NOTES ON PROPOSITIONAL MODAL LOGIC

1. Preliminaries

1.1. **Axiomatization of Propositional Logic.** Axiomatic proof systems provide a simple, elegant representation of everything that a logic has to say about validity. To provide an axiomatic system for propositional logic, we may introduce the following as *axioms*:

$$P \to (Q \to P) \tag{A1}$$

$$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R)) \tag{A2}$$

$$(\sim Q \to \sim P) \to ((\sim Q \to P) \to Q) \tag{A3}$$

To accept these as axioms means that, at any point in an axiomatic proof, you may write down (A1), (A2), or (A3).

And we introduce the following rules of inference:

> *Uniform Substitution* (*US*): You may uniformly replace any propositional letter, $\ulcorner \alpha \urcorner$, occurring in a theorem of *PL* with another wff of *PL*, $\ulcorner \psi \urcorner$.
>
> $$\text{from } \vdash_{PL} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_{PL} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$$
>
> *Modus Ponens* (*MP*): From a wff of the form $\ulcorner \phi \to \psi \urcorner$ and a wff of the form $\ulcorner \phi \urcorner$, you may infer $\ulcorner \psi \urcorner$.

Uniform Substitution tells us, for instance, that we may write down, at any time, the result of replacing 'P' with '$\sim P$' and 'Q' with '$(Q \to R)$' in (A3), getting:

$$(\sim(Q \to R) \to \sim\sim P) \to ((\sim(Q \to R) \to \sim P) \to (Q \to R))$$

It is important that *Uniform Substitution* applies only to *theorems*. Otherwise, we would be able to prove 'Q' from 'P'. It is also important that the substitution be *uniform*—that is, if you replace 'P' with '$\sim Q$' in *one* place where the wff 'P' occurs, then you must replace 'P' with '$\sim Q$' in every *other* places that 'P' occurs. It is also important that we only replace *atomic*, proposition letters with other wffs. We cannot, for instance, replace '$(Q \to P)$' in (A1) with 'Q' to get '$P \to Q$'.

Here's how an axiomatic proof works: you write things down on sequentially numbered lines, in accordance with the rules of the axiomatic proof system. The rules tell you that you are allowed to write *any* wff of *PL* down as a premise; you are allowed to write down *any* of

---

These notes are heavily indebted to G. E. Hughes and M. J. Cresswell (1996), *A New Introduction to Modal Logic*, Routledge, London; James Garson (2006), *Modal Logic for Philosophers*, 2nd ed., Cambridge University Press, Cambridge; Greg W. Fitch, *Naive Modal Logic*, unpublished lecture notes; and Theodore Sider (2010) *Logic for Philosophy*, Oxford University Press, Oxford.

the axioms (A1), (A2), or (A3); and you are allowed to write down anything allowed by the inference rules *Uniform Substitution* and *Modus Ponens*.

Here's a sample axiomatic proof:

1.	$\vdash_{PL} (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	(A2)
2.	$\vdash_{PL} (P \to ((P \to P) \to P)) \to ((P \to (P \to P)) \to (P \to P))$	1, *US*
3.	$\vdash_{PL} P \to (Q \to P)$	(A1)
4.	$\vdash_{PL} P \to ((P \to P) \to P)$	3, *US*
5.	$\vdash_{PL} (P \to (P \to P)) \to (P \to P)$	2, 4 *MP*
6.	$\vdash_{PL} P \to (P \to P)$	3, *US*
7.	$\vdash_{PL} P \to P$	5, 6, *MP*

We will say that $\ulcorner \phi \urcorner$ is a *theorem* of *PL*,

$$\vdash_{PL} \phi$$

iff there is a proof with *no* premises and whose final line is $\ulcorner \phi \urcorner$. So the above axiomatic proof establishes that $\vdash_{PL} P \to P$.

We will say that $\ulcorner \phi \urcorner$ is *PL-provable* from Γ,

$$\Gamma \vdash_{PL} \phi$$

iff there is an axiomatic proof which has only premises from Γ and whose final line is $\ulcorner \phi \urcorner$.

Then, the axiomatic proof below establishes that

$$\{\sim\sim P\} \vdash_{PL} P$$

1.	$\sim\sim P$	*Premise*
2.	$\vdash_{PL} P \to (Q \to P)$	(A1)
3.	$\vdash_{PL} \sim\sim P \to (\sim P \to \sim\sim P)$	2, *US*
4.	$\sim P \to \sim\sim P$	1, 3, *MP*
5.	$\vdash_{PL} (\sim Q \to \sim P) \to ((\sim Q \to P) \to Q)$	(A3)
6.	$\vdash_{PL} (\sim P \to \sim\sim P) \to ((\sim P \to \sim P) \to P)$	5, *US*
7.	$(\sim P \to \sim P) \to P$	4, 6, *MP*
8.	$\vdash_{PL} \sim P \to ((\sim P \to \sim P) \to \sim P)$	2, *US*
9.	$\vdash_{PL} (P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$	(A2)
10.	$\vdash_{PL} (\sim P \to ((\sim P \to \sim P) \to \sim P)) \to ((\sim P \to (\sim P \to \sim P)) \to (\sim P \to \sim P))$	9, *US*
11.	$\vdash_{PL} (\sim P \to (\sim P \to \sim P)) \to (\sim P \to \sim P)$	8, 10, *MP*
12.	$\vdash_{PL} \sim P \to (\sim P \to \sim P)$	2, *US*
13.	$\vdash_{PL} \sim P \to \sim P$	11, 12, *MP*
14.	P	7, 13, *MP*

Because *Uniform Substitution* only applies to *theorems*, it is important that we keep track of which wffs on the lines are theorems, and which were merely consequences of our assumptions in the proof. Thus, I am writing $\ulcorner \vdash_{PL} \phi \urcorner$ iff $\ulcorner \phi \urcorner$ is a theorem of *PL*. If an application of a rule of inference makes use of only theorems, then the result of applying that rule will be a theorem. If, however—as on lines 4, 7, and 14 above—an application of a rule of inference makes use of non-theorems, then the result of applying that rule will *not* be a theorem.

> **Interesting and Unexpected and Fantastic Fact**: for every wff $\ulcorner \phi \urcorner$ and every set of wffs Γ,
> $$\Gamma \vdash_{PL} \phi \quad \text{if and only if} \quad \Gamma \vDash_{PL} \phi$$

1.2. **Natural Deduction For Propositional Logic.** Axiomatic proof systems are easy to prove things *about*, but they are not very easy to prove things *in*. Therefore, I'll introduce here a more natural deduction system for propositional logic.

This proof system is known as a Fitch-style proof system, after the logician Frederic Fitch, who first introduced them.

The way the proof system works is like this: you put all of your premises $\gamma \in \Gamma$ on a line—known as the *main scope line* (*main* scope line because, as we'll see in a bit, the system allows other, subsidiary, scope lines, too). There are then a collection of rules that you are allowed to follow. If, by following these rules, you are able to write down $\ulcorner \phi \urcorner$ on the main scope line at any point,

$$\begin{array}{r|l}
1 & \gamma_1 \\
2 & \gamma_2 \\
\vdots & \vdots \\
n & \gamma_n \\
\hline
\vdots & \vdots \\
m & \phi
\end{array}$$

then ϕ is derivable from Γ according to the proof system. Notice that we number each line of the proof, for easy reference. We will also write down, on each line of the proof, our *justification* for writing down the wff that we wrote down on that line. That is to say, we will say which rule of the proof system, together with which previous lines of the proof, allowed us to write down the wff we did.

If $\ulcorner \phi \urcorner$ is derivable from Γ according to the natural deduction system, we write:

$$\Gamma \vdash_{PD} \phi$$

If ⌜ϕ⌝ is derivable from *no* assumptions, then we write:

$$\vdash_{PD} \phi$$

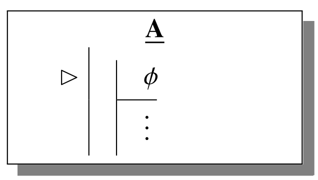
Some definitions:

(1) A line of the derivation, *m*, is ACCESSIBLE from another line, *n*, if and only if 1) line *m precedes* line *n*; and 2) the wff written on line *m* lies on a scope line which extends to line *n*.

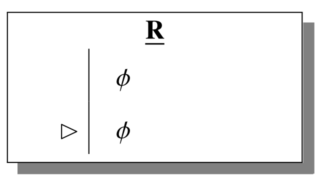
(2) An entire *subderivation*, running from lines *l*–*m*, is ACCESSIBLE from a line, *n*, if and only if 1) the subderivation *preceeds* line *n* (in particular, the subderivation is not still in progress at line *n*); and 2) the entire subderivation lies on a scope line which extends to line *n*.

Here are the rules of our natural deduction system:

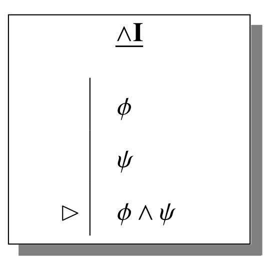
(1) Assumption: you may, at any point in a derivation, start a *new* scope line, nested within the previous scope lines. You may write *whatever you wish* on the first line of this new scope line (so long, that is, as it is a wff of *PL*).



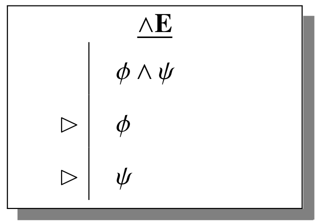
(2) Reiteration: if ⌜ϕ⌝ is written on a scope line which is *accessible* from your current line, then you can write down ⌜ϕ⌝.



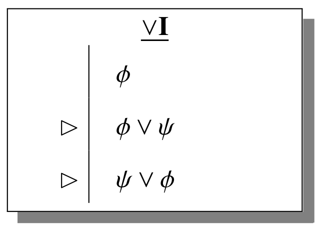
(3) Conjunction Introduction: If both ⌜ϕ⌝ and ⌜ψ⌝ are written down on lines which are accessible from your current line, then you may write down ⌜$\phi \wedge \psi$⌝.



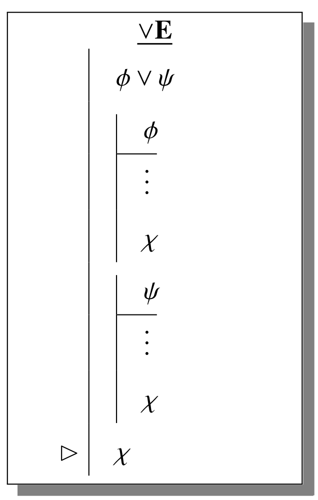
(4) Conjunction Elimination: if $\ulcorner \phi \land \psi \urcorner$ is written down on an accessible line, then you may write down $\ulcorner \phi \urcorner$ and you may write down $\ulcorner \psi \urcorner$.



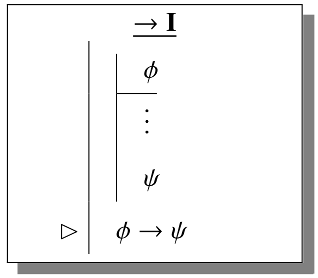
(5) Disjunction Introduction: If you have $\ulcorner \phi \urcorner$ written down on an accessible line, then you may write down $\ulcorner \phi \lor \psi \urcorner$ and you may write down $\ulcorner \psi \lor \phi \urcorner$, for any wff $\ulcorner \psi \urcorner$.



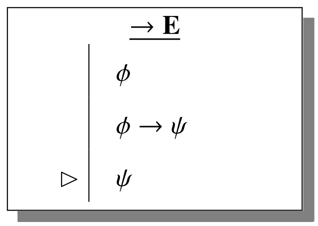
(6) Disjunction Elimination: if $\ulcorner \phi \lor \psi \urcorner$ is written on an accessible line, there is an accessible subderivation whose assumption is $\ulcorner \phi \urcorner$ and whose final line is $\ulcorner \chi \urcorner$, and there is an accessible subderivation whose assumption is $\ulcorner \psi \urcorner$ and whose final line is $\ulcorner \chi \urcorner$, then you may write down $\ulcorner \chi \urcorner$.



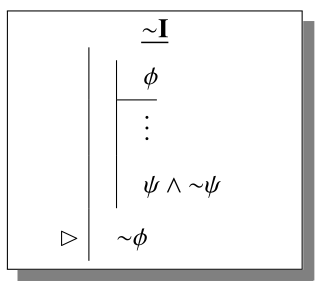
(7) Conditional Introduction: If there is an accessible subderivation whose assumption is ⌜ϕ⌝ and whose final line is ⌜ψ⌝, then you may write down ⌜$\phi \rightarrow \psi$⌝.



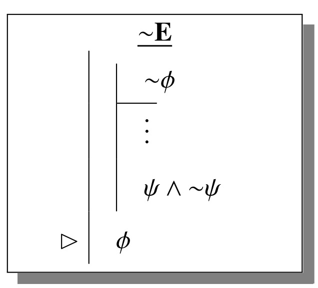
(8) Conditional Elimination: if ⌜$\phi \rightarrow \psi$⌝ appears on an accessible line and ⌜ϕ⌝ appears on an accessible line, then you may write down ⌜ψ⌝.



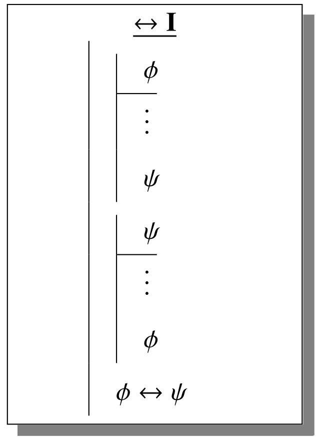
(9) Negation Introduction: If there is an accessible subderivation whose assumption is ⌜ϕ⌝ and whose final line is of the form ⌜$\psi \wedge {\sim}\psi$⌝, then you may write down ⌜${\sim}\phi$⌝.



