QL} (\forall z)((\forall x)Rxx \to Rzz)$	1, (G)
3.	$\vdash_{QL} (\forall z)((\forall x)Rxx \to Rzz) \to ((\forall x)Rxx \to (\forall z)Rzz)$	$(\forall 2)$
4.	$\vdash_{QL} (\forall x)Rxx \to (\forall z)Rzz$	2, 3 (PLR)
5.	$\vdash_{QL} (\forall z)Rzz \to Rxx$	$(\forall 1)$
6.	$\vdash_{QL} (\forall x)((\forall z)Rzz \to Rxx)$	5, (G)
7.	$\vdash_{QL} (\forall x)((\forall z)Rzz \to Rxx) \to ((\forall z)Rzz \to (\forall x)Rxx)$	$(\forall 2)$
8.	$\vdash_{QL} (\forall z)Rzz \to (\forall x)Rxx$	6, 7 (PLR)
9.	$\vdash_{QL} (\forall x)Rxx \leftrightarrow (\forall z)Rzz$	4, 8 (PLR)

Fact: For any set of wffs of QL, Γ, and any wff of QL, $\ulcorner \phi \urcorner$,

$$\Gamma \vdash_{QL} \phi \quad \text{if and only if} \quad \Gamma \models_{QL} \phi$$

1.3. **Natural Deduction for Quantificational Logic.** Axiomatic systems are easy to prove things about, but they are not particularly easy to prove things *in*. We will therefore introduce a natural deduction system for QL. To achieve this natural deduction system, we will take our natural deduction system for PL and add to it some additional rules of inference.

Firstly: we have eight rules which tell us, collectively, that pushing negations inside of quantifiers (or pulling them outside of quantifiers) flips universal quantifiers to existential quantifiers, and flips existential quantifiers to universal quantifiers.

Quantifier Negation (QN)

$(\forall\alpha)\phi$	$\lhd\rhd$	$\sim(\exists\alpha)\sim\phi$
$\sim(\forall\alpha)\phi$	$\lhd\rhd$	$(\exists\alpha)\sim\phi$
$(\exists\alpha)\phi$	$\lhd\rhd$	$\sim(\forall\alpha)\sim\phi$
$\sim(\exists\alpha)\phi$	$\lhd\rhd$	$(\forall\alpha)\sim\phi$

Using these rules, we may prove that '$(\forall x)Fx \leftrightarrow \sim(\exists x)\sim Fx$' a theorem of this natural deduction system.

1	$\quad(\forall x)Fx$	$A(\leftrightarrow I)$
2	$\quad\sim(\exists x)\sim Fx$	1, *QN*
3	$\quad\sim(\exists x)\sim Fx$	$A(\leftrightarrow I)$
4	$\quad(\forall x)Fx$	3, *QN*
5	$(\forall x)Fx \leftrightarrow \sim(\exists x)\sim Fx$	1–2, 3–4, $\leftrightarrow I$

The next rule tells us that, if we have a universally quantified wff written down on an accessible line, then you may remove the universal quantifier and go through the wff which remains, uniformly substituting all variables which the quantifier previously bound with any constant or free variable.

Universal Elimination (∀E)

$$(\forall\alpha)\phi$$
$$\rhd \quad \phi[\beta/\alpha]$$

where $\ulcorner\beta\urcorner$ is a constant; or:

$$(\forall\alpha)\phi$$
$$\rhd \quad \phi[\zeta/\alpha]$$

where $\ulcorner\zeta\urcorner$ is a variable—*provided that* $\ulcorner\zeta\urcorner$ is free in $\ulcorner\phi[\zeta/\gamma]\urcorner$.

In the above, $\ulcorner\phi[\beta/\alpha]\urcorner$ refers to the result of going through $\ulcorner\phi\urcorner$ and replacing every occurrence of $\ulcorner\alpha\urcorner$ with $\ulcorner\beta\urcorner$. This is known as a *substitution instance* of $\ulcorner\phi\urcorner$. When you use ∀*E*, you must be sure that you replace *every* occurrence of the bound variable with *the same* constant or *the same* free variable. Otherwise, what we write down won't be a substitution instance of the wff we started with.

For instance, the following *QL*-derivation is *not* legal,

$$\begin{array}{lll} 1 & (\forall y)(Fy \rightarrow Gy) & \\ 2 & Fa \rightarrow Gb & 1, \forall E \qquad \longleftarrow \textbf{MISTAKE!!!} \end{array}$$

for we replaced with first bound 'y' with 'a', and the second bound 'y' with 'b'. Similarly, the following derivation is not legal,

$$\begin{array}{lll} 1 & \forall z(Az \leftrightarrow Bz) & \\ 2 & Aa \leftrightarrow Bx & 1, \forall E \qquad \longleftarrow \textbf{MISTAKE!!!} \end{array}$$

for we replaced the first bound 'z' with 'a', and the second bound 'z' with 'x'.

These QL-derivations, on the other hand, *are* legal.

$$\begin{array}{lll} 1 & (\forall y)(Fy \rightarrow Gy) & \\ 2 & Fa \rightarrow Ga & 1, \forall E \end{array}$$

$$\begin{array}{lll} 1 & \forall z(Az \leftrightarrow Bz) & \\ 2 & Ax \leftrightarrow Bx & 1, \forall E \end{array}$$

The proviso requiring that the instantiated variable $\ulcorner \zeta \urcorner$ be free in $\ulcorner \phi[\zeta/\alpha] \urcorner$ is important. For instance, the following derivation is not legal:

$$\begin{array}{lll} 1 & (\forall y)(\exists x)Lyx & \\ 2 & (\exists x)\, Lxx & 1, \forall E \qquad \longleftarrow \textbf{MISTAKE!!!} \end{array}$$

Line 2 does not follow from line 1 because the instantiated variable, 'x', is not free in line 2. It is bound by the existential quantifier. So $\forall E$ does not allow us to write down '$(\exists x)\, Lxx$', given the wff '$(\forall y)(\exists x)Lyx$'. It's a good thing that our derivation system does not allow this, for '$(\exists x)Lxx$' does not follow from '$(\forall y)(\exists x)Lyx$'. Consider any *QL*-model like the following:

$$\begin{aligned} \mathscr{D} &= \{u_1, u_2\} \\ \mathscr{I}(L) &= \{< u_1, u_2 >, < u_2, u_1 >\} \end{aligned}$$

Because everything in the domain bears the relation L to something in the domain, '$(\forall y)(\exists x)Lyx$' is true on this interpretation. However, since nothing bears the relation L to itself, '$(\exists x)Lxx$' is false on this interpretation. Hence, $\{(\forall y)(\exists x)Lyx\} \nvDash_{QL} (\exists x)Lxx$.

The next rule tells us that, if you have a wff of *QL* according to which some *particular* thing $\ulcorner \beta \urcorner$ has a certain property, then you may infer that *something* has that property. That is, if you have a substitution instance of an existentially quantified wff, then you may write down that existentially quantified wff.

Existential Introduction ($\exists I$)

$$\begin{array}{r|l} & \phi[\beta/\alpha] \\ \triangleright & (\exists\alpha)\phi \end{array}$$

where ⌜β⌝ is a constant; or:

$$\begin{array}{r|l} & \phi[\zeta/\alpha] \\ \triangleright & (\exists\alpha)\phi \end{array}$$

where ⌜ζ⌝ is a variable.

For instance, the following are legal *QL*-derivations.

$$\begin{array}{r|ll} 1 & (\forall x)(\forall y)Bxy & \\ \hline 2 & (\forall y)Bay & 1, \forall E \\ 3 & (\exists z)(\forall y)Bzy & 2, \exists I \end{array}$$

$$\begin{array}{r|ll} 1 & (\forall x)(Px \leftrightarrow Qx) & \\ \hline 2 & Pz \leftrightarrow Qz & 1, \forall E \\ 3 & (\exists y)(Py \leftrightarrow Qy) & 2, \exists I \end{array}$$

$$\begin{array}{r|ll} 1 & (\forall x)Acx & \\ \hline 2 & Acc & 1, \forall E \\ 3 & (\exists x)Axc & 2, \exists I \end{array}$$

A potential confusion: when you *instantiate* a variable by writing down a substitution instance of a quantified wff of *QL*, you must replace every instance of the bound variable with the same term. Thus, the derivation below is *not* legal:

$$\begin{array}{r|lll} 1 & (\forall x)Rxxxx & & \\ \hline 2 & Raxxx & 1, \forall E & \longleftarrow \textbf{MISTAKE!!!} \end{array}$$

For line 2 is not a substitution instance of line 1 (all of the bound '*x*'s must be replaced with the same term in order for it to be a substitution instance).

However, when you *existentially generalize* from a substitution instance of a quantified wff to that quantified wff, you *needn't* replace every instance of the term from which you are generalizing. Thus, the derivation below *is* legal:

1	$Raaaa$	
2	$(\exists x)Rxaaa$	1, $\exists I$
3	$(\exists y)(\exists x)Rxyaa$	2, $\exists I$
4	$(\exists z)(\exists y)(\exists x)Rxyza$	3, $\exists I$

That's because line 1 is a substitution instance of line 2, line 2 is a substitution instance of line 3, and line 3 is a substitution instance of line 4.

The next new rule of inference says that, if you have an existentially quantified wff, $\ulcorner(\exists\alpha)\phi\urcorner$, then, if you begin a new subderivation starting with the assumption that you get by peeling off the quantifier and uniformly replacing all previously bound variables with a constant, *i.e.*, $\ulcorner\phi[\beta/\alpha]\urcorner$, and from this assumption, you are able to derive $\ulcorner\psi\urcorner$, then you may conclude that $\ulcorner\psi\urcorner$ outside of the scope of your subderivation—provided that the constant $\ulcorner\beta\urcorner$ that you introduce is entirely new (it doesn't appear on any previous line), and provided that it does not show up anywhere in $\ulcorner\psi\urcorner$.

Existential Elimination ($\exists E$)

$$\begin{array}{l|l} & (\exists\alpha)\phi \\ & \quad \phi[\beta/\alpha] \\ & \quad \vdots \\ & \quad \psi \\ \triangleright & \psi \end{array}$$

where $\ulcorner\beta\urcorner$ is a constant.
provided that:

(1) $\ulcorner\beta\urcorner$ does not appear on any previous line
(2) $\ulcorner\beta\urcorner$ does not appear in $\ulcorner\psi\urcorner$

It is important to keep these provisions in mind. The idea behind this rule is that, if you know that *there is* something which has a certain property, then it's o.k. to give that thing a name. *However*, you don't want to assume anything about this thing *other* than that it has the property. So you'd better give it an entirely *new* name; otherwise, you'd be assuming more about the thing than that it has the property. Similarly, you'd better get rid of the name before you leave your subderivation, since, in a *QL*-model, that name has a meaning—it refers to something in the domain $\mathscr{D}$. You don't know what that thing is, so leaving it behind at the end of the subderivation would allow you to conclude more than you know.

The following derivation is *not* legal:

1	$(\exists y)(Dy \leftrightarrow (He \vee Jy))$		
2	$\quad De \leftrightarrow (He \vee Je)$	$A(\exists E)$	
3	$\quad (\exists x)(Dx \leftrightarrow (Hx \vee Jx))$	2, $\exists I$	
4	$(\exists x)(Dx \leftrightarrow (Hx \vee Jx))$	1, 2–3, $\exists E$	⟵ **MISTAKE!!!**

The constant 'e' appears on line 1; so while the assumption made at line 2 is a substitution instance of '$(\exists y)(Dy \leftrightarrow (He \vee Jy))$', it does not utilize an entirely new name. So we may not use $\exists E$ on line 4.

This derivation, however, *is* legal:

1	$(\exists y)(Dy \leftrightarrow (He \vee Jy))$	
2	$\quad Da \leftrightarrow (He \vee Ja)$	$A(\exists E)$
3	$\quad (\exists z)(Dz \leftrightarrow (He \vee Jz))$	2, $\exists I$
4	$(\exists z)(Dz \leftrightarrow (He \vee Jz))$	1, 2–3, $\exists E$

Similarly, the following derivation is not legal:

1	$(\forall y)(Fy \rightarrow Ky)$		
2	$(\exists x)(Fx \wedge Qx)$		
3	$\quad Fe \wedge Qe$	$A(\exists E)$	
4	$\quad Fe$	3, $\wedge E$	
5	$\quad Fe \rightarrow Ke$	1, $\forall E$	
6	$\quad Ke$	4, 5, $\rightarrow E$	
7	Ke	2, 3–6, $\exists E$	⟵**MISTAKE!!!**

The constant e was a new constant introduced on line 3 for the purposes of existential elimination; however, it appears on the final line of the subderivation running from lines 3–6. $\exists E$, however, only allows you to remove a wff from a $\exists E$ subderivation if the instantiated constant does not appear in that wff.

This derivation, on the other hand, *is* legal:

1	$(\forall y)(Fy \to Ky)$	
2	$(\exists x)(Fx \land Qx)$	
3	$\quad Fe \land Qe$	$A(\exists E)$
4	$\quad Fe$	3, $\land E$
5	$\quad Fe \to Ke$	1, $\forall E$
6	$\quad Ke$	4, 5, $\to E$
7	$\quad (\exists x)Kx$	6, $\exists I$
8	$(\exists x)Kx$	2, 3–7, $\exists E$

Here is a sample *QL*-derivation:

1	$(\forall x)(Fx \land (\exists y)Gy)$	
2	$Fa \land (\exists y)Gy$	1, $\forall E$
3	$(\exists y)Gy$	2, $\land E$
4	$\quad Gc$	$A(\exists E)$
5	$\quad Fc \land (\exists y)Gy$	1, $\forall E$
6	$\quad Fc$	5, $\land E$
7	$\quad Fc \land Gc$	4, 6, $\land I$
8	$\quad (\exists z)(Fz \land Gz)$	7, $\exists I$
9	$(\exists z)(Fz \land Gz)$	3, 4–8, $\exists E$

The next rule says that, if you have a wff of *QL* in which a variable $\ulcorner \zeta \urcorner$ occurs freely, then you may *uniformly* replace it with a(nother) variable and tack on a universal quantifier out front—*provided that* $\ulcorner \zeta \urcorner$ does not occur free in either the assumptions or the first line of any accessible subderivation.

<u>*Universal Introduction* (∀*I*)</u>

$$\phi[\zeta / \alpha]$$

$$\triangleright \quad (\forall \alpha)\phi$$

where ⌜ζ⌝ is a variable.
provided that:

(1) ⌜ζ⌝ does not occur free in the assumptions

(2) ⌜ζ⌝ does not occur free in the first line of an open subderivation.

(3) ⌜ζ⌝ does not occur free in $(\forall \alpha)\phi$.

Again, it is important to keep this provisions in mind. Let us begin with the final provision. There is an important difference between *existential* introduction and *universal* introduction. With existential introduction, you are allowed to leave behind occurrences of the variable from which you existentially generalize. That is, derivations like the following are allowed:

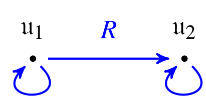
1	$(\forall z)Rzz$	
2	Rxx	1, $\forall E$
3	$(\exists z)Rxz$	2, $\exists I$

However, provision 3 above tells us that this is not allowed with *universal* introduction. The following derivation is *not* legal:

1	$(\forall z)Rzz$		
2	Rxx	1, $\forall E$	
3	$(\forall z)Rxz$	2, $\forall I$	⟵ **MISTAKE!!!**

This is very good, because $(\forall z)Rxz$ does not follow from $(\forall z)Rzz$. There are *QL*-models in which the first wff is true while the second is false. For instance, consider any *QL*-model for which:

$$\begin{aligned} \mathscr{D} &= \{u_1, u_2\} \\ \mathscr{I}(R) &= \{< u_1, u_1 >, < u_1, u_2 >, < u_2, u_2 >\} \\ \mathscr{I}(x) &= u_2 \end{aligned}$$



Both u_1 and u_2 bear the relation R to themselves, so $(\forall z)Rzz$ is true in this *QL*-model. However, u_2 does not bear the relation R to u_1, so $(\forall z)Rxz$ is false in this *QL*-model.

For another case in which failure to abide by provision 3 would lead us into trouble, consider the following *QL* derivation:

1	Rzz	$A(\rightarrow I)$	
2	Rzz	1, R	
3	$Rzz \rightarrow Rzz$	1–2, $\rightarrow I$	
4	$(\forall y)(Rzy \rightarrow Ryz)$	3, $\forall I$	⟵ **MISTAKE!!!**
5	$(\forall x)(\forall y)(Rxy \rightarrow Ryx)$	4, $\forall I$	

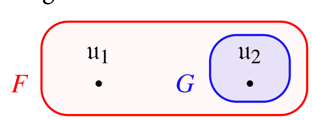
Line 4 does not follow from line 3, because occurrences of the variable z were left behind. And this is good. If this derivation were legal, then we would falsely conclude that it is a QL-tautology that every two-place relation of QL is symmetric. But that is not a QL-tautology, as the QL-model above shows ($\mathfrak{u}_1$ bears the relation R to $\mathfrak{u}_2$, but $\mathfrak{u}_2$ does not bear the relation R to $\mathfrak{u}_1$).

Provision 1 tells us that the following derivation is not legal:

1	$(\forall x)(Fx \rightarrow Gy)$		
2	$(\forall y)Fy$		
3	$Fc \rightarrow Gy$	1, $\forall E$	
4	Fc	2, $\forall E$	
5	Gy	3, 4, $\rightarrow E$	
6	$(\forall x)Gx$	5, $\forall I$	⟵ **MISTAKE!!!**

The variable y appears free in one of the assumptions of the derivation. Therefore, we may not universally generalize from that variable. This is a good thing, too, for $(\forall x)Gx$ does not follow from '$(\forall x)(Fx \rightarrow Gy)$' and '$(\forall y)Fy$'. There are QL-models in which the premises are true yet the conclusion is false. For instance, any QL-model like the following provides a QL-counterexample to the QL-validity of this argument:

$$\begin{aligned} \mathscr{D} &= \{\mathfrak{u}_1, \mathfrak{u}_2\} \\ \mathscr{I}(F) &= \{\mathfrak{u}_1, \mathfrak{u}_2\} \\ \mathscr{I}(G) &= \{\mathfrak{u}_2\} \\ \mathscr{I}(y) &= \mathfrak{u}_2 \end{aligned}$$



Had we stopped at line 5, on the other hand, our derivation *would* be legal.

1	$(\forall x)(Fx \rightarrow Gy)$	
2	$(\forall y)Fy$	
3	$Fc \rightarrow Gy$	1, $\forall E$
4	Fc	2, $\forall E$
5	Gy	3, 4, $\rightarrow E$

For an example in which provision 2 is violated, consider the following derivation:

1	Fx	$A(\rightarrow I)$	
2	$(\forall y)Fy$	1, $\forall G$	$\longleftarrow$ **MISTAKE!!!**
3	$Fx \rightarrow (\forall y)Fy$	1–2, $\rightarrow I$	
4	$(\forall z)(Fz \rightarrow (\forall y)Fy)$	3, $\forall G$	

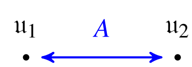
Line 2 does not follow from line 1, since the variable x appears free in the assumption of an accessible subderivation (the one starting at line 1). (Good thing, too, since '$(\forall z)(Fz \rightarrow (\forall y)Fy)$' is false in any QL-model in which one thing is F and another is not F—so it is not a QL-tautology.)