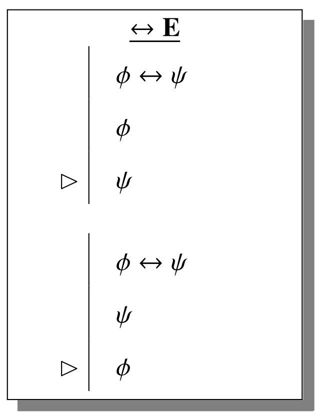
(10) Negation Elimination: If there is an accessible subderivation whose assumption is ⌜${\sim}\phi$⌝ and whose final line is is of the form ⌜$\psi \wedge {\sim}\psi$⌝, then you may write down ⌜ϕ⌝.



(11) Biconditional Introduction: If there is an accessible subderivation whose first line is $\ulcorner \phi \urcorner$ and whose final line is $\ulcorner \psi \urcorner$ and there is an accessible subderivation whose first line is $\ulcorner \psi \urcorner$ and whose final line is $\ulcorner \phi \urcorner$, then you may write down $\ulcorner \phi \leftrightarrow \psi \urcorner$.



(12) Biconditional Elimination: If you have $\ulcorner \phi \leftrightarrow \psi \urcorner$ written down on an accessible line, and you have $\ulcorner \phi \urcorner$ written down on an accessible line, then you may write down $\ulcorner \psi \urcorner$. Also: if you have $\ulcorner \phi \leftrightarrow \psi \urcorner$ written down on an accessible line and you have $\ulcorner \psi \urcorner$ written down on an accessible line, then you may write down $\ulcorner \phi \urcorner$.



Here's a sample derivation demonstrating that $\{P \to Q, {\sim}Q\} \vdash_{PD} {\sim}P$

1	$P \to Q$	
2	${\sim}Q$	
3	$\quad P$	A (${\sim}I$)
4	$\quad Q$	1, 3 ($\to I$)
5	$\quad Q \wedge {\sim}Q$	2, 4 ($\wedge I$)
6	${\sim}P$	3–5 (${\sim}I$)

Here's another demonstrating that $\{\sim(P \vee Q)\} \vdash_{PD} \sim P \wedge \sim Q$

1	$\sim(P \vee Q)$	
2	P	A ($\sim I$)
3	$P \vee Q$	2 ($\vee I$)
4	$(P \vee Q) \wedge \sim(P \vee Q)$	1, 3 ($\wedge I$)
5	$\sim P$	2–4 ($\sim I$)
6	Q	A ($\sim I$)
7	$P \vee Q$	6 ($\vee I$)
8	$(P \vee Q) \wedge \sim(P \vee Q)$	1, 7 ($\wedge I$)
9	$\sim Q$	6–8 ($\sim I$)
10	$\sim P \wedge \sim Q$	5, 9 ($\wedge I$)

Here is a derivation showing that $\vdash_{PD} P \rightarrow (P \rightarrow P)$

1	P	$A(\rightarrow I)$
2	P	$A(\rightarrow I)$
3	P	2, R
4	$P \rightarrow P$	2–3, $\rightarrow I$
5	$P \rightarrow (P \rightarrow P)$	1–4, $\rightarrow I$

In the above derivation, we were able to derive '$P \rightarrow (P \rightarrow P)$' from *no* assumptions. You can tell this because, at the end of the derivation, '$P \rightarrow (P \rightarrow P)$' lies outside of every scope line. This means that, according to the natural deduction system, '$P \rightarrow (P \rightarrow P)$' is a *theorem*.

Interesting and Unexpected and Fantastic Fact: for every wff $\ulcorner \phi \urcorner$ and every set of wffs Γ,

$$\Gamma \vdash_{PD} \phi \quad \text{if and only if} \quad \Gamma \models_{PL} \phi$$

2. The Language *PML*

(1) The vocabulary of *PML* includes

(a) An infinite number of propositional letters

$$P, Q, R, P_1, Q_1, R_1, P_2, Q_2, R_2, \ldots$$

(b) logical connectives

$$\sim, \rightarrow$$

(c) parenthases

$$(\ ,\ )$$

(d) a modal operator

$$\Box$$

(2) Any sequence of the above characters is a *formula* of *PML*. A *well-formed* formula, or *wff*, of *PML* is defined recursively

(a) Any propositional letter is a wff (known as an *atomic* wff).

(b) If $\ulcorner \phi \urcorner$ is a wff, then $\ulcorner \sim\phi \urcorner$ and $\ulcorner \Box\phi \urcorner$ are wffs.

(c) If $\ulcorner \phi \urcorner$ and $\ulcorner \psi \urcorner$ are wffs, then $\ulcorner (\phi \rightarrow \psi) \urcorner$ is a wff.

(d) Nothing else is a wff.

(3) We introduce the following stipulative definitions.

$$(\phi \vee \psi) \stackrel{\text{def}}{=} (\phi \rightarrow \psi) \rightarrow \psi$$
$$(\phi \wedge \psi) \stackrel{\text{def}}{=} \sim(\phi \rightarrow \sim\psi)$$
$$(\phi \leftrightarrow \psi) \stackrel{\text{def}}{=} \sim((\phi \rightarrow \psi) \rightarrow \sim(\psi \rightarrow \phi))$$
$$\Diamond\phi \stackrel{\text{def}}{=} \sim\Box\sim\phi$$

3. The System K

The axiomatic system *K* is characterized by one additional axiom, *K* (the distribution axiom), and one additional rule of inference, *N* (the rule of necessitation).

$$K\ :\ \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q)$$
$$N\ :\ \text{from } \vdash_K \phi, \text{ infer } \vdash_K \Box\phi$$

If we add *K* and *N* to the axioms and rules of inference for *PL*, we get the axiomatic system *K*. Since we know that every theorem of *PL* can be proven from the axioms and rules of inference for *PL*, we can make our axiomatic system *K* a bit easier to work with by allowing ourselves to write down, as an axiom, *any* theorem of *PL*, and allowing ourselves to appeal to *any* valid *PL* rule of inference. Then, the axiomatic system *K* will have the following axioms:

$$\vdash_K \phi, \text{ for all theorems of } PL, \ulcorner \phi \urcorner \qquad (PLT)$$

$$\vdash_K \Box(P \to Q) \to (\Box P \to \Box Q) \qquad (K)$$

And the following rules of inference:

all valid PL inferences $\qquad (PLR)$

from $\vdash_K \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_K \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ $\qquad (US)$

from $\vdash_K \phi$, infer $\vdash_K \Box\phi$ $\qquad (N)$

(PLR) allows you to appeal to *any* PL-valid rule of inference. However, in these notes, I will mostly confine myself to *Modus Ponens* and the following (each '$\rhd$' indicates an allowed inference):

Contraposition

$$\phi \to \psi \;\lhd\; \rhd\; \sim\psi \to \sim\phi$$

$$\phi \to \sim\psi \;\lhd\; \rhd\; \psi \to \sim\phi$$

Hypothetical Syllogism

$$\phi \to \psi, \psi \to \chi \;\rhd\; \phi \to \chi$$

Here is a sample axiomatic proof of $\Box(P \to Q)$ from the premise $\Box Q$.

1.	$\Box Q$	*Premise*
2.	$\vdash_K Q \to (P \to Q)$	(PL)
3.	$\vdash_K \Box(Q \to (P \to Q))$	2 (N)
4.	$\vdash_K \Box(P \to Q) \to (\Box P \to \Box Q)$	(K)
5.	$\vdash_K \Box(Q \to (P \to Q)) \to (\Box Q \to \Box(P \to Q))$	4, *US*
6.	$\vdash_K \Box Q \to \Box(P \to Q)$	3, 5 (PLR)
7.	$\Box(P \to Q)$	1, 6 (PLR)

Here, it is important that we keep track of which wffs on the lines are *theorems* of K, and which are merely consequences of our *assumptions* in the proof (both the rules (US) and (N) apply only to theorems). I write $\ulcorner \vdash_K \phi \urcorner$ iff $\ulcorner \phi \urcorner$ is a theorem of K. If an application of a rule of inference makes use of only theorems, then the result of applying that rule will be a theorem. If, however—as on line 7 above—an application of a rule of inference makes use of non-theorems, then the result of applying that rule will *not* be a theorem.

If $\ulcorner \phi \urcorner$ is provable in the system K from the premises in Γ, then we write

$$\Gamma \vdash_K \phi$$

Thus, the above axiomatic proof establishes that $\{\Box Q\} \vdash_K \Box(P \to Q)$.

If $\ulcorner \phi \urcorner$ is provable in the system K from no premises, then we say that $\ulcorner \phi \urcorner$ is a *theorem* of K, and we write

$$\vdash_K \phi$$

In general, for any axiomatic system S, if $\ulcorner\phi\urcorner$ is provable from the premises in Γ, then I will write

$$\Gamma \vdash_S \phi$$

And, for any axiomatic system S, if $\ulcorner\phi\urcorner$ is provable in S from no premises, then I will say that $\ulcorner\phi\urcorner$ is a *theorem* of S, and I will write

$$\vdash_S \phi$$

Here is a proof that the wff '$\Box P \to \Box{\sim}{\sim}P$' is a theorem of K:

1.	$\vdash_K P \to {\sim}{\sim}P$	(PL)
2.	$\vdash_K \Box(P \to {\sim}{\sim}P)$	1 (N)
3.	$\vdash_K \Box(P \to Q) \to (\Box P \to \Box Q)$	(K)
4.	$\vdash_K \Box(P \to {\sim}{\sim}P) \to (\Box P \to \Box{\sim}{\sim}P)$	3 (US)
5.	$\vdash_K \Box P \to \Box{\sim}{\sim}P$	2, 4 (PLR)

Given these rules of inference, we can *derive* other rules of inference. We may do this by using metavariables like $\ulcorner\phi\urcorner$ and $\ulcorner\psi\urcorner$ to stand for any wff of PML. One useful derived rule is a rule which lets us go immediately from $\vdash_K \phi \to \psi$ to $\vdash_K \Box\phi \to \Box\psi$.

We show that this inference will be valid in K, no matter which wffs $\ulcorner\phi\urcorner$ and $\ulcorner\psi\urcorner$ refer to, by providing a *schematic* derivation from $\vdash_K \phi \to \psi$ to $\vdash_K \Box\phi \to \Box\psi$ like the following:

1.	$\vdash_K \phi \to \psi$	
2.	$\vdash_K \Box(\phi \to \psi)$	1 (N)
3.	$\vdash_K \Box(P \to Q) \to (\Box P \to \Box Q)$	(K)
4.	$\vdash_K \Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	3 (US)
5.	$\vdash_K \Box\phi \to \Box\psi$	2, 4 (PLR)

Since we've proven that, whenever we have something of the form '$\vdash_K \phi \to \psi$', we can get to something of the form '$\vdash_K \Box\phi \to \Box\psi$' in 4 steps, we may introduce a new *derived rule*, which we can call 'KR' (for 'K rule'):

$$\text{from } \vdash_K \phi \to \psi, \text{ infer } \vdash_K \Box\phi \to \Box\psi \qquad (KR)$$

We may derive a similar rule for the biconditional:

1.	$\vdash_K \phi \leftrightarrow \psi$	
2.	$\vdash_K \phi \to \psi$	1 (PLR)
3.	$\vdash_K \Box\phi \to \Box\psi$	2 (KR)
4.	$\vdash_K \psi \to \phi$	1 (PLR)
5.	$\vdash_K \Box\psi \to \Box\phi$	4 (KR)
6.	$\vdash_K \Box\phi \leftrightarrow \Box\psi$	3, 5 (PLR)

(Here, I am making use of the PL-valid rule of inferences which allow you to go from a wff of the form $\ulcorner\phi \leftrightarrow \psi\urcorner$ to either $\ulcorner\phi \to \psi\urcorner$ or $\ulcorner\psi \to \phi\urcorner$; and which allows you to go from $\ulcorner\phi \to \psi\urcorner$ and $\ulcorner\psi \to \phi\urcorner$ to $\ulcorner\phi \leftrightarrow \psi\urcorner$.)

Therefore, we have the derived rule (BKR) (for 'biconditional K rule'):

$$\text{from } \vdash_K \phi \leftrightarrow \psi, \text{ infer } \vdash_K \Box\phi \leftrightarrow \Box\psi \qquad (BKR)$$

Note also that the following are theorems of PL, and therefore theorems of K:

$$(P \leftrightarrow Q) \rightarrow (\sim P \leftrightarrow \sim Q)$$
$$(P \leftrightarrow Q) \rightarrow ((P \rightarrow R) \leftrightarrow (Q \rightarrow R))$$
$$(P \leftrightarrow Q) \rightarrow ((R \rightarrow P) \leftrightarrow (R \rightarrow Q))$$

Though it may not be apparent at first, our rule (BKR), together with the above theorems of PL, tell us that, if it is a theorem that $\ulcorner \phi \leftrightarrow \psi \urcorner$, then we may substitute $\ulcorner \psi \urcorner$ for $\ulcorner \phi \urcorner$ wherever it may appear in a wff. Why? Because every wff of PML consists of the operators '$\Box$', '$\sim$', and '$\rightarrow$', built up in some way (all the other connectives are simply *defined* in terms of them). The rule (BKR) and the theorems above therefore allow us, from $\vdash_K \phi \leftrightarrow \psi$, to build up to a theorem of the form $\vdash_K \zeta[\phi] \leftrightarrow \zeta[\psi/\phi]$, for any wff $\ulcorner \zeta[\phi] \urcorner$ in which the wff $\ulcorner \phi \urcorner$ appears.

An example will help make this clearer. Suppose that we have a wff of the form $\ulcorner \sim\Box\phi \rightarrow \chi \urcorner$ at some point in our axiomatic proof, and it is a theorem that $\ulcorner \phi \leftrightarrow \psi \urcorner$. We could use the above facts to exchange $\ulcorner \psi \urcorner$ for $\ulcorner \phi \urcorner$ in $\ulcorner \sim\Box\phi \rightarrow \chi \urcorner$ by going through a procedure like the following:

1.	$\sim\Box\phi \rightarrow \chi$	
2.	$\vdash_K \phi \leftrightarrow \psi$	
3.	$\vdash_K \Box\phi \leftrightarrow \Box\psi$	2 (BKR)
4.	$\vdash_K (P \leftrightarrow Q) \rightarrow (\sim P \leftrightarrow \sim Q)$	(PL)
5.	$\vdash_K (\Box\phi \leftrightarrow \Box\psi) \rightarrow (\sim\Box\phi \leftrightarrow \sim\Box\psi)$	4 (US)
6.	$\vdash_K \sim\Box\phi \leftrightarrow \sim\Box\psi$	3, 5 (PLR)
7.	$\vdash_K (P \leftrightarrow Q) \rightarrow ((P \rightarrow R) \leftrightarrow (Q \rightarrow R))$	(PL)
8.	$\vdash_K (\sim\Box\phi \leftrightarrow \sim\Box\psi) \rightarrow ((\sim\Box\phi \rightarrow \chi) \leftrightarrow (\sim\Box\psi \rightarrow \chi))$	6 (US)
9.	$\vdash_K (\sim\Box\phi \rightarrow \chi) \leftrightarrow (\sim\Box\psi \rightarrow \chi)$	6, 8 (PLR)
10.	$\sim\Box\psi \rightarrow \chi$	1, 9 (PLR)

We could be more rigorous about this and provide a proof by mathematical induction establishing that we will *always* be able to carry out a procedure like the foregoing, but I hope the example makes it clear enough how this would go. Therefore, we have another derived rule, which I will call 'SE', for 'substitution of equivalents':

$$\text{from } \zeta[\phi] \text{ and } \vdash_K \phi \leftrightarrow \psi, \text{ infer } \zeta[\psi/\phi] \qquad (SE)$$

Before moving on, it will be helpful to get out on the table a derived rule for the relationship between the modal operators and negations. We may derive a rule which tells us that, whenever we have a wff of the form $\ulcorner \sim\Box\phi \urcorner$ appearing anywhere in a wff, we may exchange it with one of the form $\ulcorner \Diamond\sim\phi \urcorner$, and *vice versa*. Given (SE), it is enough to show this to show that $\vdash_K \sim\Box\phi \leftrightarrow \Diamond\sim\phi$ is a theorem.

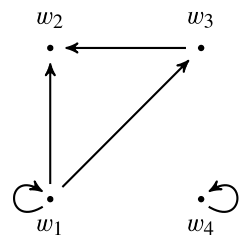


FIGURE 1. A *K-frame* consisting of the worlds w_1, w_2, w_3, and w_4, and the accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_1w_3, Rw_3w_2, and Rw_4w_4.

1.	$\vdash_K P \leftrightarrow P$	(PL)
2.	$\vdash_K \sim\Box\phi \leftrightarrow \sim\Box\phi$	1 (US)
3.	$\vdash_K P \leftrightarrow \sim\sim P$	(PL)
4.	$\vdash_K \phi \leftrightarrow \sim\sim\phi$	3 (US)
5.	$\vdash_K \sim\Box\phi \leftrightarrow \sim\Box\sim\sim\phi$	2, 4 (SE)
6.	$\vdash_K \sim\Box\phi \leftrightarrow \Diamond\sim\phi$	5 *def.* '$\Diamond$'

We may similarly derive a rule which tells us that, whenever we have a wff of the form $\ulcorner\sim\Diamond\phi\urcorner$ appearing anywhere in a wff, we may exchange it with one of the form $\ulcorner\Box\sim\phi\urcorner$, and *vice versa*. Again, given (*SE*), it is enough to show that it is a theorem that $\vdash_K \sim\Diamond\phi \leftrightarrow \Box\sim\phi$

1.	$\vdash_K \sim\sim P \leftrightarrow P$	(PL)
2.	$\vdash_K \sim\sim\Box\sim\phi \leftrightarrow \Box\sim\phi$	1 (US)
3.	$\vdash_K \sim\Diamond\phi \leftrightarrow \Box\sim\phi$	2 *def.* '$\Diamond$'

Thus, we have the derived rules '*MN*', for 'modal negation'.

$$\begin{aligned} &\text{from } \zeta[\sim\Box\phi], \text{ infer } \zeta[\Diamond\sim\phi] \\ &\text{from } \zeta[\sim\Diamond\phi], \text{ infer } \zeta[\Box\sim\phi] \\ &\text{from } \zeta[\Box\sim\phi], \text{ infer } \zeta[\sim\Diamond\phi] \\ &\text{from } \zeta[\Diamond\sim\phi], \text{ infer } \zeta[\sim\Box\phi] \end{aligned} \tag{MN}$$

3.1. **Semantics for K.** A *K-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$ (known as the 'accessibility relation'). An illustration of a K-frame is shown in figure 1.

A *K-model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of a K frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of PML and worlds $w \in \mathcal{W}$ to $\{1, 0\}$. An illustration of a K-model is displayed in figure 2.

Given a K-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in the following manner. For every world $w \in \mathcal{W}$, every atomic wff $\ulcorner\alpha\urcorner$, and every pair of wffs $\ulcorner\phi\urcorner$, $\ulcorner\psi\urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.

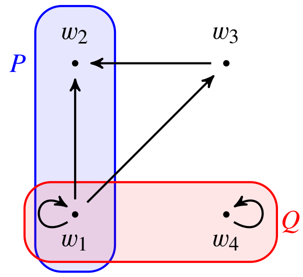


FIGURE 2. A *K-model* consisting of the frame from figure 1 together with an interpretation $\mathscr{I}$ such that P is true in w_1 and w_2 and Q is true in w_1 and w_4.

(2) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.

(3) $V_{\mathcal{M}}(\phi \to \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.

(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every $w' \in \mathcal{W}$, if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

We can show that, given this semantics, and given our definition of $\ulcorner\Diamond\phi\urcorner$,

$$V_{\mathcal{M}}(\Diamond\phi, w) = 1 \text{ iff there is some } w' \text{ such that } Rww' \text{ and } V_{\mathcal{M}}(\phi, w') = 1.$$

To do so, we'll first prove the left-to-right hand direction of the biconditional above, and then prove the right-to-left hand direction.

1.	Suppose that $V_{\mathcal{M}}(\Diamond\phi, w) = 1$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\sim\Box\sim\phi, w) = 1$.	1, *def.* '$\Diamond$'
3.	Then, $V_{\mathcal{M}}(\Box\sim\phi, w) = 0$.	2, *def.* '$\sim$'
4.	So, it is not the case that $V_{\mathcal{M}}(\Box\sim\phi, w) = 1$.	3, *bivalence*
5.	So, it is not the case that, for all w', if Rww', then $V_{\mathcal{M}}(\sim\phi, w') = 1$.	4, *def.* '$\Box$'
6.	So, there is some w' such that Rww' and it is not the case that $V_{\mathcal{M}}(\sim\phi, w') = 1$	5, *QL*
7.	So there is some w' such that Rww' and $V_{\mathcal{M}}(\sim\phi, w') = 0$	6, *bivalence*
8.	So there is some w' such that Rww' and $V_{\mathcal{M}}(\phi, w') = 1$	7, *def.* '$\sim$'

In the above semantic proof, I justified line 6 by citing Quantificational Logic, '*QL*', because quantificational logic tells us that '$\sim(\forall x)(Px \to Qx)$' is equivalent to '$(\exists x)(Px \wedge \sim Qx)$'.

We can then establish the right-to-left hand direction by just running through the semantic proof above, but in the opposite direction:

1.	Suppose there is some w' such that Rww' and $V_{\mathcal{M}}(\phi, w') = 1$	*Assumption*
2.	Then there is some w' such that Rww' and $V_{\mathcal{M}}(\sim\phi, w') = 0$	6, *def.* '$\sim$'
3.	So, there is some w' such that Rww' and it is not the case that $V_{\mathcal{M}}(\sim\phi, w') = 1$	2, *bivalence*
4.	So, it is not the case that, for all w', if Rww', then $V_{\mathcal{M}}(\sim\phi, w') = 1$.	3, *QL*
5.	So, it is not the case that $V_{\mathcal{M}}(\Box\sim\phi, w) = 1$.	4, *def.* '$\Box$'
6.	Then, $V_{\mathcal{M}}(\Box\sim\phi, w) = 0$.	5, *bivalence*

7.	Then, $V_{\mathcal{M}}(\sim\Box\sim\phi, w) = 1$.	6, *def.* '$\sim$'
8.	So $V_{\mathcal{M}}(\Diamond\phi, w) = 1$.	7, *def.* '$\Diamond$'

Now that we've proven that $V_{\mathcal{M}}(\Diamond\phi, w) = 1$ iff there is some w' such that Rww' and $V_{\mathcal{M}}(\phi, w') = 1$, we should feel free to use this fact in our semantic proofs in the future (just as we should feel free to use the fact that $V_{\mathcal{M}}(\phi \wedge \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 1$ and $V_{\mathcal{M}}(\psi, w) = 1$).

3.2. ***K*-Consequence.** We will say that $\ulcorner\phi\urcorner$ is a K-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is K-valid,

$$\Gamma \models_K \phi$$

iff there is no K-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every K-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

And we will say that a wff $\ulcorner\phi\urcorner$ is a *K-tautology*, or *K-valid*, written

$$\models_K \phi$$

if and only if $\ulcorner\phi\urcorner$ is true at every world in every K model.

> **Interesting and Unexpected and Fantastic Fact**: for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,
>
> $$\Gamma \vdash_K \phi \quad \text{if and only if} \quad \Gamma \models_K \phi$$

3.3. **Establishing Validity in K.** If we wish to show that an argument from the premises in Γ to the conclusion $\ulcorner\phi\urcorner$ is K-valid, we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\{\Diamond P\} \models_K \Diamond(Q \rightarrow P)$$

We may do so with a semantic proof like the following:

1.	Suppose that there is an arbitrary K-model $< \mathcal{W}, R, \mathscr{I} >$, with some arbitrary $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond P, w) = 1$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Diamond P, w) = 1$	1, $\wedge E$
3.	So, there is some $w' \in \mathcal{W}$ such that Rww' and $V_{\mathcal{M}}(P, w') = 1$.	2, *def.* '$\Diamond$'
4.	So, there is some $w' \in \mathcal{W}$ such that Rww' and $V_{\mathcal{M}}(Q \rightarrow P, w') = 1$	3, *def.* '$\rightarrow$'
5.	So $V_{\mathcal{M}}(\Diamond(Q \rightarrow P), w) = 1$.	4, *def.* '$\Diamond$'
6.	So, if there is an arbitrary K-model $< \mathcal{W}, R, \mathscr{I} >$ with some arbitrary $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond P, w) = 1$, then $V_{\mathcal{M}}(\Diamond(Q \rightarrow P), w) = 1$.	1–5, $\rightarrow I$
7.	So, for any K-model $< \mathcal{W}, R, \mathscr{I} >$ and any $w \in \mathcal{W}$, if $V_{\mathcal{M}}(\Diamond P, w) = 1$, then $V_{\mathcal{M}}(\Diamond(Q \rightarrow P), w) = 1$ too.	6, *QL*

3.4. **Establishing Invalidity in K.** In order to establish that an argument from the premises in Γ to the conclusion $\ulcorner\phi\urcorner$ is invalid in K, that is, that $\Gamma \not\models_K \phi$, it is enough to provide a

K-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet $\ulcorner \phi \urcorner$ is false at that world. For instance, suppose that we wish to show that

$$\{\Box P\} \not\models_K P$$

We may do so with the following K-model:

$$\mathcal{W} = \{w_1\}$$
$$R = \varnothing$$
$$\mathscr{I}(P, w_1) = 0$$

To see that $V_{\mathcal{M}}(\Box P, w_1) = 1$, note that all of the none of the worlds w' such that Rw_1w' are such that $V_{\mathcal{M}}(P, w') = 1$. Yet, $V_{\mathcal{M}}(P, w_1) = 0$, since $\mathscr{I}(P, w_1) = 0$. So this is a K-model in which '$\Box P$' is true at w_1, yet 'P' is false at w_1.

The semantics for K puts *absolutely no* constraints on the accessibility relation R. So it allows the possibility of 'blind' worlds—worlds which see no other worlds (not even themselves).

For this reason, it is also true in K that '$\Diamond P$' does not follow from '$\Box P$'.

$$\{\Box P\} \not\models_K \Diamond P$$

The K-model above provides a counterexample. For there, all of the none of the worlds which w_1 sees are P worlds, so $V_{\mathcal{M}}(\Box P, w_1) = 1$. However, it is not the case that there is a world which w_1 sees which is a P world, so $V_{\mathcal{M}}(\Diamond P, w_1) = 0$.

3.5. **Natural Deduction for K.** We can augment our natural deduction system for PL with a few new rules to achieve a natural deduction system for K.