For another example in which provision 2 is violated, consider the following derivation:

1	$(\forall x)(\exists y)Axy$		
2	$(\exists y)Azy$	1, $\forall I$	
3	Azc	$A(\exists E)$	
4	$(\forall x)Axc$	3, $\forall G$	$\longleftarrow$ **MISTAKE!!!**
5	$(\exists y)(\forall x)Axy$	4, $\exists I$	
6	$(\exists y)(\forall x)Axy$	2, 3–5, $\exists E$	

Line 4 does not follow from line 3, since, at line 4, the variable z appears free in the assumption of an open subderivation. This is a good thing, too, since '$(\exists y)(\forall x)Axy$' doesn't follow from '$(\forall x)(\exists y)Axy$'—there are QL-models on which the first is true but the second false. For instance, consider any QL-model like the following:

$$\mathscr{D} = \{u_1, u_2\}$$
$$\mathscr{I}(A) = \{< u_1, u_2 >, < u_2, u_1 >\}$$



It is also important to note that ∀I only allows you to universally generalize from *variables*. It does not allow you to universally generalize from *constants*. Thus, the following derivation is not legal:

1	$(\forall x)(Yx \wedge Zx)$		
2	$Ya \wedge Za$	1, $\forall I$	
3	Ya	2, $\wedge E$	
4	$(\forall x)Yx$	3, $\forall G$	$\longleftarrow$ **MISTAKE!!!**

This derivation, however, *is* legal.

1	$(\forall x)(Yx \wedge Zx)$	
2	$Yy \wedge Zy$	1, $\forall I$
3	Yy	2, $\wedge E$
4	$(\forall x)Yx$	3, $\forall G$

Finally, we introduce a single rule for the identity relation—known as *Identity*—which permits two new inferences.

<u>*Identity* (*Id*)</u>

$$\rhd \mid \quad \tau = \tau$$

for any term $\ulcorner \tau \urcorner$

$$\phi[\tau_1]$$
$$\tau_2 = \tau_1 (\text{ or } \tau_1 = \tau_2)$$
$$\rhd \quad \phi[\tau_2 // \tau_1]$$

for any terms $\ulcorner \tau_1 \urcorner$, $\ulcorner \tau_2 \urcorner$— *provided that*, if $\ulcorner \tau_2 \urcorner$ is a variable, then it occurs free in $\ulcorner \phi[\tau_2 // \tau_1] \urcorner$

Identity allows us to do two things. Firstly, we may, *whenever we wish*, write down an identity claim on which the identity sign is flanked by the same term of *QL* on both sides. When we do so, we should write '*Id*' on the justification line—though we needn't cite any other line of the derivation when we do so.

Note that this means that, with *Identity*, we can prove tautologies without ever having to start a subderivation. For instance, the following one-line derivation establishes that '$a = a$' is a *QL*-tautology:

1 $\quad a = a \quad$ *Id*

Similarly, the following derivation establishes that '$(\exists x)x = x$' is a *QL*-tautology:

1	$a = a$	*Id*
2	$(\exists x)x = x$	1, $\exists I$

And the following derivation establishes that '$(\forall x)x = x$' is a *QL*-tautology:

1	$z = z$	*Id*
2	$(\forall x)x = x$	1, $\forall I$

We may utilize $\forall I$ here because, even though z appears free on the first line of the derivation, it does not appear free in the derivation's *assumptions* (because the derivation has no assumptions).

Because '$(\forall x)x = x$' is a *QL*-tautology, this tells us that identity is a *reflexive* relation.

Identity tells us that, if we have a wff of *QL*, $\ulcorner \phi[\tau_1] \urcorner$, in which a term $\ulcorner \tau_1 \urcorner$ appears, and we have a wff of *QL* of the form $\ulcorner \tau_1 = \tau_2 \urcorner$, then we may replace some or all of the occurrences of $\ulcorner \tau_1 \urcorner$ in $\ulcorner \phi[\tau_1] \urcorner$ with the term $\ulcorner \tau_2 \urcorner$—***so long as*** $\ulcorner \tau_2 \urcorner$ ***doesn't end up getting bound by a quantifier when we do so***.

This provision is important. The following derivation is not legal:

1	$(\exists y)Fxy$		
2	$x = y$		
3	$(\exists y)Fyy$	1, 2, *Id*	$\longleftarrow$ **MISTAKE!!!**

However, the following derivation *is* legal:

1	$(\exists y)Fxy$	
2	$x = z$	
3	$(\exists y)Fzy$	1, 2, *Id*

The following is also a legal derivation:

1	$x = y$	$A(\rightarrow I)$
2	$x = x$	*Id*
3	$y = x$	1, 2 *Id*
4	$x = y \rightarrow y = x$	1–3, $\rightarrow I$
5	$(\forall y)(x = y \rightarrow y = x)$	4, $\forall I$
6	$(\forall x)(\forall y)(x = y \rightarrow y = x)$	5, $\forall I$

We may use $\forall I$ on lines 4 and 5 because, even though 'x' and 'y' both appear free in the assumption of the subderivation running from lines 1–2, that subderivation is no longer open at lines 4 and 5.

Thus, we may conclude that '$(\forall x)(\forall y)(x = y \rightarrow y = x)$' is a tautology of *QL*. This tells us that identity, $=$, is a *symmetric* relation.

The following *QL* derivation is also legal:

1	$\quad \vert\ x = y \wedge y = z$	$A(\rightarrow I)$
2	$\quad \vert\ x = y$	1, $\wedge E$
3	$\quad \vert\ y = z$	1, $\wedge E$
4	$\quad \vert\ x = z$	2, 3, *Id*
5	$(x = y \wedge y = z) \rightarrow x = z$	1–4, $\rightarrow I$
6	$(\forall z)((x = y \wedge y = z) \rightarrow x = z)$	5, $\forall I$
7	$(\forall y)(\forall z)((x = y \wedge y = z) \rightarrow x = z)$	6, $\forall I$
8	$(\forall x)(\forall y)(\forall z)((x = y \wedge y = z) \rightarrow x = z)$	7, $\forall I$

We may use $\forall I$ on lines 7, 8, and 9 because, even though x, y, and z all appear free in the assumption of the subderivation running from lines 1–5, that subderivation is no longer open at lines 7, 8, and 9.

Thus, we may conclude that '$(\forall x)(\forall y)(\forall z)((x = y \wedge y = z) \rightarrow x = z)$' is a tautology of *QL*. This tells us that identity is a *transitive* relation.

Though there are two *Identity* rules, you will always know which is being invoked by the number of lines cited. If an application of *Identity* cites no lines, then the first rule is being invoked. If it cites two lines, then the second rule is being invoked.

If we may derive $\ulcorner \phi \urcorner$ from the set of wffs Γ in this derivation system, then we will write

$$\Gamma \vdash_{QD} \phi$$

And if we may derive $\ulcorner \phi \urcorner$ from no assumptions in this derivation system, then we will write

$$\vdash_{QD} \phi$$

Fact: For any set of wffs of *QL* Γ and any wff of *QL* $\ulcorner \phi \urcorner$,

$$\Gamma \vdash_{QD} \phi \quad \text{if and only if} \quad \Gamma \vDash_{QL} \phi$$

Here is a *QL*-derivation establishing that

$$\{(\forall x)(x = c \rightarrow Nx)\} \vdash_{QD} Nc$$

1	$(\forall x)(x = c \rightarrow Nx)$	
2	$c = c \rightarrow Nc$	1, $\forall E$
3	$c = c$	*Id*
4	Nc	2, 3, $\rightarrow E$

And here is a *QL*-derivation showing that

$$\{\sim(\forall x)(Ax \land Bx)\} \vdash_{QD} (\exists y)(Ay \rightarrow \sim By)$$

1	$\sim(\forall x)(Ax \land Bx)$	
2	$(\exists x)\sim(Ax \land Bx)$	1, *QN*
3	$\quad \sim(Ac \land Bc)$	$A(\exists E)$
4	$\quad\quad Ac$	$A(\rightarrow I)$
5	$\quad\quad\quad Bc$	$A(\sim I)$
6	$\quad\quad\quad Ac \land Bc$	4, 5 $\land I$
7	$\quad\quad\quad (Ac \land Bc) \land \sim(Ac \land Bc)$	6, 3 $\land I$
8	$\quad\quad \sim Bc$	5–7, $\sim I$
9	$\quad Ac \rightarrow \sim Bc$	4–8, $\rightarrow I$
10	$\quad (\exists y)(Ay \rightarrow \sim By)$	9, $\exists$I
11	$(\exists y)(Ay \rightarrow \sim By)$	2, 3–10, $\exists E$

Here is one demonstrating that

$$\{(\forall z)(Fz \land Gz), (\forall x)Fx \rightarrow (\exists y)Qy, (\forall x)Hx \rightarrow (\forall y)\sim Qy\} \vdash_{QD} (\exists x)\sim Hx$$

1	$(\forall z)(Fz \wedge Gz)$	
2	$(\forall x)Fx \rightarrow (\exists y)Qy$	
3	$(\forall x)Hx \rightarrow (\forall y){\sim}Qy$	
4	$Fy \wedge Gy$	1, $\forall E$
5	Fy	4, $\wedge E$
6	$(\forall x)Fx$	5, $\forall I$
7	$(\exists y)Qy$	2, 6, $\rightarrow E$
8	$\quad(\forall x)Hx$	$A({\sim}I)$
9	$\quad(\forall y){\sim}Qy$	3, 8, $\rightarrow E$
10	$\quad{\sim}(\exists y)Qy$	9, QN
11	$\quad(\exists y)Qy \wedge {\sim}(\exists y)Qy$	7, 10 $\wedge I$
12	${\sim}(\forall x)Hx$	8–11, ${\sim}I$
13	$(\exists x){\sim}Hx$	12, QN

This derivation establishes that

$$\{Haa \rightarrow Waa, Hab, a = b\} \vdash_{QD} Wab$$

1	$Haa \rightarrow Waa$	
2	Hab	
3	$a = b$	
4	$Hab \rightarrow Wab$	1, 3, Id
5	Wab	2, 4, $\rightarrow E$

This derivation shows that

$$\{(\forall x)x = a, (\exists x)Rx\} \vdash_{QD} Ra$$

1	$(\forall x)x = a$	
2	$(\exists x)Rx$	
3	$\quad Re$	$A(\exists E)$
4	$\quad e = a$	1, $\forall E$
5	$\quad Ra$	3, 4, Id
6	Ra	2, 3–5, $\exists E$

This one shows that

$$\{Ka, \sim Kb\} \vdash_{QD} a \neq b$$

1	Ka	
2	$\sim Kb$	
3	$\quad a = b$	$A(\sim I)$
4	$\quad Kb$	1, 3, *Id*
5	$\quad Kb \wedge \sim Kb$	2, 4, $\wedge I$
6	$a \neq b$	3–5, $\sim I$

Here is a QL-derivation establishing that

$$\{(\forall x)(Px \to x = a), (\forall x)(x = c \to Qx), a = c\} \vdash_{QD} (\forall x)(Px \to Qx)$$

1	$(\forall x)(Px \to x = a)$	
2	$(\forall x)(x = c \to Qx)$	
3	$a = c$	
4	$\quad Pz$	$A(\to I)$
5	$\quad Pz \to z = a$	1, $\forall E$
6	$\quad z = a$	4, 5, $\to E$
7	$\quad z = c$	3, 6, *Id*
8	$\quad z = c \to Qz$	2, $\forall E$
9	$\quad Qz$	7, 8, $\to E$
10	$Pz \to Qz$	4–9, $\to I$
11	$(\forall x)(Px \to Qx)$	10, $\forall I$

1.4. **Free Logic.** For QL, we assumed that every constant refers to something in the domain. This assumption may fail in English—as in the sentence "Santa Claus doesn't exist". Since Santa Claus doesn't exist, the name "Santa Claus" does not refer to anything. If we attempt to translate this sentence naturally in QL, we will get '$\sim(\exists x)x = c$'. Yet, in QL, '$(\exists x)x = c$' is a theorem. Here is an axiomatic proof:

1.	$\vdash_{QL} c = c$	$(= 1)$
2.	$\vdash_{QL} (\forall x)\sim x = c \to \sim c = c$	$(\forall 1)$
3.	$\vdash_{QL} \sim(\forall x)\sim x = c$	1, 2 (PLR)

And here is a QL-derivation:

1	$c = c$	*Id*
2	$(\exists x)x = c$	1, $\exists I$

So, if we want to allow '$(\exists x)x = c$' to be false, we will have to alter our logic.

We may achieve this with a *free* logic—it is called 'free' because not all the entities we may speak of in this logic are within the scope of the quantifiers. Our interest in free logic comes from its applications in Quantified Modal Logic, '*QML*'. When we turn to *QML*, we may wish to say, for instance, that it is possible that Bob not exist. In this case, we will have to allow that $\Diamond{\sim}(\exists x)x = b$, or, equivalently, that ${\sim}\Box(\exists x)x = b$. However, if we accept that '$(\exists x)x = b$' is a theorem of *QL*, and if we accept the rule of necessitation, then we will have that '$\Box(\exists x)x = b$' is a theorem of *QML*. So we will need to revise our logic if we wish to avoid conclusions like these. Our solution will similarly be to restrict the scope of our quantifiers. So it's worthwhile to see how this works in the simple case of *QL* before turning to *QML*.

1.4.1. *Semantics for Free Logic.* To get our semantics for free logic, *FL*, we will take our semantics for *QL* and modify it slightly by introducing some subset $\mathscr{D}^*$ of our domain $\mathscr{D}$. Intuitively, $\mathscr{D}^*$ is the set of *real* things in the domain. And the rest of the domain, $\mathscr{D} - \mathscr{D}^*$, is the set of *unreal* things—things like Santa Claus and Sherlock Holmes. In free logic, the quantifiers will only range over the things in the *real* set $\mathscr{D}^*$.

At this point, we face an option: we could restrict our interpretation function $\mathscr{I}$ so that $\mathscr{I}$ only maps predicates of *QL* to tuples of *real* entities in $\mathscr{D}^*$. This amounts to the restriction that only real things have properties or bear relations. If we do this, then we will get a so-called *negative* free logic. On the other hand, we could allow our interpretation function $\mathscr{I}$ to map predicates of *QL* to tuples of *real and nonreal* entities in $\mathscr{D}$. That is: we could allow non-real entities to have properties and bear relations. If we do this, then we get a so-called *positive* free logic. Our focus here will be on *positive* free logic; but before getting to that, we'll start by defining a model for a negative free logic, *NFL*. We'll then go on to define a model for positive free logic, which we will call just '*FL*'.

An *NFL*-model $\mathcal{M}$ is a triple $< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$, such that $\mathscr{D}$ is a (non-empty) set of entities, $\mathscr{D}^* \subseteq \mathscr{D}$, and $\mathscr{I}$ is a function from the terms of *QL* to $\mathscr{D}$ and from the N-place predicates of *QL* to sets of N-tuples of entities from $\mathscr{D}^*$.

> NFL-Model:
> An NFL-MODEL $\mathcal{M}$ is a triple $< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$ of a (non-empty) *domain* $\mathscr{D}$, a subset of $\mathscr{D}$, $\mathscr{D}^* \subseteq \mathscr{D}$, and an interpretation function $\mathscr{I}$ from the terms and predicates of *QL* to (tuples of) the entities in $\mathscr{D}$. $\mathscr{I}$ maps terms of *QL* to entities in $\mathscr{D}$, and N-place predicates of *QL* to sets of N-tuples of entities in $\mathscr{D}^*$. Thus, for every term $\ulcorner\tau\urcorner$ of *QL*,
>
> $$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$

And for every N-place predicate $\ulcorner \Pi^N \urcorner$ of QL,

$$\mathscr{I}(\Pi^N) = \{\ldots, < \mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_N >, \ldots\} \subseteq \underbrace{\mathscr{D}^* \times \mathscr{D}^* \times \cdots \times \mathscr{D}^*}_{N\ times} = \mathscr{D}^{*N}$$

As I said, our primary focus here will be on a positive free logic. A positive free logic model—*i.e.*, an *FL*-model—is defined as follows:

FL-Model:
An FL-MODEL $\mathcal{M}$ is a triple $< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$ of a (non-empty) *domain* $\mathscr{D}$, a subset of $\mathscr{D}$, $\mathscr{D}^* \subseteq \mathscr{D}$, and an interpretation function $\mathscr{I}$ from the terms and predicates of QL to (tuples of) the entities in $\mathscr{D}$. $\mathscr{I}$ maps terms of QL to entities in $\mathscr{D}$, and N-place predicates of QL to sets of N-tuples of entities in $\mathscr{D}$. Thus, for every term $\ulcorner \tau \urcorner$ of QL,

$$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$

And for every N-place predicate $\ulcorner \Pi^N \urcorner$ of QL,

$$\mathscr{I}(\Pi^N) = \{\ldots, < \mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_N >, \ldots\} \subseteq \underbrace{\mathscr{D} \times \mathscr{D} \times \cdots \times \mathscr{D}}_{N\ times} = \mathscr{D}^{N}$$

Going forward, it won't matter whether we are dealing with an *NFL*-model or an *FL*-model; all the rest of the semantics are common to both negative and positive free logic.

Just as in QL, we must define the notion of a *variant FL*-model. Given an *FL*-model $\mathcal{M} =< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$, a variable $\ulcorner \alpha \urcorner$, and an entity $\mathfrak{u} \in \mathscr{D}$, we may define the variant model $\mathcal{M}_{\alpha \to \mathfrak{u}}$ as follows: the domains of $\mathcal{M}_{\alpha \to \mathfrak{u}}$ are identical to the domains of $\mathcal{M}$, and the interpretation function for $\mathcal{M}_{\alpha \to \mathfrak{u}}$ is exactly like the interpretation function for $\mathcal{M}$, *except* that $\mathscr{I}_{\alpha \to \mathfrak{u}}(\alpha) = u$. That is: a variant model $\mathcal{M}_{\alpha \to \mathfrak{u}}$ is exactly like the model $\mathcal{M}$, except that, in the variant model $\mathcal{M}_{\alpha \to \mathfrak{u}}$, the variable $\ulcorner \alpha \urcorner$ refers to the entity u.

Variant *FL*-Model:
Given a *FL*-model $\mathcal{M} =< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$, a variable of QL $\ulcorner \alpha \urcorner$, and some $\mathfrak{u} \in \mathscr{D}^*$, the VARIANT *FL*-MODEL $\mathcal{M}_{\alpha \to \mathfrak{u}} \stackrel{\text{def}}{=} < \mathscr{D}, \mathscr{D}^*, \mathscr{I}_{\alpha \to \mathfrak{u}} >$, where

$$\mathscr{I}_{\alpha \to \mathfrak{u}} \stackrel{\text{def}}{=} (\mathscr{I} - < \alpha, \mathscr{I}(\alpha) >) \cup < \alpha, u >$$

An alternative, but equivalent, definition of $\mathscr{I}_{\alpha \to \mathfrak{u}}$ is given by the following: for any N-place predicate $\ulcorner \Pi^N \urcorner$,

$$\mathscr{I}_{\alpha \to \mathfrak{u}}(\Pi^N) = \mathscr{I}(\Pi^N)$$

and, for any term $\ulcorner \tau \urcorner$,

$$\mathscr{I}_{\alpha \to \mathfrak{u}}(\tau) = \begin{cases} \mathscr{I}(\tau) & \text{if } \tau \neq \alpha \\ \mathfrak{u} & \text{if } \tau = \alpha \end{cases}$$

With this definition in hand, we may provide a definition of an *FL*-valuation:

FL-Valuation:
Given an FL-model $\mathcal{M} = < \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$, we define an FL-valuation function, $V_{\mathcal{M}}$, in the following way: for any N-place predicate $\ulcorner \Pi^N \urcorner$, any N terms $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$, any variable $\ulcorner \alpha \urcorner$, and any wffs of QL $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\Pi^N \tau_1 \tau_2 \ldots \tau_N) = 1$ iff $< \mathscr{I}(\tau_1), \mathscr{I}(\tau_2), \ldots, \mathscr{I}(\tau_N) > \in \mathscr{I}(\Pi^N)$.

(2) $V_{\mathcal{M}}(\tau_1 = \tau_2) = 1$ iff $\mathscr{I}(\tau_1) = \mathscr{I}(\tau_2)$.