What makes K natural deduction distinctive is that it introduces two new kinds of subproof, which we'll call *strict subproofs*. There are *box*-strict subproofs and *diamond*-strict subproofs. The first new rule is that you may, at any time, introduce a new box-strict subproof. For instance, here is an example of a natural deduction proof in K:

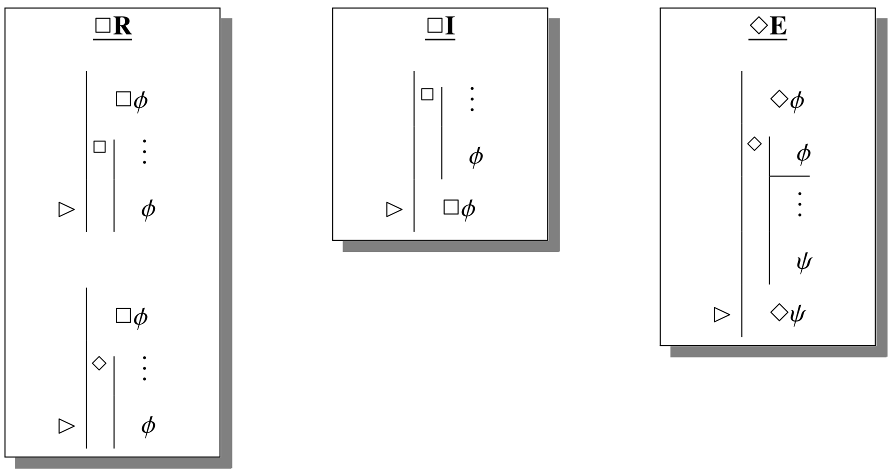
1	$\Box$ | | P	$A\ (\rightarrow I)$
2	| | P	1, R
3	| $P \rightarrow P$	1–2, $\rightarrow I$
4	$\Box(P \rightarrow P)$	1–3, $\Box I$

The subproof running from lines 1–3 is a *box-strict subproof*, as indicated by the box in the upper left-hand-corner. Here is a K-derivation which utilizes a diamond-strict subproof:

1		$\Diamond P$	
2	$\Diamond$	P	$A(\Diamond E)$
3		$P \vee Q$	2, $\vee I$
4		$\Diamond(P \vee Q)$	1, 2–3, $\Diamond E$

Note that, unlike a regular subproof, box-strict subproofs do not begin with any assumptions. Diamond-strict subproofs, on the other hand, *must* begin with an assumption.

With strict subproofs, we get three new rules:



The first rule tells us that, if we have a wff of the form ⌜$\Box\phi$⌝ outside of a strict subproof—and there are no other strict subproofs between the scope line on which ⌜$\Box\phi$⌝ appears and the strict subproof we are inside—then we may write down ⌜ϕ⌝ within our strict subproof (whether the strict subproof is a box or a diamond strict subproof). The second rule tells us that, if we have ⌜ϕ⌝ occurring on the final line of an accessible box strict subproof, then we may write down ⌜$\Box\phi$⌝ outside of the scope of the box strict subproof. The third rule says: if you have the wff ⌜$\Diamond\phi$⌝ appearing at some point in a derivation—and there are no strict subproofs between the scope line on which ⌜$\Diamond\phi$⌝ appears and your current scope line—then you may open a new diamond strict subproof with the assumption ⌜ϕ⌝. When you do so, you should write '$A(\Diamond E)$' (assumption for the purposes of $\Diamond$ elimination) in the justification for that line. If, at the bottom of that subproof, you have a wff ⌜ψ⌝, then you may write ⌜$\Diamond\psi$⌝ outside of the scope of the diamond strict subproof.

When we add strict subproofs, we must restrict the use of the reiteration rule, *R*. Wffs which lie outside of a strict subproof *may not* be reiterated within a strict subproof. For instance, the following derivation is not legal:

1	P	$A\ (\rightarrow I)$	
2	$\Box \mid P$	1, R	⟵ **MISTAKE!!!**
3	$\Box P$	2–2, $\Box I$	
4	$P \rightarrow \Box P$	1–3, $\rightarrow I$	

Nor is the following derivation legal:

1	$\Diamond P$		
2	Q		
3	$\Diamond \mid P$	$A(\Diamond E)$	
4	Q	2, R	⟵ **MISTAKE!!!**
5	$P \wedge Q$	3, 4, $\wedge I$	
6	$\Diamond(P \wedge Q)$	1, 3–5, $\Diamond E$	

This is good, because $\{\Diamond P, Q\} \not\models_K \Diamond(P \wedge Q)$ (see the homework).

Just as you cannot reiterate lines outside of a strict subproof within a strict subproof, you may not appeal to those lines for the purposes of conjunction introduction, conditional elimination, *etc.* Once you are within a strict-subproof, the only wffs which are accessible are those appearing within that strict subproof or those of the form $\ulcorner \Box\phi \urcorner$ which lie on a scope line which extends to your current line (and which is not blocked by the scope of any strict subproof other than your own). And wffs of the form $\ulcorner \Box\phi \urcorner$ outside of your strict subproof may only be appealed to with the rule $\Box R$.

For instance, the following derivation is not legal:

1	$\Box\Box P \wedge (\Box P \rightarrow (P \rightarrow Q))$		
2	$\Box\Box P$	1, $\wedge E$	
3	$\Box P \rightarrow (P \rightarrow Q)$	1, $\wedge E$	
4	$\Box \mid \Box P$	2, $\Box R$	
5	$P \rightarrow Q$	3, 4, $\rightarrow E$	⟵ **MISTAKE!!!**
6	$\Box \mid P$	4, $\Box R$	
7	Q	5, 6, $\rightarrow E$	⟵ **MISTAKE!!!**
8	$\Box Q$	6–7, $\Box I$	
9	$\Box\Box Q$	4–8, $\Box I$	

Line 3 cannot be appealed to at line 5 for the purposes of $\to E$, since line 3 lies outside of the strict subproof which begins at line 4. And, even if line 5 occurs within the strict subproof which begins at line 4, it does not occur within the strict subproof which begins at line 6, and so it is not accessible at line 7 for the purposes of $\to E$.

Note that $\Box R$ only allows you to write down $\ulcorner \phi \urcorner$ *within* a strict subproof if you have $\ulcorner \Box\phi \urcorner$ accessible just *outside of* that strict subproof. It does not allow you to write down $\ulcorner \phi \urcorner$ within a strict subproof if $\ulcorner \Box\phi \urcorner$ is accessible *within* that subproof. For instance, the following derivation is not legal:

1		$\Diamond\Box P$		
2	$\Diamond$	$\Box P$	$A(\Diamond E)$	
3		P	2, $\Box R$	$\longleftarrow$ **MISTAKE!!!**
4		$\Diamond P$	1, 2–3, $\Diamond E$	

This too is good, since $\Diamond\Box P \not\models_K \Diamond P$. (Can you think of a K-model in which '$\Diamond\Box P$' is true at a world, yet '$\Diamond P$' is false at that world?)

Note also that, while $\Box R$ allows you to place $\ulcorner \phi \urcorner$ inside of any strict subproof, so long as $\ulcorner \Box\phi \urcorner$ is accessible outside of that subproof (and there are no other strict subproofs between the scope line on which $\ulcorner \Box\phi \urcorner$ appears and the strict subproof where you write down $\ulcorner \phi \urcorner$), it *does not* allow you to place $\ulcorner \phi \urcorner$ inside of any strict subproof just because $\ulcorner \Diamond\phi \urcorner$ is accessible outside of that subproof. For instance, the following K-derivation is not legal:

1		$\Diamond P$		
2		$\Diamond Q$		
3	$\Diamond$	P	$A(\Diamond E)$	
4		Q	2, $\Box R$	$\longleftarrow$ **MISTAKE!!!**
5		$P \wedge Q$	3, 4, $\wedge I$	
6		$\Diamond(P \wedge Q)$	1, 3–5, $\Diamond E$	

This is good, because $\{\Diamond P, \Diamond Q\} \not\models_K \Diamond(P \wedge Q)$. (Can you think of a counter-model?)

Additionally, $\Box R$ only applies when there is *a single* strict subproof between the line on which $\ulcorner \Box\phi \urcorner$ appears and the line you are currently at in the derivation. The following derivation is not legal:

1	$\Box P$		
2	$\Box Q$		
3	$\Box$ | P	1, $\Box R$	
4	$\Box$ | $\Box$ | Q	2, $\Box R$	$\longleftarrow$ **MISTAKE!!!**
5	$\Box$ | $\Box Q$	4–4, $\Box I$	
6	$\Box\Box Q$	3–5, $\Box I$	

$\Box R$ only allows you to remove boxes across the border of a *single* strict subproof. It is not allowed to carry 'Q' across the borders of two strict subproofs at once. *This* derivation, on the other hand, *is* legal:

1	$\Diamond\Box P$	
2	$\Box\Box Q$	
3	$\Diamond$ | $\Box P$	$A(\Diamond E)$
4	| $\Box Q$	2, $\Box R$
5	| $\Box$ | Q	4, $\Box R$
6	| | P	3, $\Box R$
7	| | $Q \wedge P$	5, 6, $\wedge I$
8	| $\Box(Q \wedge P)$	5–7, $\Box I$
9	$\Diamond\Box(Q \wedge P)$	1, 3–8, $\Diamond E$

It is also important to understand that, in order to use $\Diamond E$, the strict subproof must have began with a wff $\ulcorner\phi\urcorner$, where $\ulcorner\Diamond\phi\urcorner$ occurs on an accessible line outside of that strict subproof. For instance, the following derivation is *not* legal:

1	| $\Box P$		
2	| $\Diamond$ | P	$A(\Diamond E)$	
3	| | P	1, $\Box R$	
4	| $\Diamond P$	2–3, $\Diamond E$	$\longleftarrow$ **MISTAKE!!!**
5	$\Box P \rightarrow \Diamond P$	1–4, $\rightarrow I$	

This is good, because '$\Box P \rightarrow \Diamond P$' is *not* a theorem of K (see above). Thus, it is important, when you are justifying a line by $\Diamond E$, that you write down both 1) the line on which a wff of the form $\ulcorner\Diamond\phi\urcorner$ appears, outside of the diamond strict subproof; and 2) the lines of a diamond strict subproof which begins with the wff $\ulcorner\phi\urcorner$. Without (1), you are *not* allowed to write down $\ulcorner\Diamond\psi\urcorner$.

If there is a legal K-derivation of $\ulcorner\phi\urcorner$ starting with the assumptions in Γ, then I will write

$$\Gamma \vdash_{KD} \phi$$

And if there is a legal K-derivation of $\ulcorner\phi\urcorner$ from no assumptions, then I will write

$$\vdash_{KD} \phi$$

In general, for the natural deduction system for S, if there is a legal S-derivation of $\ulcorner\phi\urcorner$ starting with the assumptions in Γ, I will write

$$\Gamma \vdash_{SD} \phi$$

And if there is a legal S-derivation of $\ulcorner\phi\urcorner$ from no assumptions, then I will write

$$\vdash_{SD} \phi$$

Here is a K-derivation of the K axiom.

1	$\Box(P \to Q)$	A ($\to I$)
2	$\Box P$	A ($\to I$)
3	P	2, $\Box R$
4	$P \to Q$	1, $\Box R$
5	Q	3, 4, $\to E$
6	$\Box Q$	3–5, $\Box I$
7	$\Box P \to \Box Q$	2–6, $\to I$
8	$\Box(P \to Q) \to (\Box P \to \Box Q)$	1–7, $\to I$

Here is a K-derivation establishing that $\{\Box P \wedge \Box Q\} \vdash_{KD} \Box(P \wedge Q)$:

1	$\Box P \wedge \Box Q$	
2	$\Box P$	1, $\wedge E$
3	$\Box Q$	1, $\wedge E$
4	P	2, $\Box R$
5	Q	3, $\Box R$
6	$P \wedge Q$	4, 5, $\wedge I$
7	$\Box(P \wedge Q)$	4–5, $\Box I$

Here is a K-derivation establishing that $\vdash_{KD} \Box(P \to Q) \to (\Diamond P \to \Diamond Q)$.

1	$\Box(P \to Q)$	$A(\to I)$
2	$\Diamond P$	$A(\to I)$
3	$\Diamond$ P	$A(\Diamond E)$
4	$P \to Q$	1, $\Box R$
5	Q	3, 4, $\to E$
6	$\Diamond Q$	2, 3–5, $\Diamond E$
7	$\Diamond P \to \Diamond Q$	2–6, $\to I$
8	$\Box(P \to Q) \to (\Diamond P \to \Diamond Q)$	1–7, $\to I$

And one showing that $\{\Diamond P \vee \Diamond Q\} \vdash_{KD} \Diamond(P \vee Q)$.

1	$\Diamond P \vee \Diamond Q$	
2	$\Diamond P$	$A(\vee E)$
3	$\Diamond$ P	$A(\Diamond E)$
4	$P \vee Q$	3, $\vee I$
5	$\Diamond(P \vee Q)$	2, 3–4, $\Diamond E$
6	$\Diamond Q$	$A(\vee E)$
7	$\Diamond$ Q	$A(\Diamond E)$
8	$P \vee Q$	7, $\vee I$
9	$\Diamond(P \vee Q)$	6, 7–8, $\Diamond E$
10	$\Diamond(P \vee Q)$	1, 2–5, 6–9, $\vee E$

We will also introduce eight new rules governing negations and modal operators:

MN

$\sim\Box\phi$	$\lhd\ \rhd$	$\Diamond\sim\phi$
$\sim\Diamond\phi$	$\lhd\ \rhd$	$\Box\sim\phi$
$\Box\phi$	$\lhd\ \rhd$	$\sim\Diamond\sim\phi$
$\Diamond\phi$	$\lhd\ \rhd$	$\sim\Box\sim\phi$

Here is a K-derivation establishing that $\Box P \vdash_{KD} \sim\Diamond\sim\sim\sim P$.

1	$\Box P$	
2	$\Box$ P	1, $\Box R$
3	$\sim P$	$A(\sim I)$
4	$P \wedge \sim P$	2, 3, $\wedge I$
5	$\sim\sim P$	3–4, $\sim I$
6	$\Box\sim\sim P$	2–5, $\Box I$
7	$\sim\Diamond\sim\sim\sim P$	6, MN

And here is one establishing that $\vdash_{KD} \Box((\Box(P \rightarrow Q) \wedge \Diamond\sim Q) \rightarrow \Diamond\sim P)$.

1	$\Box$ $\Box(P \rightarrow Q) \wedge \Diamond\sim Q$	$A(\rightarrow I)$
2	$\Box(P \rightarrow Q)$	1, $\wedge E$
3	$\Diamond\sim Q$	1, $\wedge E$
4	$\sim\Box Q$	3, MN
5	$\sim\Diamond\sim P$	$A(\sim E)$
6	$\Box P$	5, MN
7	$\Box$ P	6, $\Box R$
8	$P \rightarrow Q$	2, $\Box R$
9	Q	7, 8, $\rightarrow E$
10	$\Box Q$	7–9, $\Box I$
11	$\Box Q \wedge \sim\Box Q$	3, 10, $\wedge I$
12	$\Diamond\sim P$	5–11, $\sim E$
13	$(\Box(P \rightarrow Q) \wedge \Diamond\sim Q) \rightarrow \Diamond\sim P$	1–12, $\rightarrow I$
14	$\Box((\Box(P \rightarrow Q) \wedge \Diamond\sim Q) \rightarrow \Diamond\sim P)$	1–13, $\Box I$

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,

$$\Gamma \vdash_{KD} \phi \quad \text{if and only if} \quad \Gamma \vDash_{K} \phi$$

4. The System D

Suppose we add to our axiomatic system K the following axiom

$$D \;:\; \Box P \rightarrow \Diamond P$$

This gives us the system D. The axiomatic system D will have the following axioms:

$$\vdash_D \phi, \text{ for all theorems of } PL, \ulcorner \phi \urcorner \qquad (PL)$$

$$\vdash_D \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \qquad (K)$$

$$\vdash_D \Box P \rightarrow \Diamond Q \qquad (D)$$

And the following rules of inference:

$$\text{all } PL \text{ valid inferences} \qquad (PLR)$$

$$\text{from } \vdash_D \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_D \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N] \qquad (US)$$

$$\text{from } \vdash_D \phi, \text{ infer } \vdash_D \Box\phi \qquad (N)$$

Since all the axioms and rules of inference of K are axioms and rules of inference of D, we retain all the derived rules from §3.

It is automatic that, in D, unlike in K, $\{\Box P\} \vdash_D \Diamond P$:

1.	$\Box P$	*Premise*
2.	$\vdash_K \Box P \rightarrow \Diamond P$	(D)
3.	$\Diamond P$	1, 2 (PLR)

Here is an axiomatic proof showing that $\vdash_D \sim\Diamond P \rightarrow \Diamond\sim P$ (this is not a tautology in K—see the homework).

1.	$\vdash_D \Box P \rightarrow \Diamond P$	(D)
2.	$\vdash_D \Box\sim P \rightarrow \Diamond\sim P$	1 (US)
3.	$\vdash_D \sim\Diamond P \rightarrow \Diamond\sim P$	2 (MN)

4.1. **Semantics for D.** Our semantics for K imposed absolutely no constraints on the accessibility relation R. If we require that the accessibility relation R be *serial*—that is, that every world sees at least one (not necessarily distinct) world—then we get a semantics for the system D.

A *D-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *serial* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Recall,

> Seriality:
> A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is serial iff, for all $a \in \mathbf{A}$, there is some $b \in \mathbf{A}$ such that Rab.
>
> $$\forall a \, \exists b \;\; Rab$$

A sample D-frame is shown in figure 3.

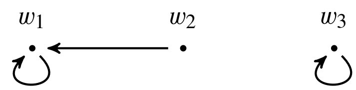


FIGURE 3. A *D-frame* consisting of the worlds w_1, w_2, and w_3, and an accessibility relation R such that Rw_1w_1, Rw_2w_1, and Rw_3w_3.

A *D model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of a D-frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of PML and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

Given a D-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in precisely the same way we did for K-models. That is: for every world $w \in \mathcal{W}$, every atomic wff $\ulcorner \alpha \urcorner$, and every pair of wffs $\ulcorner \phi \urcorner$, $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.

(2) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.

(3) $V_{\mathcal{M}}(\phi \to \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.

(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every w', if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

4.2. ***D*-Consequence.** We will say that $\ulcorner \phi \urcorner$ is a D-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is D-valid,

$$\Gamma \models_D \phi$$

iff there is no D-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every D-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

And we will say that a wff $\ulcorner \phi \urcorner$ is a *D-tautology*, or *D-valid*, written

$$\models_D \phi$$

if and only if $\ulcorner \phi \urcorner$ is true at every world in every D model.

> **Interesting and Unexpected and Fantastic Fact:** for every set of wffs Γ and every wff $\ulcorner \phi \urcorner$,
>
> $$\Gamma \vdash_D \phi \quad \text{if and only if} \quad \Gamma \models_D \phi$$

4.3. **Establishing Validity in *D*.** If we wish to show that an argument from the premises in Γ to the conclusion $\ulcorner \phi \urcorner$ is D-valid, we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\sim\Diamond P \models_D \Diamond\sim P$$

We may do so with a semantic proof like the following:

1. Suppose that there is an arbitrary D-model $< \mathcal{W}, R, \mathscr{I} >$, with some arbitrary $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\sim\Diamond P, w) = 1$. *Assumption*

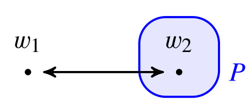
2.	Then, $V_{\mathcal{M}}(\sim\Diamond P, w) = 1$	1, $\wedge E$
3.	So, $V_{\mathcal{M}}(\Diamond P, w) = 0$.	2, *def.* '$\sim$'
4.	So, it is not the case that $V_{\mathcal{M}}(\Diamond P, w) = 1$	3, *bivalence*
5.	So, it is not the case there is some $w' \in \mathcal{W}$ such that Rww' and $V_{\mathcal{M}}(P, w') = 1$.	4, *def.* '$\Diamond$'
6.	So, for all w', if Rww', then it is not the case that $V_{\mathcal{M}}(P, w') = 1$.	5, *QL*
7.	So, for all w', if Rww', then $V_{\mathcal{M}}(P, w') = 0$.	6, *bivalence*
8.	There is some w' such that Rww'—call it 'x'.	*def.* '*D-model*'
9.	So, Rwx.	8
10.	If Rwx, then $V_{\mathcal{M}}(P, x) = 0$	7, *QL*
11.	So, $V_{\mathcal{M}}(P, x) = 0$.	9, 10, *MP*
12.	So, $V_{\mathcal{M}}(\sim P, x) = 1$.	11, *def.* '$\sim$'
13.	So, Rwx and $V_{\mathcal{M}}(\sim P, x) = 1$.	9, 12
14.	So, there is some w' such that Rww' and $V_{\mathcal{M}}(\sim P, w') = 1$.	13, $\exists G$
15.	So, $V_{\mathcal{M}}(\Diamond\sim P, w) = 1$.	14, *def.* '$\Diamond$'
16.	So, if there is an arbitrary D-model $< \mathcal{W}, R, \mathscr{I} >$ such that, for some arbitrary $w \in \mathcal{W}$, $V_{\mathcal{M}}(\sim\Diamond P, w) = 1$, then $V_{\mathcal{M}}(\Diamond\sim P, w) = 1$ as well.	1–15, $\rightarrow I$
17.	So, for any world w in any D-model $\mathcal{M}$, if $V_{\mathcal{M}}(\sim\Diamond P, w) = 1$, then $V_{\mathcal{M}}(\Diamond\sim P, w) = 1$ as well.	16, *QL*

4.4. **Establishing Invalidity in D.** In order to establish that an argument from the premises in Γ to the conclusion $\ulcorner\phi\urcorner$ is invalid in D, that is, that $\Gamma \not\models_D \phi$, it is enough to provide a D-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet $\ulcorner\phi\urcorner$ is false at that world. For instance, suppose that we wish to show that

$$\{\Diamond P\} \not\models_D \Diamond\Diamond P$$

We may do so with the following D-model:

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2\} \\ R &= \{< w_1, w_2 >, < w_2, w_1 >\} \\ \mathscr{I}(P, w_1) &= 0 \\ \mathscr{I}(P, w_2) &= 1 \end{aligned}$$



In this model, $V_{\mathcal{M}}(\Diamond P, w_1) = 1$, since Rw_1w_2 and $V_{\mathcal{M}}(P, w_2) = 1$. However, $V_{\mathcal{M}}(\Diamond P, w_2) = 0$, since w_1 is the only world which w_2 sees, and $V_{\mathcal{M}}(P, w_1) = 0$. Therefore, since w_2 is the only world which w_1 sees, $V_{\mathcal{M}}(\Diamond\Diamond P, w_1) = 0$.

Also in D,

$$\{\Box P\} \not\models_D P$$

The D-model above provides a counterexample. For in that model, $V_{\mathcal{M}}(\Box P, w_1) = 1$, since $V_{\mathcal{M}}(P, w_2) = 1$, and w_2 is the only world which w_1 sees. However, $V_{\mathcal{M}}(P, w_1) = 0$.

4.5. **Natural Deduction for D.** To achieve a natural deduction system for D, we may take the natural deduction system for K and add to it a single rule of inference:

$D\Diamond$I

$$\begin{array}{l|l} & \Box \mid \ \vdots \\ & \mid \ \phi \\ & \Diamond\phi \end{array}$$

This rule says: if you have the wff ⌜ϕ⌝ appearing on the final line of an accessible box strict subproof, then you may write ⌜$\Diamond\phi$⌝ outside the scope of that subproof.

With this new rule, here is a derivation establishing that the characteristic D axiom is a theorem of our natural deduction system, $\vdash_{DD} \Box P \to \Diamond P$

1	$\mid\ \Box P$	A ($\to I$)
2	$\mid\ \Box \mid\ P$	1, $\Box R$
3	$\mid\ \Diamond P$	2, $D\Diamond I$
4	$\Box P \to \Diamond P$	1–3, $\to I$

Here is derivation establishing that $\sim\Diamond P \vdash_{DD} \Diamond{\sim}P$.

1	$\mid\ \sim\Diamond P$	
2	$\mid\ \Box{\sim}P$	1, MN
3	$\mid\ \Box \mid\ \sim P$	2, $\Box R$
4	$\mid\ \Diamond{\sim}P$	3, $D\Diamond I$

And here is a D derivation establishing that $\vdash_{DD} \Diamond((\Box P \land \Box(P \to Q)) \to \Diamond Q)$.

1	$\Box \mid\ \mid\ \Box P \land \Box(P \to Q)$	$A(\to I)$
2	$ \mid\ \mid\ \Box P$	1, $\land E$
3	$ \mid\ \mid\ \Box(P \to Q)$	1, $\land E$
4	$ \mid\ \mid\ \Box \mid\ P$	2, $\Box R$
5	$ \mid\ \mid\ \mid\ P \to Q$	3, $\Box R$
6	$ \mid\ \mid\ \mid\ Q$	4, 5, $\to E$
7	$ \mid\ \mid\ \Diamond Q$	4–6, $D\Diamond I$
8	$ \mid\ (\Box P \land \Box(P \to Q)) \to \Diamond Q$	1–7, $\to I$
9	$\Diamond((\Box P \land \Box(P \to Q)) \to \Diamond Q)$	1–8, $D\Diamond I$

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,

$$\Gamma \vdash_{DD} \phi \quad \text{if and only if} \quad \Gamma \models_{D} \phi$$

5. The System T

Suppose we add to our axiomatic system K the following axiom

$$T \;:\; \Box P \rightarrow P$$

or, equivalently,

$$T' \;:\; P \rightarrow \Diamond P$$

This gives us the axiomatic system T. T will have the following axioms:

$$\vdash_T \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \qquad (PL)$$
$$\vdash_T \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \qquad (K)$$
$$\vdash_T \Box P \rightarrow P \qquad (T)$$

And the following rules of inference:

all PL valid inferences (PLR)

from $\vdash_T \phi[\alpha_1, \alpha_2, \ldots, \alpha_N]$, infer $\vdash_T \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N]$ (US)

from $\vdash_T \phi$, infer $\vdash_T \Box\phi$ (N)

Since all the axioms and rules of inference of K are axioms and rules of inference of T, we retain all the derived rules from §3.

I said above that adding '$P \rightarrow \Diamond P$' as an axiom was equivalent to adding '$\Box P \rightarrow P$' as an axiom. That's because, given this axiomatic framework, we can derive '$\Box P \rightarrow P$' as a theorem if we take '$P \rightarrow \Diamond P$' as an axiom, as follows:

1.	$\vdash_{T'} P \rightarrow \Diamond P$	(T')
2.	$\vdash_{T'} \sim P \rightarrow \Diamond\sim P$	1 (US)
3.	$\vdash_{T'} \sim P \rightarrow \sim\Box P$	2 (MN)
4.	$\vdash_{T'} \Box P \rightarrow P$	3 (PLR)

And we can derive '$P \rightarrow \Diamond P$' as a theorem if we take '$\Box P \rightarrow P$' as an axiom, as follows:

1.	$\vdash_T \Box P \rightarrow P$	(T)
2.	$\vdash_T \Box\sim P \rightarrow \sim P$	1 (US)
3.	$\vdash_T \sim\Diamond P \rightarrow \sim P$	2 (MN)
4.	$\vdash_T P \rightarrow \Diamond P$	3, (PLR)

Notice that I did not carry over the axiom D, '$\Box P \rightarrow \Diamond P$'. That is: (D) is not among the axioms for the system T. The reason for this is that, given (T), (D) is redundant. We can

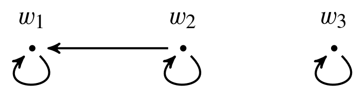


FIGURE 4. A T-*frame* consisting of the worlds w_1, w_2, and w_3, and an accessibility relation R such that Rw_1w_1, Rw_2w_2, Rw_2w_1, and Rw_3w_3.

derive (D) as a theorem within T. To see this, just extend the axiomatic proof above with an application of *hypothetical syllogism*.