(3) $V_{\mathcal{M}}(\sim\phi) = 1$ iff $V_{\mathcal{M}}(\phi) = 0$.

(4) $V_{\mathcal{M}}(\phi \rightarrow \psi) = 1$ iff $V_{\mathcal{M}}(\phi) = 0$ or $V_{\mathcal{M}}(\psi) = 1$.

(5) $V_{\mathcal{M}}((\forall\alpha)\phi) = 1$ iff, for all $\mathfrak{u} \in \mathscr{D}^*$, $V_{\mathcal{M}_{\alpha \rightarrow \mathfrak{u}}}(\phi) = 1$.

A philosophical aside: you may worry about having sets of unreal things in our semantics—given that we don't think these things exist, should we have them showing up in our semantics? This is a real worry, but the issues are a bit complicated; if we accept a positive free logic, then we should accept its use in the metalanguage, which means that we should feel comfortable saying, *e.g.*, "Sherlock Holmes is a member of $\mathscr{D} - \mathscr{D}^*$", even though we shouldn't feel comfortable saying, *e.g.*, "There is something in $\mathscr{D} - \mathscr{D}^*$". Standard set theory individuates sets via the use of *quantifiers*; it says that a set Γ and a set Δ are identical iff, for all x, $x \in \Gamma$ iff $x \in \Delta$. So standard set theory would require us, in the metalanguage, to say that our domain $\mathscr{D} = \mathscr{D}^*$, since $\mathscr{D} - \mathscr{D}^* = \varnothing$. Nevertheless, if '$\mathfrak{u}$' were our metalinguistic name for Sherlock Holmes, we could still say that $\mathfrak{u} \in \mathscr{D} - \mathscr{D}^*$. So it looks like, if we accept a free logic in the metalanguage and we accept standard set theory, then we'd have to say that Sherlock Holmes is a member of the empty set; though, of course, the empty set has no members. This is slightly odd, but positive free logicians should be comfortable with such claims. More troubling is that positive free logicians will have models within which $\mathscr{I}(F) = \{\mathfrak{u}\}$ and $\mathscr{I}(G) = \{\mathfrak{v}\}$, for two *unreal* entities $\mathfrak{u}$ and $\mathfrak{v}$. Now, a positive free logician will say that $\mathfrak{u} \in \mathscr{I}(F)$, and they will say that $\mathfrak{u} \notin \mathscr{I}(G)$. However, if they hold on to standard set theory, then they will have to say that $\mathscr{I}(F) = \mathscr{I}(G)$, since they will say that, for all x, $x \in \mathscr{I}(F)$ iff $x \in \mathscr{I}(G)$. So, if they hold on to standard set theory, the positive free logician will have to say that both $\mathfrak{u} \in \mathscr{I}(F)$ and $\mathfrak{u} \notin \mathscr{I}(F)$. And that is an outright contradiction. If they want to avoid dialetheism, then, the positive free logician will have to abandon standard set theory or adopt a different semantics.[2]

1.4.2. *Consequence for Free Logic.* A wff of QL, $\ulcorner \phi \urcorner$, is an FL-consequence of a set of wffs Γ—or, to say the same thing another way, the argument from Γ to $\ulcorner \phi \urcorner$ is FL*-valid*—written

$$\Gamma \models_{FL} \phi$$

if and only if there is no FL-model $\mathcal{M}$ such that $V_{\mathcal{M}}(\gamma) = 1$, for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi) = 0$. Or, equivalently, iff every FL-model which makes every member of Γ true makes $\ulcorner \phi \urcorner$ true as well.

[2]For an alternative semantics, see Andrew Bacon (2013) *Quantificational Logic and Empty Names*. Philosopher's Imprint (24)13.

And a wff of QL, $\ulcorner \phi \urcorner$ is an FL-tautology—or, to say the same thing in another way, $\ulcorner \phi \urcorner$ is FL-*valid*—written

$$\models_{FL} \phi$$

if and only if there is no FL-model $\mathcal{M}$ such that $V_{\mathcal{M}}(\phi) = 0$. Or, equivalently, iff every FL-model makes $\ulcorner \phi \urcorner$ true.

(This definition of consequence holds whether we are talking about positive or negative free logic.)

1.4.3. *Establishing Validity in Free Logic.* For *NFL*, it is a tautology that, if something has a property, then that thing exists,

$$\models_{NFL} Fa \rightarrow (\exists x)x = a$$

This is because, in *NFL*, only entities in the real domain, $\mathscr{D}^*$, have properties. So if a has any properties, then a must exist. We could show that this is a tautology in *NFL* with the following semantic proof:

1.	Suppose that there is some *NFL* model $\mathcal{M} =< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$ such that $V_{\mathcal{M}}(Fa \rightarrow (\exists x)x = a) \neq 1$.	*Assumption*
2.	So $V_{\mathcal{M}}(Fa \rightarrow (\exists x)x = a) = 0$.	1, *bivalence*
3.	So $V_{\mathcal{M}}(Fa) = 1$ and $V_{\mathcal{M}}((\exists x)x = a) = 0$.	2, *def* $\rightarrow$
4.	So $V_{\mathcal{M}}(Fa) = 1$	3
5.	So $\mathscr{I}(a) \in \mathscr{I}(F)$	4, *def* Π^N
6.	$\mathscr{I}(F) \subseteq \mathscr{D}^*$	*def NFL-model*
7.	So $\mathscr{I}(a) \in \mathscr{D}^*$	5, 6
8.	$V_{\mathcal{M}}((\exists x)x = a) = 0$	3
9.	So it is not the case that $V_{\mathcal{M}}((\exists x)x = a) = 1$	8, *bivalence*
10.	So it is not the case that there is some $\mathfrak{u} \in \mathscr{D}^*$ such that $V_{\mathcal{M}_{x \rightarrow \mathfrak{u}}}(x = a) = 1$.	9, *def* $\exists$
11.	So, for all $\mathfrak{u} \in \mathscr{D}^*$, it is not the case that $V_{\mathcal{M}_{x \rightarrow \mathfrak{u}}}(x = a) = 1$.	10, *QL*
12.	So, for all $\mathfrak{u} \in \mathscr{D}^*$, $\mathscr{I}_{x \rightarrow \mathfrak{u}}(x) \neq \mathscr{I}_{x \rightarrow \mathfrak{u}}(a)$.	11, *def* $=$
13.	Just to have a name, let $\mathscr{I}(a)$ be '$\mathfrak{a}$'.	
14.	Then, $\mathfrak{a} \in \mathscr{D}^*$	7, 13
15.	So, $\mathscr{I}_{x \rightarrow \mathfrak{a}}(x) \neq \mathscr{I}_{x \rightarrow \mathfrak{a}}(a)$.	12, 14, $\forall E$
16.	$\mathscr{I}_{x \rightarrow \mathfrak{a}}(x) = \mathfrak{a}$	*def variant model*
17.	$\mathscr{I}_{x \rightarrow \mathfrak{a}}(a) = \mathfrak{a}$	13, *def variant model*
18.	So $\mathscr{I}_{x \rightarrow \mathfrak{a}}(x) = \mathscr{I}_{x \rightarrow \mathfrak{a}}(a)$	16, 17
19.	Our assumption has led to a contradiction.	15, 18
20.	So there is no *NFL* model $\mathcal{M} =< \mathscr{D}, \mathscr{D}^*, \mathscr{I} >$ such that $V_{\mathcal{M}}(Fa \rightarrow (\exists x)x = a) \neq 1$.	19, $\sim I$

1.4.4. *Establishing Invalidity in Free Logic.* On the other hand, it is not a tautology in a positive free logic, *FL*, that anything which has properties exists.

$$\nvDash_{FL} Fa \to (\exists x)x = a$$

To show this, it is enough to provide a single *FL*-model in which this wff is false. The following will do.

$$\begin{aligned} \mathscr{D} &= \{\mathfrak{u}_1, \mathfrak{u}_2\} \\ \mathscr{D}^* &= \{\mathfrak{u}_1\} \\ \mathscr{I}(F) &= \{\mathfrak{u}_2\} \\ \mathscr{I}(a) &= \{\mathfrak{u}_2\} \end{aligned}$$

In this *FL*-model, 'a' refers to $\mathfrak{u}_2$, and $\mathfrak{u}_2$ has the property F. So the antecedent 'Fa' of '$Fa \to (\exists x)x = a$' is true. Nevertheless, the consequent is false, since the only real entity in $\mathscr{D}^*$ is $\mathfrak{u}_1$. And $V_{\mathcal{M}_{x\to\mathfrak{u}_1}}(x = a) = 0$, since $\mathscr{I}_{x\to\mathfrak{u}_1}(x) = \mathfrak{u}_1$, while $\mathscr{I}_{x\to\mathfrak{u}_2}(a) = \mathfrak{u}_2$. So the antecedent is true while the consequent is false. So '$Fa \to (\exists x)x = a$' is false on this *FL*-model.

This *FL*-model is not an *NFL*-model, because an *NFL*-model requires $\mathscr{I}(F)$ to be a subset of $\mathscr{D}^*$.

We may also show that, in both a positive and a negative free logic, unlike in *QL*,

$$\{(\forall x)Fx\} \nvDash_{FL} (\exists x)Fx$$

In *QL*, '$(\forall x)Fx$' entails '$(\exists x)Fx$' because we require our domain, $\mathscr{D}$, to be non-empty. So if everything has the property F, then there must be some thing in the domain which has the property F. However, because an *FL*-model allows the real domain to be empty, this argument is invalid in *FL*. The following *FL*-model (which is also an *NFL*-model) provides a counterexample.

$$\begin{aligned} \mathscr{D} &= \{\mathfrak{u}_1\} \\ \mathscr{D}^* &= \varnothing \\ \mathscr{I}(F) &= \varnothing \end{aligned}$$

All of the none of the entities in the real domain, $\mathscr{D}^*$, have the property F, so '$(\forall x)Fx$' is true in this model; however, since there is no thing in the real domain which has the property F, '$(\exists x)Fx$' is false in this model.

1.4.5. *Axioms for Free Logic.* Here, we will provide an axiomatic system for a *positive* free logic, *FL*. This axiomatic system is like our axiomatic system for *QL*, except that we have exchanged the axiom schema '$(\forall\alpha)\phi \to \phi[\tau/\alpha]$' for two new ones. We have also added an additional axiom for identity, saying that everything is identical to something.

$$\vdash_{FL} \phi, \quad \text{for all } PL\text{-valid schemata } \ulcorner\phi\urcorner \tag{PL}$$

$$\vdash_{FL} (\forall\alpha)\phi \to ((\exists\zeta)\zeta = \tau \to \phi[\tau/\alpha]) \tag{$\forall 1$}$$

provided that $\ulcorner\tau\urcorner$ *is free in* $\ulcorner\phi[\tau/\alpha]\urcorner$

$$\vdash_{FL} (\forall\alpha)(\phi \to \psi) \to (\phi \to (\forall\alpha)\psi) \tag{$\forall 2$}$$

provided that ⌜α⌝ *is not free in* ⌜ϕ⌝

$$\vdash_{FL} (\forall\zeta)((\forall\alpha)\phi \rightarrow \phi[\zeta/\alpha]) \qquad (\forall 3)$$

provided that ⌜ζ⌝ *is free in* ⌜$\phi[\zeta/\alpha]$⌝

$$\vdash_{FL} \tau = \tau \qquad (=1)$$

$$\vdash_{FL} \tau_1 = \tau_2 \rightarrow (\phi \rightarrow \phi[\tau_2 /\!/ \tau_1]) \qquad (=2)$$

provided that ⌜τ_2⌝ *is free in* ⌜$\phi[\tau_2 /\!/ \tau_1]$⌝

$$\vdash_{FL} (\forall\alpha)(\exists\zeta)\alpha = \zeta \qquad (=3)$$

And we retain the following rules of inference:

> *Propositional Logic Rules* (*PLR*): If ⌜ψ⌝ follows from ⌜ϕ⌝ according to propositional logic, then, from ⌜ϕ⌝, infer ⌜ψ⌝.
>
> *Generalization* (*G*): If a wff ⌜ϕ⌝ is a theorem of FL, then you may infer ⌜$(\forall\alpha)\phi$⌝ as a theorem of FL, where ⌜α⌝ is a variable.
>
> $$\text{from } \vdash_{FL} \phi, \text{ infer } \vdash_{FL} (\forall\alpha)\phi$$

1.4.6. *Natural Deduction for Free Logic.* This axiomatic system is difficult to work with, so we will introduce a natural deduction system for positive free logic, *FL*.

For this natural deduction system, we simply exchange the rules for quantifiers, $(\forall E)$, $(\forall I)$, $(\exists E)$, and $(\exists I)$, with four new rules. The first is a modification of $(\forall E)$.

> <u>*Universal Elimination* ($\forall E^*$)</u>
>
> | $(\forall\alpha)\phi$
>
> ▷ | $(\exists\alpha)\alpha = \beta \rightarrow \phi[\beta/\alpha]$
>
> where ⌜β⌝ is a constant; or:
>
> | $(\forall\alpha)\phi$
>
> ▷ | $(\exists\alpha)\alpha = \zeta \rightarrow \phi[\zeta/\alpha]$
>
> where ⌜ζ⌝ is a variable—*provided that* ⌜ζ⌝ is free in ⌜$\phi[\zeta/\alpha]$⌝.

That is: we can no longer infer 'Fa' from '$(\forall x)Fx$'. The reason is that the name 'a' may not refer to anything at all. For instance, from 'everything is physical', we cannot infer 'Casper the ghost is physical', since 'Casper the ghost' may not refer to anything at all—and therefore, not anything in the scope of the quantifier 'everything'. What we *can* infer is that *if* Casper the ghost exists, then Casper the ghost is physical. And this is what the new rule of inference $(\forall E^*)$ tells us.

For instance, the following is a legal *FL* derivation:

1	$(\exists y)y = b$	
2	$(\forall x)Rax$	
3	$(\exists y)y = b \rightarrow Rab$	2, $\forall E^*$
4	Rab	1, 3, $\rightarrow E$

As is the following:

1	$(\exists y)y = z$	
2	$(\forall x)Rax$	
3	$(\exists y)y = z \rightarrow Raz$	2, $\forall E^*$
4	Raz	1, 3, $\rightarrow E$

The free logic rule for Universal Introduction, $(\forall I^*)$, tells us that, while we cannot infer '$(\forall x)Fx$' from 'Fy', we *can* infer '$(\forall x)Fx$' from '$(\exists z)y = z \rightarrow Fy$' (provided that y doesn't occur free in any assumption or on the first line of an open subderivation).

Universal Introduction ($\forall I^*$)

$$(\exists \alpha)\alpha = \zeta \rightarrow \phi[\zeta/\alpha]$$

$$\triangleright \quad (\forall \alpha)\phi$$

where $\ulcorner \zeta \urcorner$ is a variable.
provided that:

(1) $\ulcorner \zeta \urcorner$ does not occur free in the assumptions

(2) $\ulcorner \zeta \urcorner$ does not occur free in the first line of an open subderivation.

(3) $\ulcorner \zeta \urcorner$ does not occur free in $(\forall \alpha)\phi$.

For instance, the following is a legal *FL*-derivation:

1	$(\forall x)(\forall y)Rxy$	
2	$(\exists y)y = z \to (\forall y)Rzy$	1 $\forall E^*$
3	$\quad (\exists y)y = z$	$A(\to I)$
4	$\quad (\forall y)Rzy$	2, 3, $\to E$
5	$\quad (\exists y)y = z \to Rzz$	4, $\forall E^*$
6	$\quad Rzz$	3, 5, $\to E$
7	$(\exists y)y = z \to Rzz$	3–6, $\to I$
8	$(\forall x)Rxx$	7, $\forall I^*$

As is the following:

1	$(\forall x)(Fx \to Gx)$	
2	$(\forall x)Fx$	
3	$\quad (\exists x)x = z$	$A(\to I)$
4	$\quad (\exists x)x = z \to (Fz \to Gz)$	1, $\forall E^*$
5	$\quad Fz \to Gz$	3, 4, $\to E$
6	$\quad (\exists x)x = z \to Fz$	2, $\forall E^*$
7	$\quad Fz$	3, 6, $\to E$
8	$\quad Gz$	5, 7, $\to E$
9	$(\exists x)x = z \to Gz$	3–8, $\to I$
10	$(\forall x)Gx$	9, $\forall I^*$

The modified rule for existential elimination says that, if you have an accessible wff of the form $\ulcorner(\exists\alpha)\phi\urcorner$, and you have an accessible subderivation beginning with $\ulcorner\phi[\beta/\alpha] \wedge (\exists\alpha)\alpha = \beta\urcorner$ and ending with $\ulcorner\psi\urcorner$ (where $\ulcorner\beta\urcorner$ doesn't appear on any previous line and $\ulcorner\psi\urcorner$ does not contain $\ulcorner\beta\urcorner$), then you may write $\ulcorner\psi\urcorner$ outside of the scope of your subderivation.

Existential Elimination ($\exists E^*$)

$$
\begin{array}{l|l}
 & (\exists\alpha)\phi \\
 & \quad \phi[\beta/\alpha] \wedge (\exists\alpha)\alpha = \beta \\
 & \quad \vdots \\
 & \quad \psi \\
\rhd & \psi
\end{array}
$$

where $\ulcorner\beta\urcorner$ is a constant.
provided that:

(1) $\ulcorner\beta\urcorner$ does not appear on any previous line
(2) $\ulcorner\beta\urcorner$ does not appear in $\ulcorner\psi\urcorner$

And the modified rule for existential introduction says that, *e.g.*, if you have both 'Fa' and '$(\exists x)x = a$' written down on accessible lines, then you may infer '$(\exists x)Fx$'.

Existential Introduction ($\exists I^*$)

$$
\begin{array}{l|l}
 & \phi[\beta/\alpha] \\
 & (\exists\alpha)\alpha = \beta \\
\rhd & (\exists\alpha)\phi
\end{array}
$$

where $\ulcorner\beta\urcorner$ is a constant; or:

$$
\begin{array}{l|l}
 & \phi[\zeta/\alpha] \\
 & (\exists\alpha)\alpha = \zeta \\
\rhd & (\exists\alpha)\phi
\end{array}
$$

where $\ulcorner\zeta\urcorner$ is a variable.