1.	$\vdash_T \Box P \to P$	(T)
⋮	⋮	⋮
4.	$\vdash_T P \to \Diamond P$	3, (PLR)
5.	$\vdash_T \Box P \to \Diamond P$	1, 4 (PLR)

5.1. **Semantics for T.** If we require that the accessibility relation R be *reflexive*—that is, that every world sees itself—then we get a semantics for the system T.

A T-*frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Recall,

REFLEXIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is REFLEXIVE iff, for all $a \in \mathbf{A}$, Raa.

$$\forall a \ Raa$$

A sample T-frame is shown in figure 4.

A T-*model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of a T-frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of PML and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

Given a T-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in precisely the same way we did for K-models and D-models. That is: for every world $w \in \mathcal{W}$, every atomic wff $\ulcorner \alpha \urcorner$, and every pair of wffs $\ulcorner \phi \urcorner$, $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.

(2) $V_{\mathcal{M}}({\sim}\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.

(3) $V_{\mathcal{M}}(\phi \to \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.

(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every w', if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

5.2. ***T*-Consequence.** We will say that $\ulcorner \phi \urcorner$ is a T-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is T-valid,

$$\Gamma \models_T \phi$$

iff there is no T-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every T-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

And we will say that a wff $\ulcorner\phi\urcorner$ is a *T-tautology*, or *T-valid*, written

$$\models_T \phi$$

if and only if $\ulcorner\phi\urcorner$ is true at every world in every T model.

> **Interesting and Unexpected and Fantastic Fact:** for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,
>
> $$\Gamma \vdash_T \phi \quad \text{if and only if} \quad \Gamma \models_T \phi$$

5.3. **Establishing Validity in T.** If we wish to show that an argument from the premises in Γ to the conclusion $\ulcorner\phi\urcorner$ is T-valid, we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\models_T \Box(\sim P \vee \Diamond P)$$

We may do so with a semantic proof like the following:

1.	Suppose that there is a T-model $< \mathcal{W}, R, \mathscr{I} >$, with some arbitrary $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\sim P \vee \Diamond P), w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Box(\sim P \vee \Diamond P), w) = 0$.	1, $\wedge E$
3.	So it is not the case that $V_{\mathcal{M}}(\Box(\sim P \vee \Diamond P), w) = 1$	2, *bivalence*
4.	So it is not the case that, for all w', if Rww', then $V_{\mathcal{M}}(\sim P \vee \Diamond P, w') = 1$.	3, *def.* '$\Box$'
5.	So, there is some w' such that $Rww'$and it is not the case that $V_{\mathcal{M}}(\sim P \vee \Diamond P, w') = 1$—call it '$x$'.	4, *QL*
6.	So Rwx and it is not the case that $V_{\mathcal{M}}(\sim P \vee \Diamond P, x) = 1$.	5
7.	So it is not the case that $V_{\mathcal{M}}(\sim P \vee \Diamond P, x) = 1$.	6, $\wedge E$
8.	So $V_{\mathcal{M}}(\sim P \vee \Diamond P, x) = 0$	7, *bivalence*
9.	So $V_{\mathcal{M}}(\sim P, x) = 0$ and $V_{\mathcal{M}}(\Diamond P, x) = 0$.	8, *def.* $\vee$
10.	So $V_{\mathcal{M}}(\sim P, x) = 0$.	9, $\wedge E$
11.	So $V_{\mathcal{M}}(P, x) = 1$.	10, *def.* '$\sim$'
12.	$V_{\mathcal{M}}(\Diamond P, x) = 0$.	9, $\wedge E$
13.	So it is not the case that $V_{\mathcal{M}}(\Diamond P, x) = 1$.	12, *bivalence*
14.	So it is not the case that there is some w' such that Rxw' and $V_{\mathcal{M}}(P, w') = 1$.	13, *def.* '$\Diamond$'
15.	So, for all w', if Rxw', then it is not the case that $V_{\mathcal{M}}(P, w') = 1$.	14, *QL*
16.	So, if Rxx, then it is not the case that $V_{\mathcal{M}}(P, x) = 1$.	15, *QL*
17.	Rxx	*def. T-model*
18.	So it is not the case that $V_{\mathcal{M}}(P, x) = 1$.	16, 17, *MP*
19.	But $V_{\mathcal{M}}(P, x) = 1$	11
20.	Our assumption that there is a T-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\sim P \vee \Diamond P), w) = 0$ has led to a contradiction.	18, 19.
21.	Therefore, there is no T-model $< \mathcal{W}, R, \mathscr{I} >$ with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Box(\sim P \vee \Diamond P), w) = 0$.	20, $\sim I$

The following semantic proof shows that $P \models_T \Diamond P$.

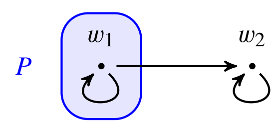
1.	Suppose that there is an arbitrary T-model $< \mathcal{W}, R, \mathscr{I} >$, with some arbitrary $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(P, w) = 1$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(P, w) = 1$.	1, $\wedge E$
3.	Rww	*def. T-model*
4.	So, Rww and $V_{\mathcal{M}}(P, w) = 1$	2, 3, $\wedge I$
5.	So, there is some w' such that Rww' and $V_{\mathcal{M}}(P, w') = 1$	4, *QL*
6.	So, $V_{\mathcal{M}}(\Diamond P, w) = 1$	5, def. '$\Diamond$'
7.	So, if there is an arbitrary T-model $< \mathcal{W}, R, \mathscr{I} >$, with some arbitrary $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(P, w) = 1$, then $V_{\mathcal{M}}(\Diamond P, w) = 1$ as well.	1–6, $\rightarrow I$
8.	So, for any T-model $< \mathcal{W}, R, \mathscr{I} >$ and any $w \in \mathcal{W}$, if $V_{\mathcal{M}}(P, w) = 1$, then $V_{\mathcal{M}}(\Diamond P, w) = 1$ as well.	7, *QL*

5.4. **Establishing Invalidity in T.** In order to establish that an argument from the premises in Γ to the conclusion $\ulcorner \phi \urcorner$ is invalid in T, that is, that $\Gamma \not\models_D \phi$, it is enough to provide a T-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet $\ulcorner \phi \urcorner$ is false at that world. For instance, suppose that we wish to show that

$$\{P\} \not\models_T \Box\Diamond P$$

We may do so with the following T-model:

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2\} \\ R &= \{< w_1, w_1 >, < w_1, w_2 >, < w_2, w_2 >\} \\ \mathscr{I}(P, w_1) &= 1 \\ \mathscr{I}(P, w_2) &= 0 \end{aligned}$$



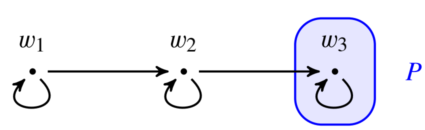
In this model, $V_{\mathcal{M}}(P, w_1) = 1$ and$V_{\mathcal{M}}(\Diamond P, w_2) = 0$, since w_2 only sees itself, and P is false at w_2. Thus, there is some world that w_1 sees at which '$\Diamond P$' is false. So it is not the case that '$\Diamond P$' is true at every world that w_1 sees, so $V_{\mathcal{M}}(\Box\Diamond P, w_1) = 0$. So '$P$' is true at w_1, yet '$\Box\Diamond P$' is false at w_1. So $\{P\} \not\models_T \Box\Diamond P$.

Also in T,

$$\{\Diamond\Diamond P\} \not\models_T \Diamond P$$

Consider the following T-model:

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2, w_3\} \\ R &= \{< w_1, w_1 >, < w_1, w_2 >, < w_2, w_2 >, \\ &\quad < w_2, w_3 >, < w_3, w_3 >\} \\ \mathscr{I}(P, w_1) &= 0 \\ \mathscr{I}(P, w_2) &= 0 \\ \mathscr{I}(P, w_3) &= 1 \end{aligned}$$



In this model, $V_{\mathcal{M}}(\Diamond P, w_2) = 1$, since w_2 sees w_3, and $V_{\mathcal{M}}(P, w_3) = 1$. Therefore, there is a world that w_1 sees at which '$\Diamond P$' is true. So '$\Diamond\Diamond P$' is true at w_1, $V_{\mathcal{M}}(\Diamond\Diamond P, w_1)$. However,

'P' is not true at any world that w_1 sees, so '$\Diamond P$' is false at w_1, $V_{\mathcal{M}}(\Diamond P, w_1) = 0$. So, at w_1, '$\Diamond\Diamond P$' is true yet '$\Diamond P$' is false.

5.5. **Natural Deduction for T.** To achieve a natural deduction system for T, we may take the natural deduction system for K and add to it the following rules of inference:

$\Box$E

	$\Box\phi$
$\triangleright$	ϕ

This rule says: if you have $\ulcorner\Box\phi\urcorner$ written down on an accessible line, then you may write down $\ulcorner\phi\urcorner$. When you do so, you should write the line that $\ulcorner\Box\phi\urcorner$ appeared on and '$\Box E$'.

$\Diamond$I

	ϕ
$\triangleright$	$\Diamond\phi$

This rule says: if you have $\ulcorner\phi\urcorner$ written down on an accessible line, then you may write down $\ulcorner\Diamond\phi\urcorner$. When you do so, you should write the line that $\ulcorner\phi\urcorner$ appeared on and '$\Diamond I$'. Note that the rule $D\Diamond I$ from the natural deduction system for D does not exist in our new natural deduction system for T. That is because it is now redundant. We can derive '$\Diamond P$' from '$\Box P$' by, first, an application of $\Box E$, and next, an application of $\Diamond I$, like so:

1	$\Box P$	
2	P	1, $\Box E$
3	$\Diamond P$	2, $\Diamond I$

Here is a T-derivation establishing that

$$P \vdash_{TD} \Diamond\Diamond\Diamond\Diamond P$$

1	P	
2	$\Diamond P$	1, $\Diamond I$
3	$\Diamond\Diamond P$	2, $\Diamond I$
4	$\Diamond\Diamond\Diamond P$	3, $\Diamond I$
5	$\Diamond\Diamond\Diamond\Diamond P$	4, $\Diamond I$

Here is a T-derivation establishing that $\vdash_{TD} \Box(\sim P \vee \Diamond P)$.

1	$\Box \mid \; \mid \sim(\sim P \vee \Diamond P)$	$A(\sim E)$
2	$\mid \; \mid \; \mid P$	$A(\sim I)$
3	$\mid \; \mid \; \mid \Diamond P$	$2, \Diamond I$
4	$\mid \; \mid \; \mid \sim P \vee \Diamond P$	$3, \vee I$
5	$\mid \; \mid \; \mid (\sim P \vee \Diamond P) \wedge \sim(\sim P \vee \Diamond P)$	$1, 4, \wedge I$
6	$\mid \; \mid \sim P$	$2\text{–}5, \sim I$
7	$\mid \; \mid \sim P \vee \Diamond P$	$6, \vee I$
8	$\mid \; \mid (\sim P \vee \Diamond P) \wedge \sim(\sim P \vee \Diamond P)$	$1, 7, \wedge I$
9	$\mid \sim P \vee \Diamond P$	$1\text{–}8, \sim E$
10	$\Box(\sim P \vee \Diamond P)$	$1\text{–}9, \Box I$

The following T-derivation establishes that $\vdash_{TD} \Box(\Box(P \to Q) \to (P \to \Diamond Q))$

1	$\Box \mid \; \mid \Box(P \to Q)$	$A(\to I)$
2	$\mid \; \mid \; \mid P$	$A(\to I)$
3	$\mid \; \mid \; \mid P \to Q$	$1, \Box E$
4	$\mid \; \mid \; \mid Q$	$2, 3, \to E$
5	$\mid \; \mid \; \mid \Diamond Q$	$4, \Diamond I$
6	$\mid \; \mid P \to \Diamond Q$	$2\text{–}5, \to I$
7	$\mid \Box(P \to Q) \to (P \to \Diamond Q)$	$1\text{–}6, \to I$
8	$\Box(\Box(P \to Q) \to (P \to \Diamond Q))$	$1\text{–}7, \Box I$

Here is one showing that $\{\Box P\} \vdash_{TD} \Diamond\Box\Diamond P$:

1	$\mid \Box P$	
2	$\mid \Box \mid P$	$1, \Box R$
3	$\mid \; \mid \Diamond P$	$2, \Diamond I$
4	$\mid \Box\Diamond P$	$2\text{–}3, \Box I$
5	$\mid \Diamond\Box\Diamond P$	$4, \Diamond I$

And here is a T-derivation establishing that $\vdash_{TD} (\Box P \to \Diamond Q) \to \Diamond(P \to Q)$.

1	$\Box P \rightarrow \Diamond Q$	$A(\rightarrow I)$
2	$\sim\Diamond(P \rightarrow Q)$	$A(\sim E)$
3	$\Box\sim(P \rightarrow Q)$	2, MN
4	$\Box$ $\sim(P \rightarrow Q)$	3, $\Box R$
5	$\sim P$	$A(\sim E)$
6	P	$A(\rightarrow I)$
7	$\sim Q$	$A(\sim E)$
8	$P \wedge \sim P$	5, 6, $\wedge I$
9	Q	7–8, $\sim E$
10	$P \rightarrow Q$	6–9, $\rightarrow I$
11	$(P \rightarrow Q) \wedge \sim(P \rightarrow Q)$	4, 10, $\wedge I$
12	P	5–11, $\sim E$
13	$\Box P$	4–12, $\Box I$
14	$\Diamond Q$	1, 13, $\rightarrow E$
15	$\Diamond$ Q	$A(\Diamond E)$
16	P	$A(\rightarrow I)$
17	Q	15, R
18	$P \rightarrow Q$	16–17, $\rightarrow I$
19	$\Diamond(P \rightarrow Q)$	14, 15–18, $\Diamond E$
20	$\Diamond(P \rightarrow Q) \wedge \sim\Diamond(P \rightarrow Q)$	2, 19, $\wedge I$
21	$\Diamond(P \rightarrow Q)$	2-20, $\sim E$
22	$(\Box P \rightarrow \Diamond Q) \rightarrow \Diamond(P \rightarrow Q)$	1–21, $\rightarrow I$

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,

$$\Gamma \vdash_{TD} \phi \quad \text{if and only if} \quad \Gamma \models_{T} \phi$$

6. The System B

Suppose we add to our axiomatic system T the following axiom

$$B \ : \ P \rightarrow \Box\Diamond P$$

or, equivalently,

$$B' \ : \ \Diamond\Box P \rightarrow P$$

This gives us the axiomatic system B. B will have the following axioms:

$$\vdash_B \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \quad (PLT)$$
$$\vdash_B \Box(P \rightarrow Q) \rightarrow (\Box P \rightarrow \Box Q) \quad (K)$$
$$\vdash_B \Box P \rightarrow P \quad (T)$$
$$\vdash_B P \rightarrow \Box\Diamond P \quad (B)$$

And the following rules of inference:

$$\text{all valid } PL \text{ inferences} \quad (PLR)$$
$$\text{from } \vdash_B \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_B \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N] \quad (US)$$
$$\text{from } \vdash_B \phi, \text{ infer } \vdash_B \Box\phi \quad (N)$$

Since all the axioms and rules of inference of K are axioms and rules of inference of B, we retain all the derived rules from §3.

I said above that adding '$\Diamond\Box P \rightarrow P$' as an axiom was equivalent to adding '$P \rightarrow \Box\Diamond P$' as an axiom. That's because, taking '$P \rightarrow \Box\Diamond P$' as an axiom, we can derive '$\Diamond\Box P \rightarrow P$':

1.	$\vdash_B P \rightarrow \Box\Diamond P$	(B)
2.	$\vdash_B {\sim}P \rightarrow \Box\Diamond{\sim}P$	1 (US)
3.	$\vdash_B {\sim}P \rightarrow \Box{\sim}\Box P$	2 (MN)
4.	$\vdash_B {\sim}P \rightarrow {\sim}\Diamond\Box P$	3 (MN)
5.	$\vdash_B \Diamond\Box P \rightarrow P$	4 (PLR)

And we can derive '$P \rightarrow \Box\Diamond P$' as a theorem if we take '$\Diamond\Box P \rightarrow P$' as an axiom, as follows:

1.	$\vdash_{B'} \Diamond\Box P \rightarrow P$	(B')
2.	$\vdash_{B'} \Diamond\Box{\sim}P \rightarrow {\sim}P$	1 (US)
3.	$\vdash_{B'} \Diamond{\sim}\Diamond P \rightarrow {\sim}P$	2 (MN)
4.	$\vdash_{B'} {\sim}\Box\Diamond P \rightarrow {\sim}P$	3 (MN)
5.	$\vdash_{B'} P \rightarrow \Box\Diamond P$	4 (PLR)

6.1. **Semantics for B.** If we require that the accessibility relation R be *reflexive* and *symmetric*—that is, that every world sees itself and every world sees all the worlds that see it—then we get a semantics for the system B.

A *B-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *symmetric* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Recall,

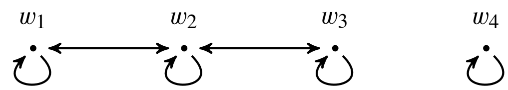


FIGURE 5. A *B-frame* consisting of the worlds $w_1, w_2,$, w_3, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_2w_2, Rw_2w_1, Rw_2w_3, Rw_3w_3, Rw_3w_2, and Rw_4w_4.

REFLEXIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is REFLEXIVE iff, for all $a \in \mathbf{A}$, Raa.

$$\forall a\ Raa$$

SYMMETRIC:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is SYMMETRIC iff, for all $a, b \in \mathbf{A}$, if Rab, then Rba.

$$\forall a \forall b\ (Rab \rightarrow Rba)$$

A sample B-frame is shown in figure 5.

A *B-model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of a B-frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of PML and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

Given a B-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in precisely the same way we did for K-models, D-models, and T-models. That is: for every world $w \in \mathcal{W}$, every atomic wff $\ulcorner \alpha \urcorner$, and every pair of wffs $\ulcorner \phi \urcorner$, $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.
(2) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.
(3) $V_{\mathcal{M}}(\phi \rightarrow \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.
(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every w', if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

6.2. ***B*-Consequence.** We will say that $\ulcorner \phi \urcorner$ is a B-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is B-valid,

$$\Gamma \models_B \phi$$

iff there is no B-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every B-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

And we will say that a wff $\ulcorner \phi \urcorner$ is a *B-tautology*, or *B-valid*,

$$\models_B \phi$$

if and only if $\ulcorner \phi \urcorner$ is true at every world in every B model.

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff ⌜ϕ⌝,

$$\Gamma \vdash_B \phi \quad \text{if and only if} \quad \Gamma \models_B \phi$$

6.3. **Establishing Validity in *B*.** If we wish to show that an argument from the premises in Γ to the conclusion ⌜ϕ⌝ is *B*-valid, then we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\models_B \Diamond\Box P \rightarrow P$$

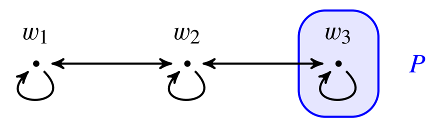
1.	Suppose that there is a *B*-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Diamond\Box P \rightarrow P, w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Diamond\Box P \rightarrow P, w) = 0$.	1, $\wedge E$
3.	So $V_{\mathcal{M}}(\Diamond\Box P, w) = 1$ and $V_{\mathcal{M}}(P, w) = 0$	2, *def.* '$\rightarrow$'
4.	So $V_{\mathcal{M}}(\Diamond\Box P, w) = 1$.	3, $\wedge E$
5.	So there is some w' such that Rww' and $V_{\mathcal{M}}(\Box P, w') = 1$—call it '$x$'.	4, *def.* '$\Diamond$'
6.	So Rwx and $V_{\mathcal{M}}(\Box P, x) = 1$.	5
7.	So $V_{\mathcal{M}}(\Box P, x) = 1$.	6, $\wedge E$
8.	So, for all w', if Rxw', then $V_{\mathcal{M}}(P, w') = 1$.	7, *def.* '$\Box$'
9.	So, if Rxw, then $V_{\mathcal{M}}(P, w) = 1$	8, $\forall I$
10.	Rwx	6, $\wedge E$
11.	If Rwx, then Rxw	*def. B-model*
12.	So Rxw	10, 11, *MP*
13.	So $V_{\mathcal{M}}(P, w) = 1$	9, 12, *MP*
14.	But $V_{\mathcal{M}}(P, w) = 0$	3, $\wedge E$
15.	So it is not the case that $V_{\mathcal{M}}(P, w) = 1$	14, *bivalence*
16.	Our assumption that there is a *B*-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Diamond\Box P \rightarrow P, w) = 0$ has led to a contradiction.	13, 15
17.	So there is no *B*-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\Diamond\Box P \rightarrow P, w) = 0$.	16, $\sim I$

6.4. **Establishing Invalidity in *B*.** In order to establish that an argument from the premises in Γ to the conclusion ⌜ϕ⌝ is invalid in *B*, that is, that $\Gamma \not\models_D \phi$, it is enough to provide a *B*-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet ⌜ϕ⌝ is false at that world. For instance, suppose that we wish to show that

$$\{\Diamond P\} \not\models_B \Box\Diamond P$$

We may do so with the following B-model:

$$\begin{aligned}
\mathcal{W} &= \{w_1, w_2, w_3\} \\
R &= \{< w_1, w_1 >, < w_2, w_2 >, < w_3, w_3 >, \\
&\quad < w_1, w_2 >, < w_2, w_1 >, < w_2, w_3 >, < w_3, w_2 >\} \\
\mathscr{I}(P, w_1) &= 0 \\
\mathscr{I}(P, w_2) &= 0 \\
\mathscr{I}(P, w_3) &= 1
\end{aligned}$$



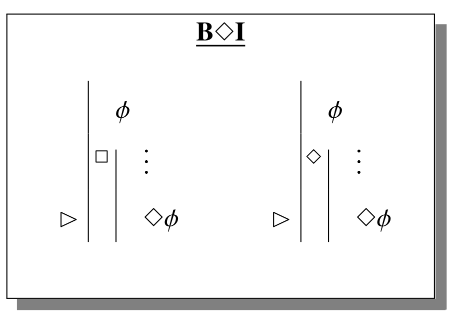
In this model, $V_{\mathcal{M}}(P, w_3) = 1$ and w_2 sees w_3, so $V_{\mathcal{M}}(\Diamond P, w_2) = 1$. So '$\Diamond P$' is true at w_2. However, 'P' is false at w_1 and w_2—$V_{\mathcal{M}}(P, w_1) = 0$ and $V_{\mathcal{M}}(P, w_2) = 0$—and w_1 and w_2 are the only worlds that w_1 sees. So '$\Diamond P$' is false at w_1—$V_{\mathcal{M}}(\Diamond P, w_1) = 0$. Therefore, there is some world that w_2 sees at which '$\Diamond P$' is false. So '$\Box\Diamond P$' is false at w_2—$V_{\mathcal{M}}(\Box\Diamond P, w_2) = 0$. So w_2 is a world at which '$\Diamond P$' is true, yet '$\Box\Diamond P$' is false.

The same model shows that

$$\{\Diamond\Diamond P\} \not\models_{B} \Diamond P$$

For '$\Diamond P$' is true at w_2, and w_1 sees w_2, so '$\Diamond\Diamond P$' is true at w_1. However, 'P' is not true at any world that w_1 sees, so '$\Diamond P$' is false at w_1. So w_1 is a world at which '$\Diamond\Diamond P$' is true, yet '$\Diamond P$' is false.

6.5. **Natural Deduction for B.** To achieve a natural deduction system for B, we may take our natural deduction system for T and add to it a single new rule of inference:



This rule says: if you have a wff $\ulcorner\phi\urcorner$ written down outside of the scope of a strict subproof, then you may write down $\ulcorner\Diamond\phi\urcorner$ within the scope of that strict subproof.

Here is a B-derivation establishing that the B-axiom is a theorem in this derivation system.

$$\vdash_{BD} P \to \Box\Diamond P$$

1	P	$A(\to I)$
2	$\Box$ $\Diamond P$	1, $B\Diamond I$
3	$\Box\Diamond P$	2–2, $\Box I$
4	$P \to \Box\Diamond P$	1–3, $\to I$

Here is a B-derivation establishing that $\vdash_{BD} \Diamond\Box P \to P$.

1	$\Diamond\Box P$	$A(\to I)$
2	$\sim P$	$A(\sim E)$
3	$\Box$ $\Diamond\sim P$	2, $B\Diamond I$
4	$\sim\Box P$	3, MN
5	$\Box\sim\Box P$	3–4, $\Box I$
6	$\sim\Diamond\Box P$	5, MN
7	$\Diamond\Box P \wedge \sim\Diamond\Box P$	1, 6, $\wedge I$
8	P	2–7, $\sim E$
9	$\Diamond\Box P \to P$	1–8, $\to I$

And here is one showing that $\{\Diamond P, \Box Q\} \vdash_{BD} \Box(\Diamond\Diamond P \wedge Q)$.

1	$\Diamond P$	
2	$\Box Q$	
3	$\Box$ Q	2, $\Box R$
4	$\Diamond\Diamond P$	1, $B\Diamond I$
5	$\Diamond\Diamond P \wedge Q$	3, 4, $\wedge I$
6	$\Box(\Diamond\Diamond P \wedge Q)$	3–5, $\Box I$

7. The System S4

Suppose we add to our axiomatic system T the following axiom

$$\text{S4} : \Box P \to \Box\Box P$$

or, equivalently,

$$\text{S4}' : \Diamond\Diamond P \to \Diamond P$$

This gives us the axiomatic system S4. S4 will have the following axioms:

$$\vdash_{S4} \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \quad (PL)$$

$$\vdash_{S4} \Box(P \to Q) \to (\Box P \to \Box Q) \quad (K)$$

$$\vdash_{S4} \Box P \to P \quad (T)$$

$$\vdash_{S4} \Box P \to \Box\Box P \quad (\text{S4})$$

And the following rules of inference:

$$\text{all } PL\text{-valid inferences} \quad (PLR)$$

$$\text{from } \vdash_{S4} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_{S4} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N] \quad (US)$$

$$\text{from } \vdash_{S4} \phi, \text{ infer } \vdash_{S4} \Box\phi \quad (N)$$

Since all the axioms and rules of inference of K are axioms and rules of inference of S4, we retain all the derived rules from §3.