Then, here is an *FL*-derivation showing that '$(\exists x)x = x$' follows, in *FL*, from '$(\exists x)x = a$':

$$
\begin{array}{r|l l}
1 & (\exists x)x = a & \\
2 & \quad a = a \wedge (\exists x)x = a & A(\exists E^*) \\
3 & \quad a = a & 2, \wedge E \\
4 & \quad (\exists x)x = x & 1, 3, \exists I^* \\
5 & (\exists x)x = x & 1, 2\text{–}4, \exists E^*
\end{array}
$$

And here is an *FL*-derivation establishing that '$(\forall y)(\exists x)Rxy$' follows from '$(\exists x)(\forall y)Rxy$':

1	$(\exists x)(\forall y)Rxy$	
2	$(\forall y)Ray \wedge (\exists x)x = a$	$A(\exists E^*)$
3	$(\forall y)Ray$	2, $\wedge E$
4	$(\exists x)x = a$	2, $\wedge E$
5	$(\exists x)x = z \rightarrow Raz$	3, $\forall E^*$
6	$(\exists x)x = z$	$A(\rightarrow I)$
7	Raz	5, 6, $\rightarrow E$
8	$(\exists x)Rxz$	4, 7, $\exists I^*$
9	$(\exists x)x = z \rightarrow (\exists x)Rxz$	6–8, $\rightarrow I$
10	$(\forall y)(\exists x)Rxy$	9, $\forall I^*$
11	$(\forall y)(\exists x)Rxy$	1, 2–10, $\exists E^*$

If $\ulcorner \phi \urcorner$ is derivable from the wffs in Γ within this natural deduction system, then we will write:

$$\Gamma \vdash_{FD} \phi$$

And if $\ulcorner \phi \urcorner$ is derivable from no assumptions in this system, then we will write:

$$\vdash_{FD} \phi$$

Here is an *FL*-derivation to show that

$$\vdash_{FD} (\forall x)(Fx \rightarrow (\exists y)Fy)$$

1	$(\exists x)x = z$	$A(\rightarrow I)$
2	Fz	$A(\rightarrow I)$
3	$(\exists y)Fy$	1, 2 $\exists I^*$
4	$Fz \rightarrow (\exists y)Fy$	2–3, $\rightarrow I$
5	$(\exists x)x = z \rightarrow (Fz \rightarrow (\exists y)Fy)$	1–4, $\rightarrow I$
6	$(\forall x)(Fx \rightarrow (\exists y)Fy)$	5, $\forall I^*$

And here is one to show that

$$\{(\exists x)Rxx\} \vdash_{FD} (\exists x)(\exists y)Rxy$$

1	$(\exists x)Rxx$	
2	$\quad Raa \wedge (\exists x)x = a$	$A(\exists E^*)$
3	$\quad (\exists x)x = a$	2, $\wedge E$
4	$\quad Raa$	2, $\wedge E$
5	$\quad (\exists y)Ray$	3, 4 $\exists I^*$
6	$\quad (\exists x)(\exists y)Rxy$	3, 5, $\exists I^*$
7	$(\exists x)(\exists y)Rxy$	1, 2–6, $\exists E^*$

And here's a derivation showing that

$$\{(\forall x)Fx, (\exists x)x = x\} \vdash_{FD} (\exists x)Fx$$

1	$(\forall x)Fx$	
2	$(\exists x)x = x$	
3	$\quad a = a \wedge (\exists x)x = a$	$A(\exists E^*)$
4	$\quad (\exists x)x = a$	3, $\wedge E$
5	$\quad (\exists x)x = a \rightarrow Fa$	1, $\forall E^*$
6	$\quad Fa$	4, 5 $\rightarrow E$
7	$\quad (\exists x)Fx$	4, 6, $\exists I^*$
8	$(\exists x)Fx$	2, 3–7, $\exists E^*$

Finally, here's a derivation showing that

$$\{(\forall x)(Fx \rightarrow (\exists y)Rxy)\} \vdash_{FD} (\forall x)(\exists y)(Fx \rightarrow Rxy)$$

1	$(\forall x)(Fx \to (\exists y)Rxy)$	
2	$(\exists x)x = w \to (Fw \to (\exists y)Rwy)$	1, $\forall E^*$
3	$(\exists x)x = w$	$A(\to I)$
4	$Fw \to (\exists y)Rwy$	2, 3, $\to E$
5	$\sim(\exists y)(Fw \to Rwy)$	$A(\sim E)$
6	$(\forall y)\sim(Fw \to Rwy)$	5, QN
7	$(\exists x)x = w \to \sim(Fw \to Rww)$	6, $\forall E^*$
8	$\sim(Fw \to Rww)$	3, 7, $\to E$
9	$\sim Fw$	$A(\sim E)$
10	Fw	$A(\to I)$
11	$\sim Rww$	$A(\sim E)$
12	$Fw \wedge \sim Fw$	9, 10, $\wedge I$
13	Rww	11–12, $\to I$
14	$Fw \to Rww$	10–13, $\to I$
15	$(Fw \to Rww) \wedge \sim(Fw \to Rww)$	8, 14, $\wedge I$
16	Fw	9–15, $\sim E$
17	$(\exists y)Rwy$	4, 16, $\to E$
18	$Rwb \wedge (\exists x)x = b$	$A(\exists E^*)$
19	$(\exists x)x = b$	18, $\wedge E$
20	$(\exists x)x = b \to \sim(Fw \to Rwb)$	6, $\forall E^*$
21	$\sim(Fw \to Rwb)$	19, 20 $\to E$
22	Fw	$A(\to I)$
23	Rwb	18, $\wedge E$
24	$Fw \to Rwb$	22–23, $\to I$
25	$\sim(Ga \wedge \sim Ga)$	$A(\sim E)$
26	$(Fw \to Rwb) \wedge \sim(Fw \to Rwb)$	21, 24, $\wedge I$
27	$Ga \wedge \sim Ga$	25–26, $\sim E$
28	$Ga \wedge \sim Ga$	17, 18–27, $\exists E^*$
29	$(\exists y)(Fw \to Rwy)$	5–28, $\sim E$
30	$(\exists x)x = w \to (\exists y)(Fw \to Rwy)$	3–29, $\to I$
31	$(\forall x)(\exists y)(Fx \to Rxy)$	30, $\forall I^*$

2. The Language of Quantified Modal Logic

2.1. **Syntax for QML.** The vocabulary for the language QML consists of the following:

(1) An infinite number of *constants*—lowercase letters from the start of the alphabet, potentially with subscripts:

$$a, b, c, d, e, a_1, b_1, c_1, d_1, e_1, a_2, b_2, \ldots$$

(2) An infinite number of *variables*—lowercase letters from the end of the alphabet, potentially with subscripts:

$$w, x, y, z, w_1, x_1, y_1, z_1, w_2, \ldots$$

(3) For every natural number $N \geq 1$, an infinite number of n-place predicates—capital letters, potentially with subscripts:

$$\begin{array}{cccccccc} A^1, & B^1, & \cdots, & Y^1, & Z^1, & A^1_1 & B^1_1, & \cdots \\ A^2, & B^2, & \cdots, & Y^2, & Z^2, & A^2_1 & B^2_1, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A^N, & B^N, & \cdots, & Y^N, & Z^N, & A^N_1 & B^N_1, & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{array}$$

(4) The identity relation:

$$=$$

(5) Logical operators:

$$\forall,\ \sim,\ \rightarrow,\ \Box$$

(6) Parentheses:

$$(,\ )$$

Nothing else is included in the vocabulary of QML.

Terminology: we call both constants and variables *terms* of QML.

2.2. **Rules for Wffs.** We specify what it is for a string of symbols from the vocabulary of QML to constitute a *well-formed formula*, of *wff* of QML recursively with the following.

Π) If $\ulcorner \Pi^N \urcorner$ is an N-place predicate and $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$ are N terms, then $\ulcorner \Pi^N \tau_1 \tau_2 \ldots \tau_N \urcorner$ is a wff—known as an *atomic* wff.

=) If $\ulcorner \tau_1 \urcorner$ and $\ulcorner \tau_2 \urcorner$ are terms, then $\ulcorner \tau_1 = \tau_2 \urcorner$ is a wff—also known as an *atomic* wff.

∼) If $\ulcorner \phi \urcorner$ is a wff, then $\ulcorner \sim\phi \urcorner$ is a wff.

→) If $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$ are wffs, then $\ulcorner (\phi \rightarrow \psi) \urcorner$ is a wff.

∀) If $\ulcorner \phi \urcorner$ is a wff and $\ulcorner \alpha \urcorner$ is a variable, then $\ulcorner (\forall \alpha)\phi \urcorner$ is a wff.

□) if $\ulcorner \phi \urcorner$ is a wff, then $\ulcorner \Box\phi \urcorner$ is a wff.

– Nothing else is a wff.

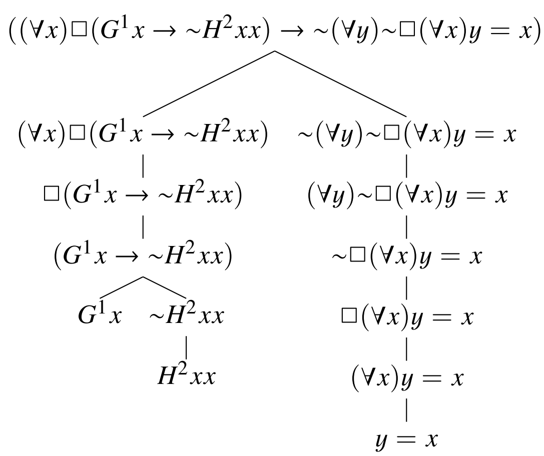
For instance, the following is a wff of QML:

$$((\forall x)\Box(G^1x \rightarrow {\sim}H^2xx) \rightarrow {\sim}(\forall y){\sim}\Box(\forall x)y = x)$$

We could show this by providing the following proof, which appeals to the rules for wffs above:

1.	'G^1x' is a wff.	(Π)
2.	'H^2xx' is a wff.	(Π)
3.	'$y = x$' is a wff.	$(=)$
4.	So, '${\sim}H^2xx$' is a wff.	2, $({\sim})$
5.	So, '$(G^1x \rightarrow {\sim}H^2xx)$' is a wff.	1, 4 $(\rightarrow)$
6.	So, '$\Box(G^1x \rightarrow {\sim}H^2xx)$' is a wff.	5, $(\Box)$
7.	So, '$(\forall x)\Box(G^1x \rightarrow {\sim}H^2xx)$' is a wff.	6, $(\forall)$
8.	So, '$(\forall x)y = x$' is a wff.	3, $(\forall)$
9.	So, '$\Box(\forall x)y = x$' is a wff.	8, $(\Box)$
10.	So '${\sim}\Box(\forall x)y = x$' is a wff.	9, $({\sim})$
11.	So '$(\forall y){\sim}\Box(\forall x)y = x$' is a wff.	10, $(\forall)$
12.	So '${\sim}(\forall y){\sim}\Box(\forall x)y = x$' is a wff.	11, $({\sim})$
13.	So '$((\forall x)\Box(G^1x \rightarrow {\sim}H^2xx) \rightarrow {\sim}(\forall y){\sim}\Box(\forall x)y = x)$' is a wff.	7, 12 $(\rightarrow)$

Another way of notating a proof like this is with a syntax tree like the following:



2.3. **Definitions.** We introduce the following stipulative definitions, for any wffs $\ulcorner\phi\urcorner$, $\ulcorner\psi\urcorner$ of QL, and any variable $\ulcorner\alpha\urcorner$ of QL:

(1) $\ulcorner(\phi \vee \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner((\phi \rightarrow \psi) \rightarrow \psi)\urcorner$

(2) $\ulcorner(\phi \wedge \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner{\sim}(\phi \rightarrow {\sim}\psi)\urcorner$

(3) $\ulcorner(\phi \leftrightarrow \psi)\urcorner \stackrel{\text{def}}{=} \ulcorner{\sim}((\phi \rightarrow \psi) \rightarrow {\sim}(\psi \rightarrow \phi))\urcorner$

(4) $\ulcorner(\exists\alpha)\,\phi\urcorner \stackrel{\text{def}}{=} \ulcorner{\sim}(\forall\alpha){\sim}\phi\urcorner$

(5) $\ulcorner\tau_1 \neq \tau_2\urcorner \stackrel{\text{def}}{=} \ulcorner{\sim}\tau_1 = \tau_2\urcorner$

(6) $\ulcorner\Diamond\phi\urcorner \stackrel{\text{def}}{=} \ulcorner{\sim}\Box{\sim}\phi\urcorner$

2.4. **Conventions.** As a matter of convention, we will omit the outermost parentheses, and suppress the superscripts on the predicates of QL. Thus, rather than writing

$$((\forall x)\Box(G^1x \to {\sim}\Box H^2xx) \to (\exists y)\Diamond(\forall x)y = x)$$

we could instead simply write

$$(\forall x)\Box(Gx \to {\sim}\Box Hxx) \to (\exists y)\Diamond(\forall x)y = x$$

3. The Simple Semantics for QML

For Quantificational Modal Logic, we will begin with a simple semantics; we will see some reasons to want to complicate this semantics later on. But, for now, we will introduce the notion of an *SQML-model* (for 'simple quantified modal logic model'), which is a four-tuple of a set of worlds, $\mathcal{W}$, a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$, called the accessibility relation, a set of some things—called the *domain*, $\mathscr{D}$, of the model—and an interpretation function $\mathscr{I}$—which is a function from terms of QML to the things in the domain $\mathscr{D}$, and from pairs of worlds and N-place predicates of QML to N-tuples of the things in $\mathscr{D}$.

> SQML-Model:
> A SQML-MODEL $\mathcal{M}$ is a 4-tuple $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ of a (non-empty) set of worlds, $\mathcal{W}$, a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$, a (non-empty) *domain* of entities, $\mathscr{D}$, and an interpretation function, $\mathscr{I}$. $\mathscr{I}$ maps a term $\ulcorner\tau\urcorner$ of QML to an entity in $\mathscr{D}$, and it maps a pair of a world w and an N-place predicate $\ulcorner\Pi^N\urcorner$ of QML to a set of N-tuples of entities in $\mathscr{D}$. Thus, for every term $\ulcorner\tau\urcorner$ of QML,
>
> $$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$
>
> And for every N-place predicate $\ulcorner\Pi^N\urcorner$ of QML, and every world $w \in \mathcal{W}$,
>
> $$\mathscr{I}(\Pi^N, w) = \{\ldots, < \mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_N >, \ldots\} \subseteq \underbrace{\mathscr{D} \times \mathscr{D} \times \cdots \times \mathscr{D}}_{N\ times} = \mathscr{D}^N$$

We will use an $SQML$-model to construct an $SQML$-*valuation* function which maps us from pairs of wffs of QML and worlds to $\{0, 1\}$.

Before getting to that, however, we must define the notion of a *variant* $SQML$-model. Given a $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, a variable $\ulcorner\alpha\urcorner$, and an entity $\mathfrak{u} \in \mathscr{D}$, we may define the variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$ as follows: the set of worlds of $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is just the set of worlds $\mathcal{W}$; the accessibility relation is just R; the domain of $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is just the domain $\mathscr{D}$, and the interpretation function for $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is exactly like the interpretation function for $\mathcal{M}$, *except* that $\mathscr{I}_{\alpha\to\mathfrak{u}}(\alpha) = \mathfrak{u}$. That is: a variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$ is exactly like the model $\mathcal{M}$, except that, in the variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$, the variable $\ulcorner\alpha\urcorner$ refers to the entity $\mathfrak{u}$.

Variant SQML-Model:
Given an $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, a variable of QML, $\ulcorner \alpha \urcorner$, and some $\mathfrak{u} \in \mathscr{D}$, the variant $SQML$-model $\mathcal{M}_{\alpha \to \mathfrak{u}} \stackrel{\text{def}}{=} < \mathcal{W}, R, \mathscr{D}, \mathscr{I}_{\alpha \to \mathfrak{u}} >$ where:

$$\mathscr{I}_{\alpha \to \mathfrak{u}} \stackrel{\text{def}}{=} (\mathscr{I} - < \alpha, \mathscr{I}(\alpha) >) \cup < \alpha, \mathfrak{u} >$$

An alternative, but equivalent, definition of $\mathscr{I}_{\alpha \to \mathfrak{u}}$ is given by the following: for any N-place predicate $\ulcorner \Pi^N \urcorner$,

$$\mathscr{I}_{\alpha \to \mathfrak{u}}(\Pi^N, w) = \mathscr{I}(\Pi^N, w)$$

and, for any term $\ulcorner \tau \urcorner$,

$$\mathscr{I}_{\alpha \to \mathfrak{u}}(\tau) = \begin{cases} \mathscr{I}(\tau) & \text{if } \tau \neq \alpha \\ \mathfrak{u} & \text{if } \tau = \alpha \end{cases}$$

With this definition in hand, we may provide a definition of an $SQML$-valuation:

SQML-Valuation:
Given an $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, we define an $SQML$-valuation function, $V_{\mathcal{M}}$, in the following way: for every world $w \in \mathcal{W}$, any N-place predicate $\ulcorner \Pi^N \urcorner$, any N terms $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$, any variable $\ulcorner \alpha \urcorner$, and any wffs of QML $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\Pi^N \tau_1 \tau_2 \ldots \tau_N, w) = 1$ iff $< \mathscr{I}(\tau_1), \mathscr{I}(\tau_2), \ldots, \mathscr{I}(\tau_N) > \in \mathscr{I}(\Pi^N, w)$.

(2) $V_{\mathcal{M}}(\tau_1 = \tau_2, w) = 1$ iff $\mathscr{I}(\tau_1) = \mathscr{I}(\tau_2)$.

(3) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.

(4) $V_{\mathcal{M}}(\phi \to \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.

(5) $V_{\mathcal{M}}((\forall \alpha)\phi, w) = 1$ iff, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{\alpha \to \mathfrak{u}}}(\phi, w) = 1$.

(6) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for all $w' \in \mathcal{W}$, if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

By placing the familiar restrictions on the accessibility relation R, we arrive at the following defintions of $SQML$-models for each of the systems we encountered when looking at propositional modal logic.

(D) A $SQML$-D-model is a $SQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is serial.

(T) A $SQML$-T-model is a $SQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive.

(B) A $SQML$-B-model is a $SQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and symmetric.

(S4) A $SQML$-S4-model is a $SQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and transitive.

(S5) A $SQML$-S5-model is a $SQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and euclidean (and therefore, reflexive, symmetric, and transitive).

3.1. *SQML*-**Consequence.** We will say that $\ulcorner\phi\urcorner$ is an *SQML*-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is *SQML*-valid,

$$\Gamma \models_{\text{SQML}} \phi$$

iff there is no *SQML*-model $< \mathcal{W}, R, \mathcal{D}, \mathcal{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every *SQML*-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

And we will say that a wff $\ulcorner\phi\urcorner$ is an *SQML-tautology*, or *SQML-valid*, written

$$\models_{\text{SQML}} \phi$$

if and only if $\ulcorner\phi\urcorner$ is true at every world in every *SQML* model.

Similarly, we'll say that the argument from Γ to $\ulcorner\phi\urcorner$ is *SQML-D*-valid,

$$\Gamma \models_{\text{SQML-D}} \phi$$

iff there is no world in any *SQML-D*-model at which all the wffs in Γ are true, yet $\ulcorner\phi\urcorner$ is false. And we'll say that $\ulcorner\phi\urcorner$ is an *SQML-D*-tautology, or *SQML-D*-valid,

$$\models_{\text{SQML-D}} \phi$$

iff there is no world in any *SQML-D*-model at which $\ulcorner\phi\urcorner$ is false.

We may provide similar definitions of consequence for each of the other systems of *SQML*:

(*T*) $\Gamma \models_{\text{SQML-T}} \phi$ iff there is no world in any *SQML-T*-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{SQML-T}} \phi$ iff there is no world in any *SQML-T*-model at which $\ulcorner\phi\urcorner$ is false;

(*B*) $\Gamma \models_{\text{SQML-B}} \phi$ iff there is no world in any *SQML-B*-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{SQML-B}} \phi$ iff there is no world in any *SQML-B*-model at which $\ulcorner\phi\urcorner$ is false;

(S4) $\Gamma \models_{\text{SQML-4}} \phi$ iff there is no world in any *SQML*-S4-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{SQML-4}} \phi$ iff there is no world in any *SQML*-S4-model at which $\ulcorner\phi\urcorner$ is false;

(S5) $\Gamma \models_{\text{SQML-5}} \phi$ iff there is no world in any *SQML*-S5-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{SQML-5}} \phi$ iff there is no world in any *SQML*-S5-model at which $\ulcorner\phi\urcorner$ is false.

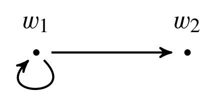
4. Establishing Invalidity in *SQML*

Suppose that we wish to show that

$$\{(\forall x)\Box(Px \vee Qx)\} \not\models_{\text{SQML}} \Box(\exists x)Px \vee \Box(\exists x)Qx$$

It is enough to provide a single *SQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}((\forall x)\Box(Px \lor Qx), w) = 1$, while $V_{\mathcal{M}}(\Box(\exists x)Px \lor \Box(\exists x)Qx, w) = 0$. The following *SQML*-model will suffice:

$$\begin{aligned}
\mathcal{W} &= \{w_1, w_2\} \\
R &= \{< w_1, w_1 >, < w_1, w_2 >\} \\
\mathscr{D} &= \{\text{u}_1\} \\
\mathscr{I}(P, w_1) &= \{\text{u}_1\} \\
\mathscr{I}(P, w_2) &= \varnothing \\
\mathscr{I}(Q, w_1) &= \varnothing \\
\mathscr{I}(Q, w_2) &= \{\text{u}_1\}
\end{aligned}$$



To see that $V_{\mathcal{M}}((\forall x)\Box(Px \lor Qx), w_1) = 1$, note that there is only one entity, u_1, in the domain $\mathscr{D}$. So the only variant model we need to consider is $\mathcal{M}_{x\to\text{u}_1}$. Then, we need to check whether $V_{\mathcal{M}_{x\to\text{u}_1}}(\Box(Px \lor Qx), w_1) = 1$. There are two worlds which are accessible from w_1, so we need to check both of them. $V_{\mathcal{M}_{x\to\text{u}_1}}(Px \lor Qx, w_1) = 1$ because $V_{\mathcal{M}_{x\to\text{u}_1}}(Px, w_1) = 1$. And $V_{\mathcal{M}_{x\to\text{u}_1}}(Px \lor Qx, w_2) = 1$ because $V_{\mathcal{M}_{x\to\text{u}_1}}(Qx, w_2) = 1$. Since $V_{\mathcal{M}_{x\to\text{u}_1}}(Px \lor Qx, w') = 1$, for all w' such that Rw_1w', $V_{\mathcal{M}_{x\to\text{u}_1}}(\Box(Px \lor Qx), w_1) = 1$. And therefore, $V_{\mathcal{M}}((\forall x)\Box(Px \lor Qx), w_1) = 1$.

To see that $V_{\mathcal{M}}(\Box(\exists x)Px \lor \Box(\exists x)Qx, w_1) = 0$, we must show both that $V_{\mathcal{M}}(\Box(\exists x)Px, w_1) = 0$ and that $V_{\mathcal{M}}(\Box(\exists x)Qx, w_1) = 0$.

(1) Begin with '$\Box(\exists x)Px$'. In order for $V_{\mathcal{M}}(\Box(\exists x)Px, w_1) = 1$, it must be that both $V_{\mathcal{M}}((\exists x)Px, w_1) = 1$ and that $V_{\mathcal{M}}((\exists x)Px, w_2) = 1$, since Rw_1w_1 and Rw_1w_2. However, $V_{\mathcal{M}}((\exists x)Px, w_2) = 1$ iff $V_{\mathcal{M}_{x\to\text{u}_1}}(Px, w_2) = 1$, which is so iff $\mathscr{I}_{x\to\text{u}_1}(x) \in \mathscr{I}(P, w_2)$. Since $\mathscr{I}(P, w_2) = \varnothing$, $\mathscr{I}_{x\to\text{u}_1}(x) \notin \mathscr{I}(P, w_2)$. So $V_{\mathcal{M}}((\exists x)Px, w_2) = 0$. So $V_{\mathcal{M}}(\Box(\exists x)Px, w_1) = 0$.

(2) Next consider '$\Box(\exists x)Qx$'. In order for $V_{\mathcal{M}}(\Box(\exists x)Qx, w_1) = 1$, it must be that both $V_{\mathcal{M}}((\exists x)Qx, w_1) = 1$ and that $V_{\mathcal{M}}((\exists x)Qx, w_2) = 1$, since Rw_1w_1 and Rw_1w_2. However, $V_{\mathcal{M}}((\exists x)Qx, w_1) = 1$ iff $V_{\mathcal{M}_{x\to\text{u}_1}}(Qx, w_1) = 1$, which is so iff $\mathscr{I}_{x\to\text{u}_1}(x) \in \mathscr{I}(Q, w_1)$. Since $\mathscr{I}(Q, w_1) = \varnothing$, $\mathscr{I}_{x\to\text{u}_1}(x) \notin \mathscr{I}(Q, w_1)$. So $V_{\mathcal{M}}((\exists x)Qx, w_1) = 0$. So $V_{\mathcal{M}}(\Box(\exists x)Qx, w_1) = 0$.

So both disjuncts of '$\Box(\exists x)Px \lor \Box(\exists x)Qx$' are false at w_1. So the disjunction is false at w_1.

So this is an *SQML*-model such that there is some world (w_1) in that model at which the premise, '$(\forall x)\Box(Px \lor Qx)$', is true while the conclusion, '$\Box(\exists x)Px \lor \Box(\exists x)Qx$', is false. So this argument is *SQML*-invalid.

5. Establishing Validity in *SQML*

Suppose that we wish to show that

$$\models_{\text{SQML}} (\forall x)(\forall y)(x = y \rightarrow \Box x = y)$$

This says that it is a tautology of *SQML* that identity is necessary. If Hesperus is identical to Phosphorus, then it is necessary that Hesperus is identical to Phosphorus. And if water is identical to H_2O, then it is necessary that water is identical to H_2O.

We may show that this is a theorem of *SQML* (without any constraints placed on the accessibility relation at all) by providing a semantic proof like the following:

1.	Suppose that there is some *SQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}((\forall x)(\forall y)(x = y \rightarrow \Box x = y), w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}((\forall x)(\forall y)(x = y \rightarrow \Box x = y), w) = 0$	1
3.	So, it is not the case that $V_{\mathcal{M}}((\forall x)(\forall y)(x = y \rightarrow \Box x = y), w) = 1$.	2, *bivalence*
4.	So it is not the case that, for all $\mathfrak{u} \in \mathscr{D}$, $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\forall y)(x = y \rightarrow \Box x = y), w) = 1$.	2, *def.* $\forall$
5.	So there is some $\mathfrak{u} \in \mathscr{D}$—call it '$\mathfrak{u}$'—such that it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\forall y)(x = y \rightarrow \Box x = y), w) = 1$	4, *QL*
6.	So it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\forall y)(x = y \rightarrow \Box x = y), w) = 1$	5
7.	So, it is not the case that, for all $\mathfrak{v} \in \mathscr{D}$, $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y \rightarrow \Box x = y, w) = 1$.	6, *def.* $\forall$
8.	So there is some $\mathfrak{v} \in \mathscr{D}$—call it '$\mathfrak{v}$'—such that it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y \rightarrow \Box x = y, w) = 1$	7, *QL*
9.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y \rightarrow \Box x = y, w) = 0$.	8 *bivalence*
10.	So, $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y, w) = 1$ and $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(\Box x = y, w) = 0$.	3, *def.* $\rightarrow$
11.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y, w) = 1$.	10
12.	So $\mathscr{I}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}(x) = \mathscr{I}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}(y)$.	11, *def.* '='
13.	$\mathscr{I}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}(x) = \mathfrak{u}$.	*def. var. model*
14.	$\mathscr{I}_{x\rightarrow y, y\rightarrow\mathfrak{v}}(y) = \mathfrak{v}$.	*def. var. model*
15.	So $\mathfrak{u} = \mathfrak{v}$.	12, 13, 14
16.	$V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(\Box x = y, w) = 0$	10
17.	So it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(\Box x = y, w) = 1$.	16, *bivalence*
18.	So it is not the case that, for all w', if Rww', then $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y, w') = 1$.	17, *def* $\Box$
19.	So there is some w'—call it 'w''—such that Rww' and it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y, w') = 1$.	18, *QL*
20.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(x = y, w') = 0$.	19, *bivalence*
21.	So $\mathscr{I}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}(x) \neq \mathscr{I}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}(y)$	20, *def* =
22.	So $\mathfrak{u} \neq \mathfrak{v}$.	13, 14, 21
23.	Our assumption has led to a contradiction.	15, 22
24.	So there is no *SQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}((\forall x)(\forall y)(x = y \rightarrow \Box x = y), w) = 0$.	23

The necessity of identity turns out to be a tautology of $SQML$ because we set up our interpretation function in such a way that which entities in the domain a given term refers to does not depend upon which world we are at. Many philosophers are happy to accept the necessity of identity—though not all are happy to accept so-called *trans-world identity*. That is: some philosophers think that, for all x and y, if x exists in world w_1 and y exists in a different world $w_2 \neq w_1$, then x and y are not identical. The question of the necessity of identity and the question of transworld identity are distinct. One could think that, if Mark Twain is Samuel Clemens, then necessarily, Twain is Clemens; even though there is no other possible world in which either Twain or Clemens exist. Those who deny trans-world identity will have to say something about how to understand claims like "Twain is necessarily human". David Lewis denies trans-world identity, but he paraphrases expressions like "Twain is necessarily human" by means of a (contextually variable) *counterpart relation*. Though Twain does not exist at any world other than the actual world, he nevertheless has various *counterparts* at other worlds. On Lewis's approach, to say that Twain is necessarily human is to say that all of his (contextually salient) *counterparts* are human—that is: for all things that exist at any world, if that thing is a counterpart of Twain (according to our contextually salient counterpart relation), then that thing is human. Such a view of *de re* modal predication is sometimes called *Abelardian*, after the medieval philosopher Abelard.

While the necessity of identity is not too controversial, there are other tautologies of $SQML$ which *are* rather controversial. First, consider

$$\models_{\text{SQML}} \Box(\exists x)x = x$$

'$\Box(\exists x)x = x$' says that it is necessary that something exist. Equivalently: it is not possible for there to be nothing at all. To see that this is a tautology of $SQML$, consider the following semantic proof:

1.	Suppose that there is an $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathcal{D}, \mathcal{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\exists x)x = x, w) = 0$.	*Assumption*
2.	Then, it is not the case that $V_{\mathcal{M}}(\Box(\exists x)x = x, w) = 1$.	1, *bivalence*
3.	So it is not the case that, for all w' such that Rww', $V_{\mathcal{M}}((\exists x)x = x, w') = 1$.	2, *def* $\Box$
4.	So there is some w'—call it 'w''—such that Rww' and it is not the case that $V_{\mathcal{M}}((\exists x)x = x, w') = 1$.	3 *QL*
5.	So it is not the case that $V_{\mathcal{M}}((\exists x)x = x, w') = 1$.	4
6.	So it is not the case that there is some $\mathfrak{u} \in \mathcal{D}$ such that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(x = x, w') = 1$.	5 *def* $\exists$
7.	So, for all $\mathfrak{u} \in \mathcal{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(x = x, w') \neq 1$.	6, *QL*
8.	There is some $\mathfrak{u} \in \mathcal{D}$—call it '$\mathfrak{u}_1$'.	*def SQML-model*
9.	So $V_{\mathcal{M}_{x\to\mathfrak{u}_1}}(x = x, w') \neq 1$.	7, 8, $\forall E$
10.	So $\mathcal{I}_{x\to\mathfrak{u}_1}(x) \neq \mathcal{I}_{x\to\mathfrak{u}_1}(x)$.	9, *def* $=$
11.	Our assumption has led to a contradiction.	10
12.	So there is no $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathcal{D}, \mathcal{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\exists x)x = x, w) = 0$.	11, $\sim I$

So *SQML* tells us that it is not possible for there to be nothing at all. Perhaps this is an acceptable consequence. Perhaps, for instance, we think that mathematical truths are made true by the existence of certain *abstracta* bearing certain properties and relations to one another; then, if you think that mathematical truths are necessary, you would have to think that these abstracta exist at every possible world. However, such a view of the nature of mathematical truths is controversial—and perhaps we think that it ought not be settled as a matter of logic alone.

Here is another tautology of *SQML* which is even more controversial:

$$\models_{\text{SQML}} \Box(\forall x)\Box(\exists y)y = x$$

Since '$(\exists y)y = a$' says that a exists, the above wff says that necessarily, everything necessarily exists. Given the T axiom, this claim entails, for instance, that if you exist, then you necessarily exist. In K, it entails that, if it is *possible* for something to exist, then that thing *necessarily* exists.

To see that '$\Box(\forall x)\Box(\exists y)y = x$' is a tautology of *SQML*, consider the following semantic proof:

1.	Suppose that there is some *SQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathcal{D}, \mathcal{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w) = 0$.	1
3.	So it is not the case that $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w) = 1$	2, *bivalence*
4.	So it is not the case that, for all w', if Rww', then $V_{\mathcal{M}}((\forall x)\Box(\exists y)y = x, w') = 1$.	3, *def.* $\Box$
5.	So there is some w'—call it 'w''—such that Rww' and it is not the case that $V_{\mathcal{M}}((\forall x)\Box(\exists y)y = x, w') = 1$.	4, *QL*
6.	So it is not the case that $V_{\mathcal{M}}((\forall x)\Box(\exists y)y = x, w') = 1$	5
7.	So it is not the case that, for all $\mathfrak{u} \in \mathcal{D}$, $V_{\mathcal{M}_{x\to\mathfrak{u}}}(\Box(\exists y)y = x, w') = 1$.	6, *def* $\forall$
8.	So there is some $\mathfrak{u} \in \mathcal{D}$—call it '$\mathfrak{u}$'—such that it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(\Box(\exists y)y = x, w') = 1$.	7, *QL*
9.	So it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(\Box(\exists y)y = x, w') = 1$	8
10.	So it is not the case that, for all w'', if $Rw'w''$, then $V_{\mathcal{M}_{x\to\mathfrak{u}}}((\exists y)y = x, w'') = 1$.	9, *def.* $\Box$
11.	So there is some w''—call it 'w'''—such that $Rw'w''$ and it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}((\exists y)y = x, w'') = 1$.	10, *QL*
12.	So it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u}}}((\exists y)y = x, w'') = 1$	11
13.	So it is not the case that there is some $\mathfrak{v} \in \mathcal{D}$ such that $V_{\mathcal{M}_{x\to\mathfrak{u},y\to\mathfrak{v}}}(y = x, w'') = 1$.	12, *def* $\exists$
14.	So, for all $\mathfrak{v} \in \mathcal{D}$, it is not the case that $V_{\mathcal{M}_{x\to\mathfrak{u},y\to\mathfrak{v}}}(y = x, w'') = 1$.	13, *QL*
15.	$\mathfrak{u} \in \mathcal{D}$	8
16.	So $V_{\mathcal{M}_{x\to\mathfrak{u},y\to\mathfrak{u}}}(y = x, w'') \neq 1$.	14, 15, $\forall E$
17.	So $\mathcal{I}_{x\to\mathfrak{u},y\to\mathfrak{u}}(y) \neq \mathcal{I}_{x\to\mathfrak{u},y\to\mathfrak{u}}(x)$.	12

18.	$\mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{u}}(y) = \mathfrak{u}$.	*def. variant model*
19.	$\mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{u}}(x) = \mathfrak{u}$.	*def. variant model*
20.	So $\mathfrak{u} \neq \mathfrak{u}$.	17, 18, 19
21.	Our assumption has led to a contradiction.	20
22.	So there is no $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w) = 0$.	21, $\sim I$

This thesis that $\Box(\forall x)\Box(\exists y)y = x$ is called *necessitism*. The denial of necessitism is *contingentism*. Paraphrasing '$\Box(\forall x)\Box(\exists y)y = x$' into English, it says that necessarily, everything necessarily exists. Or, in terms of our semantics: at every possible world, everything that exists at that world exists at every world possible from it. That means that there are no contingent objects. Everything which possibly could exist *necessarily* exists. Given the symmetry of the accessibility relation, this means, if you think that Queen Elizabeth *could* have had a son—that is, if it is possible that there is somebody who is Queen Elizabeth's son—then *there is actually* something which is possibly Queen Elizabeth's son. That is, if 'Sx' means 'x is the son of Queen Elizabeth', then:

$$\models_{\text{SQML}} \Diamond(\exists x)Sx \to (\exists x)\Diamond Sx$$

This is an instance of a more general schema known as the *Barcan Formula*:

$$\Diamond(\exists \alpha)\phi \to (\exists \alpha)\Diamond\phi \qquad (BF)$$

By contraposing and making use of the equivalences $\ulcorner\sim(\exists\alpha)\phi \leftrightarrow (\forall\alpha)\sim\phi\urcorner$ and $\ulcorner\sim\Diamond\phi \leftrightarrow \Box\sim\phi\urcorner$, the Barcan Formula may be written (equivalently) as:

$$(\forall \alpha)\Box\phi \to \Box(\forall \alpha)\phi \qquad (BF')$$

The Barcan formula says that there are no merely possible individuals; like, for instance, Queen Elizabeth's merely possible son. Every instance of the Barcan Formula is an $SQML$-tautology. For instance, here is a semantic proof demonstrating that '$\Diamond(\exists x)Fx \to (\exists x)\Diamond Fx$' is an $SQML$-tautology:

1.	Suppose that there is an $SQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Diamond(\exists x)Fx \to (\exists x)\Diamond Fx, w) = 0$.	*Assumption*
2.	So $V_{\mathcal{M}}(\Diamond(\exists x)Fx, w) = 1$ and $V_{\mathcal{M}}((\exists x)\Diamond Fx, w) = 0$.	1, *def* $\to$
3.	So $V_{\mathcal{M}}(\Diamond(\exists x)Fx, w) = 1$.	2
4.	So there is some w'—call it 'w''—such that Rww' and $V_{\mathcal{M}}((\exists x)Fx, w') = 1$.	3 *def* $\Diamond$
5.	So $V_{\mathcal{M}}((\exists x)Fx, w') = 1$.	4
6.	So there is some $\mathfrak{u} \in \mathscr{D}$—call it '$\mathfrak{u}_1$'—such that $V_{\mathcal{M}_{x\to\mathfrak{u}_1}}(Fx, w') = 1$.	5, *def* $\exists$
7.	So $V_{\mathcal{M}_{x\to\mathfrak{u}_1}}(Fx, w') = 1$	6
8.	And $V_{\mathcal{M}}((\exists x)\Diamond Fx, w) = 0$.	2
9.	So it is not the case that $V_{\mathcal{M}}((\exists x)\Diamond Fx, w) = 1$.	8, *bivalence*
10.	So it is not the case that there is some $\mathfrak{u} \in \mathscr{D}$ such that $V_{\mathcal{M}_{x\to\mathfrak{u}}}(\Diamond Fx, w) = 1$.	9, *def* $\exists$

11.	So, for all $\mathfrak{u} \in \mathscr{D}$, it is not the case that $V_{\mathcal{M}_{x \to \mathfrak{u}}}(\Diamond Fx, w) = 1$.	10, *QL*
12.	$\mathfrak{u}_1 \in \mathscr{D}$.	6
13.	So it is not the case that $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(\Diamond Fx, w) = 1$	11, 12 $\forall E$
14.	So it is not the case that there is some w'' such that Rww'' and $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx, w'') = 1$.	13, *def* $\Diamond$
15.	So, for all w'', if Rww'', then it is not the case that $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx, w'') = 1$.	14, *QL*
16.	So, if Rww', then it is not the case that $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx, w') = 1$.	15, $\forall E$
17.	Rww'	4
18.	So it is not the case that $V_{\mathcal{M}_{x \to \mathfrak{u}_1}}(Fx, w') = 1$.	16, 17, $\to E$
19.	Lines 7 and 18 contradict.	7, 18
20.	So our assumption has led to a contradiction.	19
21.	So there is no *SQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$, with a $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Diamond(\exists x)Fx \to (\exists x)\Diamond Fx, w) = 0$.	20, $\sim I$

While many have found the Barcan formula counterintuitive, fewer have found the *converse* Barcan formula as counterintuitive. The converse Barcan formula is just the converse of the Barcan formula, namely:

$$(\exists \alpha)\Diamond\phi \to \Diamond(\exists \alpha)\phi \qquad (CBF)$$

Or, equivalently (given the duality of necessity and possibility and the duality of universal and existential quantification):

$$\Box(\forall \alpha)\phi \to (\forall \alpha)\Box\phi \qquad (CBF')$$

The converse Barcan formula says that, if something is possibly ϕ, then it is possible that something is ϕ. So, for instance: if there is something which could possibly travel faster than the speed of light, then it is possible that something travel faster than the speed of light. Or, equivalently: if it is necessary that everything travels at or below the speed of light, then everything necessarily travels at or below than the speed of light.

Nevertheless, there are some more controversial instances of the converse Barcan formula. Consider, for instance, the wff we get if we let $\ulcorner \alpha \urcorner =$ 'x' and we let $\ulcorner \phi \urcorner =$ '$\sim(\exists y)y = x$':

$$(\exists x)\Diamond\sim(\exists y)y = x \to \Diamond(\exists x)\sim(\exists y)y = x$$

This says that, if there exists something contingent—*i.e.*, something which is possibly not identical to anything—then it is possible that there is something which isn't identical to anything. The contingentist will accept this antecedent, since they think that there are things which exist which nevertheless could fail to exist; yet they will deny its consequent, since it is necessary that everything that exists exists.

6. Axioms for *SQML*

Suppose that we take the axiom schemata of *QL* and add to them the axiom schemata of our various propositional modal logic theorems. For the modal system *K*, this would give us the following axioms:

$$\vdash_{\text{SQML-K}} \phi, \quad \text{for all } PL\text{-valid schemata } \ulcorner\phi\urcorner \qquad (PL)$$

$$\vdash_{\text{SQML-K}} \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \qquad (K)$$

$$\vdash_{\text{SQML-K}} (\forall\alpha)\phi \rightarrow \phi[\tau/\alpha] \qquad (\forall 1)$$

provided that $\ulcorner\tau\urcorner$ *is free in* $\ulcorner\phi[\tau/\alpha]\urcorner$

$$\vdash_{\text{SQML-K}} (\forall\alpha)(\phi \rightarrow \psi) \rightarrow (\phi \rightarrow (\forall\alpha)\psi) \qquad (\forall 2)$$

provided that $\ulcorner\alpha\urcorner$ *is not free in* $\ulcorner\phi\urcorner$

$$\vdash_{\text{SQML-K}} \tau = \tau \qquad (=1)$$

$$\vdash_{\text{SQML-K}} \tau_1 = \tau_2 \rightarrow (\phi \rightarrow \phi[\tau_2 /\!/ \tau_1]) \qquad (=2)$$

provided that $\ulcorner\tau_2\urcorner$ *is free in* $\ulcorner\phi[\tau_2 /\!/ \tau_1]\urcorner$

and the following rules of inference:

Propositional Logic Rules (*PLR*): If $\ulcorner\psi\urcorner$ follows from $\ulcorner\phi\urcorner$ according to propositional logic, then, from $\ulcorner\phi\urcorner$, infer $\ulcorner\psi\urcorner$.

Generalization (*G*): If a wff $\ulcorner\phi\urcorner$ is a theorem, then you may infer $\ulcorner(\forall\alpha)\phi\urcorner$ as a theorem, where $\ulcorner\alpha\urcorner$ is a variable.

$$\text{from } \vdash_{\text{SQML-K}} \phi, \text{ infer } \vdash_{\text{SQML-K}} (\forall\alpha)\phi$$

Necessitation (*N*): If a wff $\ulcorner\phi\urcorner$ is a theorem, then you may infer $\ulcorner\Box\phi\urcorner$ as a theorem.

$$\text{from } \vdash_{\text{SQML-K}} \phi, \text{ infer } \vdash_{\text{SQML-K}} \Box\phi$$

For the system *SQML-D*, we would add the axiom schema $\ulcorner\Box\phi \rightarrow \Diamond\phi\urcorner$; for the system *SQML-T*, we would add the axiom schema $\ulcorner\Box\phi \rightarrow \phi\urcorner$; for the system *SQML-B*, we would add the axiom schema $\ulcorner\phi \rightarrow \Box\Diamond\phi\urcorner$; for the system *SQML*-S4, we would add $\ulcorner\Box\Box\phi \rightarrow \Box\phi\urcorner$; and, for the system *SQML*-S5, we would add $\ulcorner\Diamond\phi \rightarrow \Box\Diamond\phi\urcorner$.

If we take this simple approach, an unexpected result follows. While all instances of the Barcan formula, (*BF′*), are valid in the systems *SQML-B* and *SQML*-S5, they are *not* all valid in the systems *SQML-K*, *SQML-D*, *SQML-T*, and *SQML*-S4.

$$(\forall\alpha)\Box\phi \rightarrow \Box(\forall\alpha)\phi \qquad (BF')$$

This seems like an odd result on its own; since it appears that the Barcan formula should either be valid in all of the systems or none of them. It also spells disaster if we want our axiomatic system to be both sound and complete for the semantics *SQML*—since the Barcan formula is true in *all* *SQML*-models.

Similarly, while the necessity of identity, '$(\forall x)(\forall y)(x = y \rightarrow \Box x = y)$' is a theorem of all of these systems, the necessity of *distinctness*, ($\Box \neq$),

$$(\forall x)(\forall y)(x \neq y \rightarrow \Box x \neq y) \qquad (\Box \neq)$$

is only a theorem of *SQML-B* and *SQML*-S5. Again, this seems like an odd result on its own. It appears that ($\Box \neq$) should either be a theorem of all of the systems or none of them. This, too, spells disaster if we want our axiomatic system to be both sound and complete for the semantics *SQML*, since ($\Box \neq$) is true in *all SQML*-models.

The solution is to add both the Barcan formula and the necessity of distinctness as axiom schemata in all of the systems (though they will be redundant in *B* and S5):

With the addition of (BF') and ($\Box \neq$) as axiom schemata, the axiomatic systems *QML-K*, *QML-D*, *QML-T*, *QML-B*, *QML*-S4, and *QML*-S5 will all be sound and complete for the semantics *SQML*.

7. Natural Deduction for *SQML*

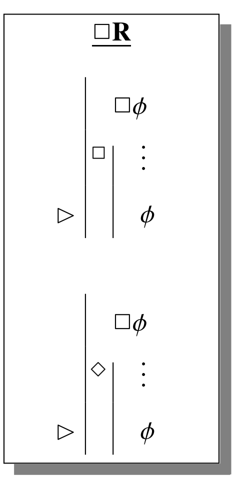
The axiomatic systems for *SQML* are difficult to work with. So we introduce a *natural deduction* system for the various systems of *SQML*. For these natural deduction systems, we will take all the rules of our natural deduction system for *QL*, all of the rules from the natural deduction systems for the systems of propositional modal logic, and add to them two new rules.

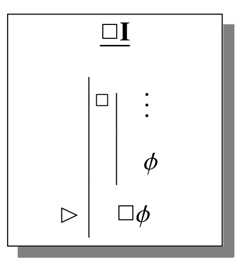
To refresh your memory, every natural deduction system for *PML* has the following rules for quantifier negation:

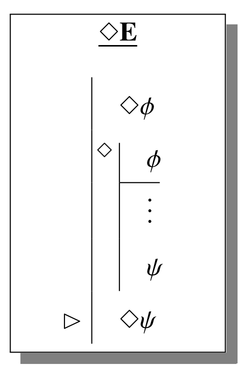
MN

$\sim\Box\phi$	$\lhd\ \rhd$	$\Diamond\sim\phi$
$\sim\Diamond\phi$	$\lhd\ \rhd$	$\Box\sim\phi$
$\Box\phi$	$\lhd\ \rhd$	$\sim\Diamond\sim\phi$
$\Diamond\phi$	$\lhd\ \rhd$	$\sim\Box\sim\phi$

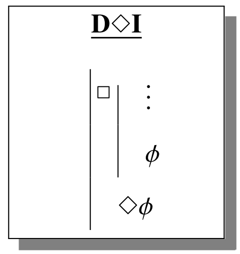
Here are the natural deduction rules for the system *K*:



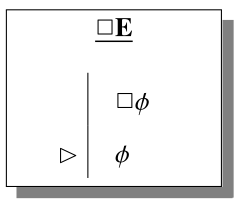


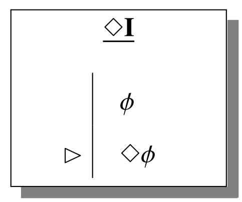


For *D*, we add to the rules from *K* the following:

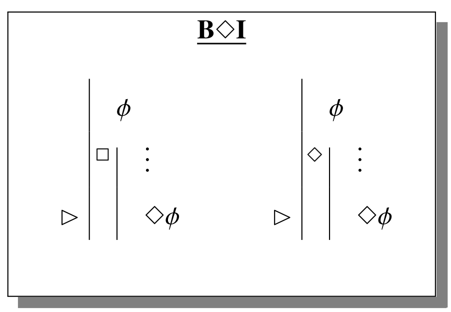


For *T*, we add the following rules of inference to those from *K*:

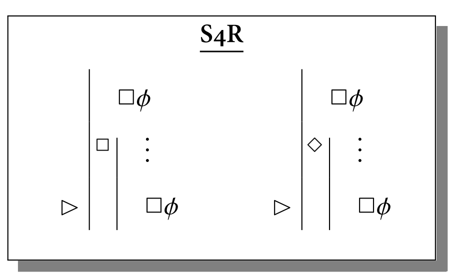




For *B*, we add to the rules from *T* the following:



For S4, we add the following to the natural deduction rules for *T*:



For S5, we add the following rules to the ones from S4:

S5R

	$\Diamond\phi$	
$\Box$	$\vdots$	
$\triangleright$	$\Diamond\phi$	

	$\Diamond\phi$	
$\Diamond$	$\vdots$	
$\triangleright$	$\Diamond\phi$	

Now, to get a natural deduction system for *SQML*, we will simply add to the foregoing rules (plus the rules of *PL* and the rules of *QL*), two new ones:

BF

	$(\forall\alpha)\Box\phi$
$\triangleright$	$\Box(\forall\alpha)\phi$

$\Box \neq$

	$\tau_1 \neq \tau_2$
$\triangleright$	$\Box\tau_1 \neq \tau_2$

This gives us a natural deduction system for each of *SQML-K*, *SQML-D*, *SQML-T*, *SQML-B*, *SQML*-S4, and *SQML*-S5. ***However***, we must qualify the rule $(\forall I)$ slightly once we have strict subproofs in the mix. When we reiterate into a strict subproof with the rules like $\Box R$, any constants or free variables which appear in the wffs we write down must count as assumptions for the purposes of the rule $(\forall I)$.

For instance, the following is *not* a legal derivation of an instance of the Barcan formula in *SQML-K*:

1	$(\forall x)\Box Fx$	$A(\rightarrow I)$	
2	$\sim\Box(\forall x)Fx$	$A(\sim E)$	
3	$\Box Fz$	1, $\forall E$	
4	$\Box$ Fz	3, $\Box R$	
5	$(\forall x)Fx$	4, $\forall I$	$\longleftarrow$ **MISTAKE!!!**
6	$\Box(\forall x)Fx$	4–5, $\Box I$	
7	$\Box(\forall x)Fx \wedge \sim\Box(\forall x)Fx$	2, 6, $\wedge I$	
8	$\Box(\forall x)Fx$	2–7, $\sim E$	
9	$(\forall x)\Box Fx \rightarrow \Box(\forall x)Fx$	1–8, $\rightarrow I$	

The variable 'z' appears free when it is reiterated within the strict subproof on line 4; because 'Fz' is written down within a strict subproof with an application of $\Box R$, it counts as an assumption for the rule $\forall I$. Therefore, we may not universally generalize from the variable 'z' on line 5; for that variable appears free in an assumption of an open (strict) subderivation.

In order to derive this instance of the Barcan formula in *SQML-K*, we must make use of the rule (BF), like so:

1	$(\forall x)\Box Fx$	$A(\rightarrow I)$
2	$\Box(\forall x)Fx$	1, BF
3	$(\forall x)\Box Fx \rightarrow \Box(\forall x)Fx$	1–2, $\rightarrow I$

If we can prove $\ulcorner\phi\urcorner$ from the set of wff Γ in this natural deduction system for K plus the rules of QL and (BF) and $(\Box \neq)$, then we will write

$$\Gamma \vdash^{D}_{SQML\text{-}K} \phi$$

And if we can prove $\ulcorner\phi\urcorner$ from no assumptions in this natural deduction system, then we will write

$$\vdash^{D}_{SQML\text{-}K} \phi$$

Corresponding subscripts will be added for the various modal systems beyond K. For instance, if $\ulcorner\phi\urcorner$ can be proven from no assumptions in the natural deduction system that we get by taking the rules for system B from *PML* together the the other rules, then we will write

$$\vdash^{D}_{SQML\text{-}B} \phi$$

For instance, here is a natural deduction proof, in *SQML-K*, of the necessity of identity:

$$\vdash^{D}_{SQML\text{-}K} (\forall x)(\forall y)(x = y \rightarrow \Box x = y)$$

1	$x = y$	$A(\rightarrow I)$
2	$\Box$ $x = x$	*Id*
3	$\Box x = x$	2–2, $\Box I$
4	$\Box x = y$	1, 3 *Id*
5	$x = y \rightarrow \Box x = y$	1–4, $\rightarrow I$
6	$(\forall y)(x = y \rightarrow \Box x = y)$	5, $\forall I$
7	$(\forall x)(\forall y)(x = y \rightarrow \Box x = y)$	6, $\forall I$

And here is a derivation, in *SQML-K*, establishing the necessity of *distinctness*:

$$\vdash^{D}_{\text{SQML-K}} (\forall x)(\forall y)(x \neq y \rightarrow \Box x \neq y)$$

1	$\mid\ x \neq y$	$A(\rightarrow I)$
2	$\mid\ \Box x \neq y$	$1, \Box \neq$
3	$x \neq y \rightarrow \Box x \neq y$	$1\text{–}2, \rightarrow I$
4	$(\forall y)(x \neq y \rightarrow \Box x \neq y)$	$3, \forall I$
5	$(\forall x)(\forall y)(x \neq y \rightarrow \Box x \neq y)$	$4, \forall I$

Here is a derivation, in *SQML-K*, establishing that an instance of the converse Barcan formula is a theorem of our natural deduction system for *SQML-K*:

$$\vdash^{D}_{\text{SQML-K}} (\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx$$

1	$\mid\ (\exists x)\Diamond Fx$	$A(\rightarrow I)$
2	$\mid\ \mid\ \Diamond Fa$	$A(\exists E)$
3	$\mid\ \mid\ \Diamond \mid\ Fa$	$A(\Diamond E)$
4	$\mid\ \mid\ \mid\ (\exists x)Fx$	$3, \exists I$
5	$\mid\ \mid\ \Diamond(\exists x)Fx$	$3\text{–}4, \Diamond E$
6	$\mid\ \Diamond(\exists x)Fx$	$1, 2\text{–}5, \exists E$
7	$(\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx$	$1\text{–}6, \rightarrow I$

And here is a derivation, in *SQML-K*, of the necessitist thesis that necessarily, everything necessarily exists:

$$\vdash^{D}_{\text{SQML-K}} \Box(\forall x)\Box(\exists y)y = x$$

1	$\Box \mid \Box \mid\ z = z$	Id
2	$\mid\ \mid\ (\exists y)y = z$	$1, \exists I$
3	$\mid\ \Box(\exists y)y = z$	$1\text{–}2, \Box I$
4	$\mid\ (\forall x)\Box(\exists y)y = x$	$3, \forall I$
5	$\Box(\forall x)\Box(\exists y)y = x$	$1\text{–}4, \Box I$

Here is a derivation, in *SQML-K*, of an instance of the existential form of the Barcan formula:

$$\overset{\text{D}}{\underset{\text{SQML-K}}{\vdash}} \Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx$$

1	$\Diamond(\exists x)Fx$	$A(\rightarrow I)$
2	$\quad \sim(\exists x)\Diamond Fx$	$A(\sim E)$
3	$\quad (\forall x)\sim\Diamond Fx$	2, QN
4	$\quad (\forall x)\Box\sim Fx$	3, MN
5	$\quad \Box(\forall x)\sim Fx$	4, BF
6	$\quad \Diamond \quad (\exists x)Fx$	$A(\Diamond E)$
7	$\qquad (\forall x)\sim Fx$	5, $\Box R$
8	$\qquad \sim(\exists x)Fx$	7, QN
9	$\qquad\quad \sim Fc$	$A(\sim E)$
10	$\qquad\quad (\exists x)Fx \wedge \sim(\exists x)Fx$	6, 8, $\wedge I$
11	$\qquad Fc$	9–10, $\sim E$
12	$\quad \Diamond Fc$	1, 6–11, $\Diamond E$
13	$\quad (\exists x)\Diamond Fx$	12, $\exists I$
14	$\quad (\exists x)\Diamond Fx \wedge \sim(\exists x)\Diamond Fx$	2, 13, $\wedge I$
15	$(\exists x)\Diamond Fx$	2–14, $\sim E$
16	$\Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx$	1–15, $\rightarrow I$

In the above proof, we made use of the rule (BF) on line 5. This is necessary to prove the Barcan formula in *SQML-K*. We learned above, however, that the Barcan formula was a theorem of *QML-B without* the addition of (BF') as an axiom schema. We can show something similar with our natural deduction system by providing a derivation of an instance of the Barcan formula, '$\Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx$' in the natural deduction system for *SQML-B* which *does not* make use of the rule (BF):

1	$\Diamond(\exists x)Fx$	$A(\to I)$
2	$\sim(\exists x)\Diamond Fx$	$A(\sim E)$
3	$(\forall x)\sim\Diamond Fx$	2, QN
4	$\Diamond$ $(\exists x)Fx$	$A(\Diamond E)$
5	$\Diamond(\forall x)\sim\Diamond Fx$	3, $B\Diamond I$
6	Fc	$A(\exists E)$
7	$\Diamond$ $(\forall x)\sim\Diamond Fx$	$A(\Diamond E)$
8	$\sim\Diamond Fc$	7, $\forall E$
9	$\Diamond\sim\Diamond Fc$	5, 7–8, $\Diamond E$
10	$\sim\Box\Diamond Fc$	9, MN
11	$\Box$ $\Diamond Fc$	6, $B\Diamond I$
12	$\Box\Diamond Fc$	11–11, $\Box I$
13	$\sim Fd$	$A(\sim E)$
14	$\Box\Diamond Fc \wedge \sim\Box\Diamond Fc$	10, 12, $\wedge I$
15	Fd	13–14, $\sim I$
16	Fd	4, 6–15, $\exists E$
17	$\Diamond Fd$	1, 4–16, $\Diamond E$
18	$\sim\Diamond Fd$	3, $\forall E$
19	$\Diamond Fd \wedge \sim\Diamond Fd$	17, 18, $\wedge I$
20	$(\exists x)\Diamond Fx$	2–19, $\sim E$
21	$\Diamond(\exists x)Fx \to (\exists x)\Diamond Fx$	1–20 $\to I$

The above proof proves this instance of the Barcan formula in *SQML-B* *without* the rule (*BF*). A similar proof is possible in *SQML*-S5.

8. Variable Domain Quantified Modal Logic

If we find the theorems of *SQML* unpalatable, then there is a solution. *SQML*-models had a single domain which the quantifiers ranged over, no matter what the world of evaluation was. The solution is to drop this assumption and have the domain of quantification vary from world to world. To do this, we will introduce a new piece of apparatus to our models: a *function* $\mathscr{Q}$, which takes as input a world and provides as output a subset of our domain $\mathscr{D}$. The interpretation is that $\mathscr{Q}(w)$ contains those entities which exist at world w.

When we build up our variable-domain quantified modal logic models, we will then provide a set of worlds $\mathcal{W}$, a binary relation R amongst those worlds, a (non-empty) set of entities $\mathscr{D}$, a function $\mathscr{Q}$ from worlds to subsets of $\mathscr{D}$, and an interpretation function $\mathscr{I}$. With the interpretation function, however, we now face an option. With $SQML$-models, we simply required $\mathscr{I}(\Pi^N, w)$ to be a subset of $\mathscr{D}^N$; this was because the domain $\mathscr{D}$ was the same at every possible world. We might want to say that, if an entity doesn't exist at a world—and if, for that reason, a constant $\ulcorner \alpha \urcorner$ does not refer at that world—then that entity does not have any properties at that world. An entity must exist at a world in order to have properties at that world. This would amount to the requirement that $\mathscr{I}(\Pi^N, w)$ be a subset of $\mathscr{Q}(w)^N$. If we impose such a requirement, then we get a *negative* variable domain quantified modal logic-model—or what we'll call a '$NVDQML$-model'.

> NVDQML-Model:
> A NVDQML-MODEL $\mathcal{M}$ is a 5-tuple $< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$ of a (non-empty) set of worlds, $\mathcal{W}$, a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$, a (non-empty) *domain* of entities, $\mathscr{D}$, a function, $\mathscr{Q}$, from worlds in $\mathcal{W}$ to subsets of $\mathscr{D}$, and an interpretation function, $\mathscr{I}$. $\mathscr{I}$ maps a term $\ulcorner \tau \urcorner$ of QML to an entity in $\mathscr{D}$, and it maps a pair of a world w and an N-place predicate $\ulcorner \Pi^N \urcorner$ of QML to a set of N-tuples of entities in $\mathscr{Q}(w)$. Thus, for every term $\ulcorner \tau \urcorner$ of QML,
>
> $$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$
>
> And for every N-place predicate $\ulcorner \Pi^N \urcorner$ of QML, and every world $w \in \mathcal{W}$,
>
> $$\mathscr{I}(\Pi^N, w) = \{\ldots, < \mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_N >, \ldots\} \subseteq \underbrace{\mathscr{Q}(w) \times \mathscr{Q}(w) \times \cdots \times \mathscr{Q}(w)}_{N\ times} = \mathscr{Q}(w)^N$$

On the other hand, we may wish to allow entities to have properties even at worlds where they do not exist. If we wish to allow this, then we may simply allow $\mathscr{I}(\Pi^N, w)$ to be any subset of $\mathscr{D}^N$. If we do this, then we will get a *positive* variable domain quantified modal logic model—or just a '$VDQML$-model'.

> VDQML-Model:
> A VDQML-MODEL $\mathcal{M}$ is a 5-tuple $< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$ of a (non-empty) set of worlds, $\mathcal{W}$, a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$, a (non-empty) *domain* of entities, $\mathscr{D}$, a function, $\mathscr{Q}$, from worlds in $\mathcal{W}$ to subsets of $\mathscr{D}$, and an interpretation function, $\mathscr{I}$, from pairs of worlds and terms or predicates of QML to (tuples of) the entities in $\mathscr{D}$. $\mathscr{I}$ maps a term $\ulcorner \tau \urcorner$ of QML to an entity in $\mathscr{D}$, and it maps a pair of a world w and an N-place predicate $\ulcorner \Pi^N \urcorner$ of QML to a set of N-tuples of entities in $\mathscr{D}$. Thus, for every term $\ulcorner \tau \urcorner$ of QML,
>
> $$\mathscr{I}(\tau) = \mathfrak{u} \in \mathscr{D}$$
>
> And for every N-place predicate $\ulcorner \Pi^N \urcorner$ of QML, and every world $w \in \mathcal{W}$,
>
> $$\mathscr{I}(\Pi^N, w) = \{\ldots, < \mathfrak{u}_1, \mathfrak{u}_2, \ldots, \mathfrak{u}_N >, \ldots\} \subseteq \underbrace{\mathscr{D} \times \mathscr{D} \times \cdots \times \mathscr{D}}_{N\ times} = \mathscr{D}^N$$

Note the similarity between our discussion here and our discussion of positive and negative free logics. In the case of *FL*, we were motivated by the thought that it should not be a truth of logic that Santa Claus exists. That is, if 'c' is name for Santa Clause, then '$(\exists x)x = c$' should not be a logical truth—it should not be a truth of logic that Santa Claus exists simply because we have a name for Santa Claus. In the case of *VDQML*, we were motivated in part by the thought that it should not be a logical truth that *Adele* necessarily exists; that is, if 'a' is a name for Adele, then it should not be a truth of logic that $\Box(\exists x)x = a$. It should not be a truth of logic that Adele necessarily exists simply because Adele actually exists. In the case of *FL*, the solution was to allow names that don't refer to anything which exists. In the case of *VDQML*, our solution is to allow names that don't refer to anything which exists *at a world*. And, just as in *FL*, we faced a choice about whether or not to allow things which don't exist to have properties, in *VDQML*, we face a choice about whether or not to allow things which don't exist *at a world* to have properties *at that world*.

Going forward, it won't matter whether we are dealing with a *NVDQML*-model or a *VDQML*-model; all the rest of the semantics are common to both.

We will define a *variant VDQML*-model in precisely the same way that we defined a variant *QL* model and a variant *SQML*-model—we simply take the interpretation function $\mathscr{I}$, and have it map a variable $\ulcorner \alpha \urcorner$ to some new entity $\mathfrak{u} \in \mathscr{D}$.

> Variant VDQML-Model:
> Given an *VDQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, a variable of *QML*, $\ulcorner \alpha \urcorner$, and some $\mathfrak{u} \in \mathscr{D}$, the variant *VDQML*-model $\mathcal{M}_{\alpha\to\mathfrak{u}} \stackrel{\text{def}}{=} < \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I}_{\alpha\to\mathfrak{u}} >$ where:
> $$\mathscr{I}_{\alpha\to\mathfrak{u}} \stackrel{\text{def}}{=} (\mathscr{I} - < \alpha, \mathscr{I}(\alpha) >) \cup < \alpha, \mathfrak{u} >$$

An alternative, but equivalent, definition of $\mathscr{I}_{\alpha\to\mathfrak{u}}$ is given by the following: for any N-place predicate $\ulcorner \Pi^N \urcorner$ and any world w,

$$\mathscr{I}_{\alpha\to\mathfrak{u}}(\Pi^N, w) = \mathscr{I}(\Pi^N, w)$$

and, for any term $\ulcorner \tau \urcorner$,

$$\mathscr{I}_{\alpha\to\mathfrak{u}}(\tau) = \begin{cases} \mathscr{I}(\tau) & \text{if } \tau \neq \alpha \\ \mathfrak{u} & \text{if } \tau = \alpha \end{cases}$$

We may now provide a definition of a *valuation function* for *VDQML*. Everything from *SQML* carries over *except* the clause for '$\forall$', which now tells us that a wff $\ulcorner (\forall\alpha)\phi \urcorner$ is true at a world w iff $\ulcorner \phi \urcorner$ is true at world w in the variant model $\mathcal{M}_{\alpha\to\mathfrak{u}}$ *for every* $\mathfrak{u} \in \mathscr{Q}(w)$. That is: $\ulcorner (\forall\alpha)\phi \urcorner$ is true at w iff $\ulcorner \phi \urcorner$ is true, no matter what we let $\ulcorner \alpha \urcorner$ refer to *at world* w. So, at a world w, '$(\forall x)Fx$' says that everything *at* w is F.

> VDQML-Valuation:
> Given a *VDQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, we define a *VDQML*-valuation function, $V_{\mathcal{M}}$, in the following way: for every world $w \in \mathcal{W}$, any

N-place predicate $\ulcorner \Pi^N \urcorner$, any N terms $\ulcorner \tau_1 \urcorner, \ulcorner \tau_2 \urcorner, \ldots, \ulcorner \tau_N \urcorner$, any variable $\ulcorner \alpha \urcorner$, and any wffs of QML $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\Pi^N \tau_1 \tau_2 \ldots \tau_N, w) = 1$ iff $< \mathscr{I}(\tau_1), \mathscr{I}(\tau_2), \ldots, \mathscr{I}(\tau_N) > \in \mathscr{I}(\Pi^N, w)$.

(2) $V_{\mathcal{M}}(\tau_1 = \tau_2, w) = 1$ iff $\mathscr{I}(\tau_1) = \mathscr{I}(\tau_2)$.

(3) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.

(4) $V_{\mathcal{M}}(\phi \rightarrow \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.

(5) $V_{\mathcal{M}}((\forall\alpha)\phi, w) = 1$ iff, for all $\mathfrak{u} \in \mathscr{Q}(w)$, $V_{\mathcal{M}_{\alpha \rightarrow \mathfrak{u}}}(\phi, w) = 1$.

(6) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for all $w' \in \mathcal{W}$, if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

By placing the familiar restrictions on the accessibility relation R, we arrive at the following defintions of $VDQML$-models for each of the systems we encountered when looking at propositional modal logic.

(D) A $VDQML$-D-model is a $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is serial.

(T) A $VDQML$-T-model is a $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive.

(B) A $VDQML$-B-model is a $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and symmetric.

(S4) A $VDQML$-S4-model is a $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and transitive.

(S5) A $VDQML$-S5-model is a $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{I} >$ such that R is reflexive and euclidean (and therefore, reflexive, symmetric, and transitive).

8.1. *VDQML*-**Consequence.** We will say that $\ulcorner \phi \urcorner$ is a $VDQML$-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is $VDQML$-valid,

$$\Gamma \models_{\text{VDQML}} \phi$$

iff there is no $VDQML$-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every $VDQML$-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

And we will say that a wff $\ulcorner \phi \urcorner$ is a *VDQML-tautology*, or *VDQML-valid*, written

$$\models_{\text{VDQML}} \phi$$

if and only if $\ulcorner \phi \urcorner$ is true at every world in every $VDQML$ model.

Similarly, we'll say that the argument from Γ to $\ulcorner \phi \urcorner$ is $VDQML$-D-valid,

$$\Gamma \models_{\text{VDQML-D}} \phi$$

iff there is no world in any $VDQML$-D-model at which all the wffs in Γ are true, yet $\ulcorner\phi\urcorner$ is false. And we'll say that $\ulcorner\phi\urcorner$ is a $VDQML$-D-tautology, or $VDQML$-D-valid,

$$\models_{\text{VDQML-D}} \phi$$

iff there is no world in any $VDQML$-D-model at which $\ulcorner\phi\urcorner$ is false.

We may provide similar definitions of consequence for each of the other systems of $VDQML$:

(T) $\Gamma \models_{\text{VDQML-T}} \phi$ iff there is no world in any $VDQML$-T-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{VDQML-T}} \phi$ iff there is no world in any $VDQML$-T-model at which $\ulcorner\phi\urcorner$ is false;

(B) $\Gamma \models_{\text{VDQML-B}} \phi$ iff there is no world in any $VDQML$-B-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{VDQML-B}} \phi$ iff there is no world in any $VDQML$-B-model at which $\ulcorner\phi\urcorner$ is false;

(S4) $\Gamma \models_{\text{VDQML-4}} \phi$ iff there is no world in any $VDQML$-S4-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{VDQML-4}} \phi$ iff there is no world in any $VDQML$-S4-model at which $\ulcorner\phi\urcorner$ is false;

(S5) $\Gamma \models_{\text{VDQML-5}} \phi$ iff there is no world in any $VDQML$-S5-model at which all the members of Γ are true, yet $\ulcorner\phi\urcorner$ is false; $\models_{\text{VDQML-5}} \phi$ iff there is no world in any $VDQML$-S5-model at which $\ulcorner\phi\urcorner$ is false.

9. Establishing Invalidity in $VDQML$

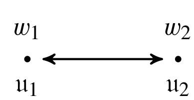
To show that an argument is $VDQML$-invalid, it is enough to provide a $VDQML$-model such that the premises are all true at some world in that model, yet the conclusion is false at that world. To show that a wff is $VDQML$-invalid, it is enough to provide a $VDQML$-model such that the wff is false at some world in that model.

For instance, we may show that the necessitist thesis that $\Box(\forall x)\Box(\exists y)y = x$ is not a $VDQML$-tautology

$$\not\models_{\text{VDQML}} \Box(\forall x)\Box(\exists y)y = x$$

by providing the following $VDQML$-model:

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2\} \\ R &= \{< w_1, w_2 >, < w_2, w_1 >\} \\ \mathcal{D} &= \{u_1, u_2\} \\ \mathcal{Q}(w_1) &= \{u_1\} \\ \mathcal{Q}(w_2) &= \{u_2\} \end{aligned}$$



In this model, '$\Box(\forall x)\Box(\exists y)y = x$' is false at w_1. For $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w_1) = 1$ iff $V_{\mathcal{M}}((\forall x)\Box(\exists y)y = x, w_2) = 1$, since w_2 is the only world which w_1 sees. And

$V_{\mathcal{M}}((\forall x)\Box(\exists y)y = x, w_2) = 1$ iff $V_{\mathcal{M}_{x\to\mathfrak{u}_2}}(\Box(\exists y)y = x, w_2) = 1$, since $\mathfrak{u}_2$ is the only entity which exists at w_2. And $V_{\mathcal{M}_{x\to\mathfrak{u}_2}}(\Box(\exists y)y = x, w_2) = 1$ iff $V_{\mathcal{M}_{x\to\mathfrak{u}_2}}((\exists y)y = x, w_1) = 1$, since w_1 is the only world which w_2 sees. And $V_{\mathcal{M}_{x\to\mathfrak{u}_2}}((\exists y)y = x, w_1) = 1$ iff $V_{\mathcal{M}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}}(y = x, w_1) = 1$, since $\mathfrak{u}_1$ is the only entity which exists at w_1. But $V_{\mathcal{M}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}}(y = x, w_1) \neq 1$, because $V_{\mathcal{M}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}}(y = x, w_1) = 1$ iff $\mathscr{I}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}(y) = \mathscr{I}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}(x)$, but $\mathscr{I}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}(y) = \mathfrak{u}_1$ while $\mathscr{I}_{x\to\mathfrak{u}_2, y\to\mathfrak{u}_1}(x) = \mathfrak{u}_2$. Since $\mathfrak{u}_1 \neq \mathfrak{u}_2$, $V_{\mathcal{M}}(\Box(\forall x)\Box(\exists y)y = x, w_1) = 0$.

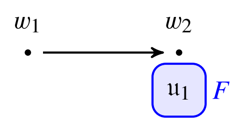
Because there were no predicates in this wff, the above model also shows that the necessitist thesis is not a $NVDQML$-tautology.

Similarly, we may show that an instance of the Barcan Formula is not a $VDQML$-tautology

$$\not\models_{\text{VDQML}} \Diamond(\exists x)Fx \to (\exists x)\Diamond Fx$$

by providing the following $VDQML$-model:

$$\begin{aligned}
\mathcal{W} &= \{w_1, w_2\} \\
R &= \{< w_1, w_2 >\} \\
\mathscr{D} &= \{\mathfrak{u}_1\} \\
\mathscr{Q}(w_1) &= \varnothing \\
\mathscr{Q}(w_2) &= \{\mathfrak{u}_1\} \\
\mathscr{I}(F, w_1) &= \varnothing \\
\mathscr{I}(F, w_2) &= \{\mathfrak{u}_1\}
\end{aligned}$$



In this model, the antecedent '$\Diamond(\exists x)Fx$' of our wff is true at w_1; yet its consequent '$(\exists x)\Diamond Fx$' is false at w_1. So the conditional is false at w_1.

To see that '$\Diamond(\exists x)Fx$' is true at w_1, note that there is a world which w_1 sees—namely, w_2—at which '$(\exists x)Fx$' is true—since $\mathfrak{u}_1$ exists at w_2 and is F at w_2.

To see that '$(\exists x)\Diamond Fx$' is false at w_1, just note that nothing exists at w_1. So there is nothing at w_1 which is possibly F—because there's nothing at all at w_1.

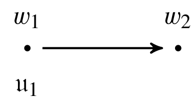
Since nothing has any properties at worlds at which it doesn't exist in this model, it additionally shows us that this instance of the Barcan formula is not a tautology of $NVDQML$.

Note, however, that some seemingly good instances of the *converse* Barcan formula additionally comes out as invalid on $VDQML$:

$$\not\models_{\text{VDQML}} (\exists x)\Diamond Fx \to \Diamond(\exists x)Fx$$

To see this, consider the following $VDQML$-model:

$$
\begin{aligned}
\mathcal{W} &= \{w_1, w_2\} \\
R &= \{< w_1, w_2 >\} \\
\mathscr{D} &= \{\mathfrak{u}_1\} \\
\mathscr{Q}(w_1) &= \{\mathfrak{u}_1\} \\
\mathscr{Q}(w_2) &= \varnothing \\
\mathscr{I}(F, w_1) &= \varnothing \\
\mathscr{I}(F, w_2) &= \{\mathfrak{u}_1\}
\end{aligned}
$$



In this model, it is true at w_1 that $(\exists x)\Diamond Fx$. For $\mathfrak{u}_1$ exists at w_1, and $\mathfrak{u}_1$ has the property F at w_2 (though it doesn't exist there). So, at w_1, there is something which is possibly F. Nevertheless, it is not possible at w_1 that there is something which is F. For the only world which w_1 sees is w_2, and at w_2, there is nothing which is F (since there is nothing at all). So the antecedent of '$(\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx$' is true at w_1; yet its consequent is false at w_1. So the conditional is false at w_1.

Note, however, that while this is a *VDQML*-model, it is not a *NVDQML*-model.

10. Establishing Validity in *VDQML*

The instance of the converse Barcan formula '$(\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx$' is a tautology of a *negative* variable domain quantified modal logic, *NVDQML*. To show this, we may provide the following semantic proof:

1.	Assume that there is some *NVDQML*-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}((\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx, w) = 0$.	*Assumption*
2.	So $V_{\mathcal{M}}((\exists x)\Diamond Fx, w) = 1$ and $V_{\mathcal{M}}(\Diamond(\exists x)Fx, w) = 0$.	1, *def* $\rightarrow$
3.	So $V_{\mathcal{M}}((\exists x)\Diamond Fx, w) = 1$.	2
4.	So there is some entity in $\mathscr{Q}(w)$—call it '$\mathfrak{u}$'—such that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(\Diamond Fx, w) = 1$.	3, *def* $\exists$
5.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(\Diamond Fx, w) = 1$	4
6.	So there is some world—call it 'w''—such that Rww' and $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(Fx, w') = 1$.	5, *def* $\Diamond$
7.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(Fx, w') = 1$.	6
8.	So $\mathscr{I}_{x\rightarrow\mathfrak{u}}(x) \in \mathscr{I}_{x\rightarrow\mathfrak{u}}(F, w')$	7, *def* Π
9.	$\mathscr{I}_{x\rightarrow\mathfrak{u}}(x) = \mathfrak{u}$	*def var model*
10.	$\mathscr{I}_{x\rightarrow\mathfrak{u}}(F, w') = \mathscr{I}(F, w')$	*def var model*
11.	So $\mathfrak{u} \in \mathscr{I}(F, w')$.	8, 9, 10
12.	So $\mathfrak{u} \in \mathscr{Q}(w')$	11, *NVDQML-model*
13.	It is not the case that $V_{\mathcal{M}}(\Diamond(\exists x)Fx, w) = 1$.	2, *bivalence*
14.	So it is not the case that there is some world w'' such that Rww'' and $V_{\mathcal{M}}((\exists x)Fx, w'') = 1$.	13, *def.* $\Diamond$
15.	So, for all w'', if Rww'', then it is not the case that $V_{\mathcal{M}}((\exists x)Fx, w'') = 1$.	14, *QL*

16.	So, if Rww', then it is not the case that $V_{\mathcal{M}}((\exists x)Fx, w') = 1$	15, $\forall E$
17.	Rww'.	6
18.	So it is not the case that $V_{\mathcal{M}}((\exists x)Fx, w') = 1$.	16, 17 $\rightarrow E$
19.	So it is not the case that there is some entity $\mathfrak{v} \in \mathscr{Q}(w')$ such that $V_{\mathcal{M}_{x\rightarrow\mathfrak{v}}}(Fx, w') = 1$.	18, *QL*
20.	So, for all $\mathfrak{v} \in \mathscr{Q}(w')$, it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{v}}}(Fx, w') = 1$.	19, *QL*
21.	$\mathfrak{u} \in \mathscr{Q}(w')$.	12
22.	So it is not the case that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(Fx, w') = 1$.	20, 21 $\rightarrow E$
23.	Lines 7 and 22 contradict.	7, 22
24.	So there is no *NVDQML*-model $< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}((\exists x)\Diamond Fx \rightarrow \Diamond(\exists x)Fx, w) = 0$	23, $\sim I$

Additionally, in any *VDQML*-model (and therefore, in any *NVDQML*-model, since every *NVDQML*-model is a *VDQML*-model), a restricted converse Barcan formula will hold:

$$\models_{\text{VDQML}} \Box(\forall x)Fx \rightarrow (\forall x)\Box((\exists y)y = x \rightarrow Fx)$$

This wff says that, if everything is F at every possible world, then everything is F at every world at which it exists. To show that this is a tautology of *VDQML*, we may provide the following semantic proof:

1.	Suppose that there is a *VDQML*-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\forall x)Fx \rightarrow (\forall x)\Box((\exists y)y = x \rightarrow Fx), w) = 0$.	*Assumption*
2.	So, $V_{\mathcal{M}}(\Box(\forall x)Fx, w) = 1$ and $V_{\mathcal{M}}((\forall x)\Box((\exists y)y = x \rightarrow Fx), w) = 0$.	1, *def.* $\rightarrow$
3.	So it is not the case that $V_{\mathcal{M}}((\forall x)\Box((\exists y)y = x \rightarrow Fx), w) = 1$.	2, *bivalence*
4.	So it is not the case that, for all $\mathfrak{u} \in \mathscr{Q}(w)$, $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(\Box((\exists y)y = x \rightarrow Fx), w) = 1$.	3, *def* $\forall$
5.	So there is some $\mathfrak{u} \in \mathscr{Q}(w)$—call it '$\mathfrak{u}$'—such that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(\Box((\exists y)y = x \rightarrow Fx), w) \neq 1$.	4, *QL*
6.	So it is not the case that, for all w', if Rww', then $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\exists y)y = x \rightarrow Fx, w') = 1$.	5, $\Box$
7.	So there is some world—call it 'w''—such that Rww' and $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\exists y)y = x \rightarrow Fx, w') \neq 1$.	6, *QL*
8.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\exists y)y = x, w') = 1$ and $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(Fx, w') = 0$.	7, *def*$\rightarrow$
9.	So $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}(Fx, w') = 0$.	8
10.	So $\mathscr{I}_{x\rightarrow\mathfrak{u}}(x) \notin \mathscr{I}_{x\rightarrow\mathfrak{u}}(F, w')$	9, *def* Π
11.	$\mathscr{I}_{x\rightarrow\mathfrak{u}}(x) = \mathfrak{u}$	*var. model*
12.	$\mathscr{I}_{x\rightarrow\mathfrak{u}}(F, w') = \mathscr{I}(F, w')$	*var. model*
13.	So $\mathfrak{u} \notin \mathscr{I}(F, w')$	10, 11, 12
14.	And $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}}}((\exists y)y = x, w') = 1$	8
15.	So there is some $\mathfrak{v} \in \mathscr{Q}(w')$—call it '$\mathfrak{v}$'—such that $V_{\mathcal{M}_{x\rightarrow\mathfrak{u}, y\rightarrow\mathfrak{v}}}(y = x, w') = 1$.	14, *def.* $\exists$

16.	So $\mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{v}}(y) = \mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{v}}(x)$.	15 *def.* =
17.	$\mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{v}}(y) = \mathfrak{v}$.	*var. model*
18.	$\mathscr{I}_{x\to\mathfrak{u},y\to\mathfrak{v}}(x) = \mathfrak{u}$.	*var. model*
19.	So $\mathfrak{v} = \mathfrak{u}$.	16, 17, 18
20.	$\mathfrak{v} \in \mathscr{Q}(w')$	15
21.	So $\mathfrak{u} \in \mathscr{Q}(w')$.	19, 20
22.	$V_{\mathcal{M}}(\Box(\forall x)Fx, w) = 1$.	2
23.	So, for all w'', if Rww'', then $V_{\mathcal{M}}((\forall x)Fx, w'') = 1$.	22, *def.* $\Box$
23.	So, if Rww', then $V_{\mathcal{M}}((\forall x)Fx, w') = 1$.	23, $\forall E$
24.	Rww'.	7
25.	So $V_{\mathcal{M}}((\forall x)Fx, w') = 1$.	23, 24, $\to E$
26.	So, for all $\mathfrak{t} \in \mathscr{Q}(w')$, $V_{\mathcal{M}_{x\to\mathfrak{t}}}((Fx, w') = 1$.	25, *def.* $\forall$
27.	So $V_{\mathcal{M}_{x\to\mathfrak{u}}}((Fx, w') = 1$	21, 26, $\forall E$
28.	So $\mathscr{I}_{x\to\mathfrak{u}}(x) \in \mathscr{I}_{x\to\mathfrak{u}}(F, w')$	28, *def.* Π
29.	So $\mathfrak{u} \in \mathscr{I}(F, w')$.	11, 12, 28
30.	Lines 13 and 29 contradict.	13, 29
31.	So there is no $VDQML$-model $\mathcal{M} =< \mathcal{W}, R, \mathscr{D}, \mathscr{Q}, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\forall x)Fx \to (\forall x)\Box((\exists y)y = x \to Fx), w) = 0$.	30, $\sim I$

We may also show that, in any $VDQML$-model (and therefore, any $NVDQML$-model), the necessity of identity will be true at every world. That is:

$$\models_{\text{VDQML}} (\forall x)(\forall y)(x = y \to \Box x = y)$$

The proof provided on page 44 will, with the necessary changes, provide a suitable semantic proof in $VDQML$.

11. Axiomatization of $VDQML$

Here, we will provide an axiomatization of the *positive* $VDQML$. For this axiomatic system, we will simply take the axioms of positive free logic and add to them the K-axiom and the necessity of distinctness, ($\Box \neq$), plus the rule of necessitation.

Then, our axiomatic system contains the following axiom schemata:

$$\vdash_{\text{VDQML}} \phi, \quad \text{for all } PL\text{-valid schemata } \ulcorner\phi\urcorner \qquad (PL)$$

$$\vdash_{\text{VDQML}} (\forall\alpha)\phi \to ((\exists\zeta)\zeta = \tau \to \phi[\tau/\alpha]) \qquad (\forall 1)$$

provided that $\ulcorner\tau\urcorner$ *is free in* $\ulcorner\phi[\tau/\alpha]\urcorner$

$$\vdash_{\text{VDQML}} (\forall\alpha)(\phi \to \psi) \to (\phi \to (\forall\alpha)\psi) \qquad (\forall 2)$$

provided that $\ulcorner\alpha\urcorner$ *is not free in* $\ulcorner\phi\urcorner$

$$\vdash_{\text{VDQML}} (\forall\zeta)((\forall\alpha)\phi \to \phi[\zeta/\alpha]) \qquad (\forall 3)$$

provided that $\ulcorner\zeta\urcorner$ *is free in* $\ulcorner\phi[\zeta/\alpha]\urcorner$

$$\vdash_{\text{VDQML}} \tau = \tau \qquad (=1)$$

$$\vdash_{\text{VDQML}} \tau_1 = \tau_2 \rightarrow (\phi \rightarrow \phi[\tau_2 // \tau_1]) \qquad (=2)$$

provided that $\ulcorner \tau_2 \urcorner$ *is free in* $\ulcorner \phi[\tau_2 // \tau_1] \urcorner$

$$\vdash_{\text{VDQML}} (\forall \alpha)(\exists \zeta)\alpha = \zeta \qquad (=3)$$

$$\vdash_{\text{VDQML}} \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi) \qquad (K)$$

$$\vdash_{\text{VDQML}} (\forall \alpha)(\forall \zeta)(\alpha \neq \zeta \rightarrow \Box \alpha \neq \zeta) \qquad (\Box \neq)$$

And the following rules of inference:

Propositional Logic Rules (*PLR*): If $\ulcorner \psi \urcorner$ follows from $\ulcorner \phi \urcorner$ according to propositional logic, then, from $\ulcorner \phi \urcorner$, infer $\ulcorner \psi \urcorner$.

Generalization (*G*): If a wff $\ulcorner \phi \urcorner$ is a theorem of *VDQML*, then you may infer $\ulcorner (\forall \alpha)\phi \urcorner$ as a theorem of *VDQML*, where $\ulcorner \alpha \urcorner$ is a variable of *VDQML*.

$$\text{from } \vdash_{\text{VDQML}} \phi, \text{ infer } \vdash_{\text{VDQML}} (\forall \alpha)\phi$$

Necessitation (*N*): if a wff $\ulcorner \phi \urcorner$ is a theorem of *VDQML*, then you may infer $\ulcorner \Box\phi \urcorner$ as a theorem of *VDQML*.

$$\text{from } \vdash_{\text{VDQML}} \phi, \text{ infer } \vdash_{\text{VDQML}} \Box\phi$$

12. Natural Deduction for *VDQML*

Again, this axiomatic system is rather difficult to prove things in, so we will introduce a natural deduction system for the *positive VDQML* (though not for the negative *VDQML*).

To achieve this natural deduction system, we need only combine the natural deduction rules for positive free logic, *FL*, with the relevant natural deduction systems for each of our propositional modal systems, together with the rule $\Box \neq$ from the natural deduction system for *SQML* (but *not*, importantly, the rule *BF*).

If we can prove $\ulcorner \phi \urcorner$ from the set of wff Γ in this natural deduction system, then we will write

$$\Gamma \overset{\text{D}}{\vdash}_{\text{VDQML-K}} \phi$$

And if we can prove $\ulcorner \phi \urcorner$ from no assumptions in this natural deduction system, then we will write

$$\overset{\text{D}}{\vdash}_{\text{VDQML-K}} \phi$$

Additional subscripts will be added for the various modal systems beyond *K*. For instance, if $\ulcorner \phi \urcorner$ can be proven from no assumptions in the natural deduction system that we get by taking the rules for system *B* from *PML*, then we will write

$$\overset{\text{D}}{\vdash}_{\text{VDQML-B}} \phi$$

Here is a *VDQML*-derivation showing that

$$\overset{\text{D}}{\underset{\text{VDQML-K}}{\vdash}} (\forall x)\Box(\exists y)y = x \to (\Box(\forall x)Fx \to (\forall x)\Box Fx)$$

1	$\vert\ (\forall x)\Box(\exists y)y = x$	$A(\to I)$
2	$\vert\ \vert\ \Box(\forall x)Fx$	$A(\to I)$
3	$\vert\ \vert\ (\exists x)x = z \to \Box(\exists y)y = z$	1, $\forall E^*$
4	$\vert\ \vert\ \vert\ (\exists x)x = z$	$A(\to I)$
5	$\vert\ \vert\ \vert\ \Box(\exists y)y = z$	3, 4, $\to E$
6	$\vert\ \vert\ \vert\ {}^{\Box}\vert\ (\forall x)Fx$	2, $\Box R$
7	$\vert\ \vert\ \vert\ \ \ \vert\ (\exists y)y = z$	5, $\Box R$
8	$\vert\ \vert\ \vert\ \ \ \vert\ (\exists y)y = z \to Fz$	6, $\forall E^*$
9	$\vert\ \vert\ \vert\ \ \ \vert\ Fz$	7, 8, $\to E$
10	$\vert\ \vert\ \vert\ \Box Fz$	6–9, $\Box I$
11	$\vert\ \vert\ (\exists x)x = z \to \Box Fz$	4–10, $\to I$
12	$\vert\ \vert\ (\forall x)\Box Fx$	11, $\forall I^*$
13	$\vert\ \Box(\forall x)Fx \to (\forall x)\Box Fx$	2–12, $\to I$
14	$(\forall x)\Box(\exists y)y = x \to (\Box(\forall x)Fx \to (\forall x)\Box Fx)$	1–13, $\to I$

And here is one demonstrating that, in *VDQML*-S4,

$$\{(\exists x)\Box Px\} \overset{\text{D}}{\underset{\text{VDQML-4}}{\vdash}} (\exists x)\Box\Box Px$$

1	$\vert\ (\exists x)\Box Px$	
2	$\vert\ \vert\ \Box Pa \wedge (\exists y)y = a$	$A(\exists E^*)$
3	$\vert\ \vert\ \Box Pa$	2, $\wedge E$
4	$\vert\ \vert\ {}^{\Box}\vert\ \Box Pa$	3, S4R
5	$\vert\ \vert\ \Box\Box Pa$	4–4, $\Box I$
6	$\vert\ \vert\ (\exists y)y = a$	2, $\wedge E$
7	$\vert\ \vert\ (\exists x)\Box\Box Px$	5, 6, $\exists I^*$
8	$\vert\ (\exists x)\Box\Box Px$	1, 2–7, $\exists E^*$

Here's a sample *VDQML-T* derivation showing that

$$\{\Box(\exists x)\Diamond(Fx \vee Gx)\} \vdash^{\mathrm{D}}_{\mathrm{VDQML\text{-}T}} (\exists x)(\Diamond Fx \vee \Diamond Gx)$$

1	$\Box(\exists x)\Diamond(Fx \vee Gx)$	
2	$(\exists x)\Diamond(Fx \vee Gx)$	1, $\Box E$
3	$\Diamond(Fa \vee Ga) \wedge (\exists y)y = a$	$A(\exists E^*)$
4	$\Diamond(Fa \vee Ga)$	3, $\wedge E$
5	$\sim(\Diamond Fa \vee \Diamond Ga)$	$A(\sim E)$
6	$Fa \vee Ga$	$A(\Diamond E)$
7	Fa	$A(\vee E)$
8	Fa	7, R
9	Ga	$A(\vee E)$
10	$\Diamond Ga$	9, $\Diamond I$
11	$\Diamond Fa \vee \Diamond Ga$	9, $\vee I$
12	$\sim Fa$	$A(\sim E)$
13	$(\Diamond Fa \vee \Diamond Ga) \wedge \sim(\Diamond Fa \vee \Diamond Ga)$	5, 11 $\wedge I$
14	Fa	12–13, $\sim E$
15	Fa	6, 7–8, 9–14, $\vee E$
16	$\Diamond Fa$	4, 6–15, $\Diamond E$
17	$\Diamond Fa \vee \Diamond Ga$	16, $\vee I$
18	$(\Diamond Fa \vee \Diamond Ga) \wedge \sim(\Diamond Fa \vee \Diamond Ga)$	5, 17, $\wedge I$
19	$\Diamond Fa \vee \Diamond Ga$	5–18, $\sim E$
20	$(\exists y)y = a$	3, $\wedge E$
21	$(\exists x)(\Diamond Fx \vee \Diamond Gx)$	19, 20 $\exists I^*$
22	$(\exists x)(\Diamond Fx \vee \Diamond Gx)$	2, 3–21, $\exists E^*$

And here is one showing that, in *VDQML-B*,

$$\{\Box(\forall x)\Box(\exists y)y = x\} \vdash^{D}_{VDQML\text{-}B} (\Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx)$$

1	$\Box(\forall x)\Box(\exists y)y = x$	
2	$\quad \Diamond(\exists x)Fx$	$A(\rightarrow I)$
3	$\quad\quad \sim(\exists x)\Diamond Fx$	$A(\sim E)$
4	$\quad\quad (\forall x)\sim\Diamond Fx$	3, QN
5	$\quad\quad \Diamond\ \quad (\exists x)Fx$	$A(\Diamond E)$
6	$\quad\quad\quad (\forall x)\Box(\exists y)y = x$	1, $\Box R$
7	$\quad\quad\quad \Diamond(\forall x)\sim\Diamond Fx$	4, $B\Diamond I$
8	$\quad\quad\quad\quad Fc \wedge (\exists y)y = c$	$A(\exists E^{*})$
9	$\quad\quad\quad\quad \Diamond\ \quad (\forall x)\sim\Diamond Fx$	$A(\Diamond E)$
10	$\quad\quad\quad\quad\quad (\exists y)y = c \rightarrow \sim\Diamond Fc$	9, $\forall E^{*}$
11	$\quad\quad\quad\quad \Diamond((\exists y)y = c \rightarrow \sim\Diamond Fc)$	9–10, $\Diamond E$
12	$\quad\quad\quad\quad (\exists y)y = c \rightarrow \Box(\exists y)y = c$	6, $\forall E^{*}$
13	$\quad\quad\quad\quad (\exists y)y = c$	8, $\wedge E$
14	$\quad\quad\quad\quad \Box(\exists y)y = c$	12, 13, $\rightarrow E$
15	$\quad\quad\quad\quad \Diamond\ \quad (\exists y)y = c \rightarrow \sim\Diamond Fc$	$A(\Diamond E)$
16	$\quad\quad\quad\quad\quad (\exists y)y = c$	14, $\Box R$
17	$\quad\quad\quad\quad\quad \sim\Diamond Fc$	15, 16, $\rightarrow E$
18	$\quad\quad\quad\quad \Diamond\sim\Diamond Fc$	11, 15–17, $\Diamond E$
19	$\quad\quad\quad\quad \sim\Box\Diamond Fc$	18, MN
20	$\quad\quad\quad\quad Fc$	8, $\wedge E$
21	$\quad\quad\quad\quad \Box\ \quad \Diamond Fc$	20, $B\Diamond I$
22	$\quad\quad\quad\quad \Box\Diamond Fc$	21–21, $\Box I$
23	$\quad\quad\quad\quad\quad \sim(Fd \wedge (\exists y)y = d)$	$A(\sim E)$
24	$\quad\quad\quad\quad\quad \Box\Diamond Fc \wedge \sim\Box\Diamond Fc$	19, 22, $\wedge I$
25	$\quad\quad\quad\quad Fd \wedge (\exists y)y = d$	23–24, $\sim E$
26	$\quad\quad\quad Fd \wedge (\exists y)y = d$	5, 8–25, $\exists E^{*}$
27	$\quad\quad \Diamond(Fd \wedge (\exists y)y = d)$	2, 5–26, $\Diamond E$

28	$\Diamond$ $Fd \wedge (\exists y)y = d$	$A(\Diamond E)$
29	$(\exists y)y = d$	28, $\wedge E$
30	$(\forall x)\Box(\exists y)y = x$	1, $\Box R$
31	$(\exists y)y = d \rightarrow \Box(\exists y)y = d$	30, $\forall E^*$
32	$\Box(\exists y)y = d$	29, 21, $\rightarrow E$
33	$\Diamond\Box(\exists y)y = d$	27, 28–32, $\Diamond E$
34	$\sim(\exists y)y = d$	$A(\sim E)$
35	$\Box$ $\Diamond\sim(\exists y)y = d$	34, $B\Diamond I$
36	$\sim\Box(\exists y)y = d$	35, MN
37	$\Box\sim\Box(\exists y)y = d$	35–36, $\Box I$
38	$\sim\Diamond\Box(\exists y)y = d$	37, MN
39	$\Diamond\Box(\exists y)y = d \wedge \sim\Diamond\Box(\exists y)y = d$	33, 38, $\wedge I$
40	$(\exists y)y = d$	34–39, $\sim E$
41	$(\exists y)y = d \rightarrow \sim\Diamond Fd$	4, $\forall E^*$
42	$\sim\Diamond Fd$	40, 41, $\rightarrow E$
43	$\Diamond$ $Fd \wedge (\exists y)y = d$	$A(\Diamond E)$
44	Fd	43, $\wedge E$
45	$\Diamond Fd$	43–44, $\Diamond E$
46	$\Diamond Fd \wedge \sim\Diamond Fd$	42, 45, $\wedge I$
47	$(\exists x)\Diamond Fx$	3–46, $\sim E$
48	$\Diamond(\exists x)Fx \rightarrow (\exists x)\Diamond Fx$	2–47, $\rightarrow I$

This shows that, in *VDQML-B*, necessitism suffices for the Barcan formula.