Notice that we *did not* include the axiom (B). Nor is (B) a theorem of this system. For each of the previous axiomatic systems, we have been *enlarging* the number of theorems. That is, for any $\ulcorner\phi\urcorner$ of PML, if $\ulcorner\phi\urcorner$ is a theorem of K, then it is a theorem of D; if $\ulcorner\phi\urcorner$ is a theorem of D, then it is a theorem of T; and if $\ulcorner\phi\urcorner$ is a theorem of T, then it is a theorem of B.

$$\vdash_K \phi \;\Rightarrow\; \vdash_D \phi \;\Rightarrow\; \vdash_T \phi \;\Rightarrow\; \vdash_B \phi$$

This is not true of B and S4. Not every theorem of B is a theorem of S4,

$$\vdash_B \phi \;\not\Rightarrow\; \vdash_{S4} \phi$$

and not every theorem of S4 is a theorem of B

$$\vdash_{S4} \phi \;\not\Rightarrow\; \vdash_B \phi$$

Nevertheless, if $\ulcorner\phi\urcorner$ is a theorem of T, then it is a theorem of S4:

$$\vdash_K \phi \;\Rightarrow\; \vdash_D \phi \;\Rightarrow\; \vdash_T \phi \;\Rightarrow\; \vdash_{S4} \phi$$

I said above that adding '$\Diamond\Diamond P \to \Diamond P$' as an axiom was equivalent to adding '$\Box P \to \Box\Box P$' as an axiom. That's because, given this axiomatic framework, we can derive '$\Diamond\Diamond P \to \Diamond P$' as a theorem if we take '$\Box P \to \Box\Box P$' as an axiom, as follows:

1.	$\vdash_{S4} \Box P \to \Box\Box P$	(S4)
2.	$\vdash_{S4} \Box{\sim}P \to \Box\Box{\sim}P$	1 (US)
3.	$\vdash_{S4} {\sim}\Diamond P \to \Box{\sim}\Diamond P$	2 (MN)
4.	$\vdash_{S4} {\sim}\Diamond P \to {\sim}\Diamond\Diamond P$	3 (MN)
5.	$\vdash_{S4} \Diamond\Diamond P \to \Diamond P$	4 (PLR)

And we can derive '$\Box P \to \Box\Box P$' as a theorem if we take '$\Diamond\Diamond P \to \Diamond P$' as an axiom:

1.	$\vdash_{S4'} \Diamond\Diamond P \rightarrow \Diamond P$	(S4′)
2.	$\vdash_{S4'} \Diamond\Diamond{\sim}P \rightarrow \Diamond{\sim}P$	1 (*US*)
3.	$\vdash_{S4'} \Diamond{\sim}\Box P \rightarrow {\sim}\Box P$	2 (*MN*)
4.	$\vdash_{S4'} {\sim}\Box\Box P \rightarrow {\sim}\Box P$	3 (*MN*)
5.	$\vdash_{S4'} \Box P \rightarrow \Box\Box P$	4 (*PLR*)

With the axiomatic system S4, the addition of extra modal operators of the same kind makes no difference. That is: if you have a wff of the form $\ulcorner\Diamond\Diamond\cdots\Diamond\phi\urcorner$, that wff is equivalent to one of the form $\ulcorner\Diamond\phi\urcorner$. And if you have a wff of the form $\ulcorner\Box\Box\cdots\Box\phi\urcorner$, that wff is equivalent to one of the form $\ulcorner\Box\phi\urcorner$. That is: for any wff of *PML* $\ulcorner\phi\urcorner$,

$$\vdash_{S4} \Diamond\Diamond\cdots\Diamond\phi \leftrightarrow \Diamond\phi$$

and

$$\vdash_{S4} \Box\Box\cdots\Box\phi \leftrightarrow \Box\phi$$

To see this, note that, for any wff $\ulcorner\phi\urcorner$, the following are theorems of S4:

$$\vdash_{S4} \Box\Box\phi \leftrightarrow \Box\phi \qquad (\Box C)$$

$$\vdash_{S4} \Diamond\Diamond\phi \leftrightarrow \Diamond\phi \qquad (\Diamond C)$$

(I've named these theorem schemata '($\Box C$)' and '($\Diamond C$)' for 'box collapse' and 'diamond collapse', respectively.) Here are the proofs of these schemata (for any $\ulcorner\phi\urcorner$):

1.	$\vdash_{S4} \Box P \rightarrow \Box\Box P$	(S4)
2.	$\vdash_{S4} \Box\phi \rightarrow \Box\Box\phi$	1 (*US*)
3.	$\vdash_{S4} \Box P \rightarrow P$	(*T*)
4.	$\vdash_{S4} \Box\Box\phi \rightarrow \Box\phi$	3 (*US*)
5.	$\vdash_{S4} \Box\Box\phi \leftrightarrow \Box\phi$	2, 4 (*PLR*)

1.	$\vdash_{S4} \Diamond\Diamond P \rightarrow \Diamond P$	(S4′)
2.	$\vdash_{S4} \Diamond\Diamond\phi \rightarrow \Diamond\phi$	1 (*US*)
3.	$\vdash_{S4} P \rightarrow \Diamond P$	(*T′*)
4.	$\vdash_{S4} \Diamond\phi \rightarrow \Diamond\Diamond\phi$	3 (*US*)
5.	$\vdash_{S4} \Diamond\Diamond\phi \leftrightarrow \Diamond\phi$	2, 4 (*PLR*)

To see how this allows us to show that any wff of the form $\ulcorner\Diamond\Diamond\cdots\Diamond\phi\urcorner$ is equivalent to one of the form $\ulcorner\Diamond\phi\urcorner$, let's walk through an example (hopefully this will make it clear how to provide a more rigorous proof by mathematical induction, but I won't bother to provide it here): Using instances of the theorem schema ($\Diamond C$), we may show that '$\Diamond\Diamond\Diamond\Diamond P$' is equivalent to '$\Diamond P$':

1.	$\vdash_{S4} \Diamond\Diamond P \leftrightarrow \Diamond P$	($\Diamond C$)
2.	$\vdash_{S4} \Diamond\Diamond\Diamond P \leftrightarrow \Diamond P$	1 (*SE*)
3.	$\vdash_{S4} \Diamond\Diamond\Diamond\Diamond P \leftrightarrow \Diamond P$	1, 2 (*SE*)

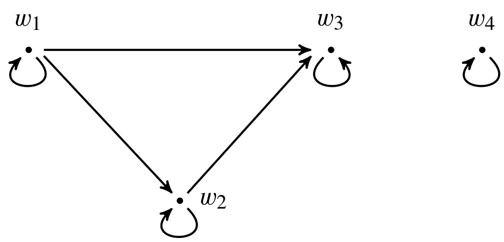


FIGURE 6. An *S4-frame* consisting of the worlds $w_1, w_2,, w_3$, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_2w_2, Rw_2w_3, Rw_1w_3, Rw_3w_3, and Rw_4w_4.

In line 2 of this proof, I applied the rule (*SE*) to line 1 of the proof by replacing the occurrence of '$\Diamond P$' on the left-hand-side with '$\Diamond\Diamond P$'. This is allowed by (*SE*) since line 1 tells us that '$\Diamond P$' and '$\Diamond\Diamond P$' are equivalent.

7.1. **Semantics for S4.** If we require that the accessibility relation R be *reflexive* and *transitive*—that is, that every world sees itself and every world sees the worlds seen by the worlds it sees—then we get a semantics for the system S4.

An *S4-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *transitive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Recall,

REFLEXIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is REFLEXIVE iff, for all $a \in \mathbf{A}$, Raa.

$$\forall a \; Raa$$

TRANSITIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is TRANSITIVE iff, for all $a, b, c \in \mathbf{A}$, if Rab and Rbc, then Rac.

$$\forall a \, \forall b \, \forall c \; ((Rab \wedge Rbc) \rightarrow Rac)$$

A sample S4-frame is shown in figure 7.

An *S4-model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of an S4-frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of *PML* and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

Given an S4-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in the usual way: for every world $w \in \mathcal{W}$, every atomic wff $\ulcorner \alpha \urcorner$, and every pair of wffs $\ulcorner \phi \urcorner, \ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.
(2) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.
(3) $V_{\mathcal{M}}(\phi \rightarrow \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.
(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every w', if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

7.2. **S4-Consequence.** We will say that $\ulcorner\phi\urcorner$ is an S4-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner\phi\urcorner$ is S4-valid,

$$\Gamma \models_{S4} \phi$$

iff there is no S4-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every S4-model at which all the premises in Γ are true, $\ulcorner\phi\urcorner$ is true as well.

And we will say that a wff $\ulcorner\phi\urcorner$ is an *S4-tautology*, or *S4-valid*,

$$\models_{S4} \phi$$

if and only if $\ulcorner\phi\urcorner$ is true at every world in every S4 model.

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff $\ulcorner\phi\urcorner$,

$$\Gamma \vdash_{S4} \phi \quad \text{if and only if} \quad \Gamma \models_{S4} \phi$$

7.3. **Establishing Validity in S4.** If we wish to show that an argument from the premises in Γ to the conclusion $\ulcorner\phi\urcorner$ is S4-valid, then we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\models_{S4} \Diamond\Diamond P \rightarrow \Diamond P$$

1.	Suppose that there is an S4 model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Diamond P \rightarrow \Diamond P, w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Diamond\Diamond P \rightarrow \Diamond P, w) = 0$	1, $\wedge E$
3.	So $V_{\mathcal{M}}(\Diamond\Diamond P, w) = 1$ and $V_{\mathcal{M}}(\Diamond P, w) = 0$	2, *def.* $\rightarrow$
4.	So $V_{\mathcal{M}}(\Diamond\Diamond P, w) = 1$.	3, $\wedge E$
5.	So there is some w' such that Rww' and $V_{\mathcal{M}}(\Diamond P, w') = 1$—call it '$x$'	4, *def.* $\Diamond$
6.	So Rwx and $V_{\mathcal{M}}(\Diamond P, x) = 1$.	5
7.	So $V_{\mathcal{M}}(\Diamond P, x) = 1$.	6, $\wedge E$
8.	So there is some w' such that Rxw' and $V_{\mathcal{M}}(P, w') = 1$—call it '$y$'	7, *def.* $\Diamond$
9.	So Rxy and $V_{\mathcal{M}}(P, y) = 1$.	8
10.	Rwx	6 $\wedge E$
11.	Rxy	9 $\wedge E$
12.	So Rwx and Rwy	10, 11 $\wedge I$
13.	If Rwx and Rxy, then Rwy.	*def. S4-model*
14.	So Rwy	12, 13 *MP*
15.	And $V_{\mathcal{M}}(P, y) = 1$.	9 $\wedge E$
16.	So Rwy and $V_{\mathcal{M}}(P, y) = 1$.	14, 15 $\wedge I$
17.	So there is some w' such that Rww' and $V_{\mathcal{M}}(P, w') = 1$.	16 $\exists G$
18.	So $V_{\mathcal{M}}(\Diamond P, w) = 1$.	17, *def.* $\Diamond$
19.	But $V_{\mathcal{M}}(\Diamond P, w) = 0$.	3, $\wedge E$

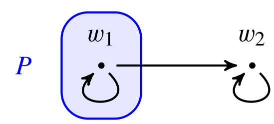
20.	So it is not the case that $V_{\mathcal{M}}(\Diamond P, w) = 1$.	19, *bivalence*
21.	Our assumption that there is an S4 model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Diamond P \to \Diamond P, w) = 0$ has led to a contradiction.	18, 20
22.	So there is no S4 model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Diamond P \to \Diamond P, w) = 0$.	21, ~*I*

7.4. **Establishing Invalidity in S4.** In order to establish that an argument from the premises in Γ to the conclusion $\ulcorner \phi \urcorner$ is invalid in S4, that is, that $\Gamma \not\models_{S4} \phi$, it is enough to provide an S4-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet $\ulcorner \phi \urcorner$ is false at that world. For instance, suppose that we wish to show that

$$\{P\} \not\models_{S4} \Box\Diamond P$$

We may do so with the following S4-model (which is just the *T*-model from §5.4):

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2\} \\ R &= \{< w_1, w_1 >, < w_1, w_2 >, < w_2, w_2 >\} \\ \mathscr{I}(P, w_1) &= 1 \\ \mathscr{I}(P, w_2) &= 0 \end{aligned}$$



'P' is true at w_1 in this model. However, since w_2 sees only itself, and 'P' is false at w_2, '$\Diamond P$' is false at w_2. And w_1 sees w_2, so w_1 sees a world at which '$\Diamond P$' is false. So '$\Diamond P$' is not true at every world that w_1 sees. So '$\Box\Diamond P$' is false at w_1. So the premise of our argument is true at w_1, yet the conclusion is false at w_1. This is an S4-model, so the argument is S4-invalid.

The same model shows that

$$\{\Diamond P\} \not\models_{S4} \Box\Diamond P$$

For '$\Diamond P$' is true at w_1, since w_1 sees itself. However, '$\Box\Diamond P$' is still false at w_1, since it sees the world w_2, where '$\Diamond P$' is false.

7.5. **Natural Deduction for S4.** To get a natural deduction system for S4, we will take our natural deduction system for *T* and add to it a single rule of inference. (Note: we *are not* free to use the rule $B\Diamond I$ from the natural deduction system for *B*—we are adding S4*R* to the natural deduction system for *T*; and *not* the natural deduction system for *B*).

S4R

$\Box\phi$	$\Box\phi$
$\Box$ | $\vdots$	$\Diamond$ | $\vdots$
$\triangleright$ | $\Box\phi$	$\triangleright$ | $\Box\phi$

This rule says: if you have a wff of the form $\ulcorner\Box\phi\urcorner$ outside of the scope of a strict subproof, then you may reiterate $\ulcorner\Box\phi\urcorner$ within that strict subproof (whether it is a box or a diamond strict subproof).

Here is an S4-derivation showing that the S4 axiom '$\Box P \rightarrow \Box\Box P$' is a theorem of this natural deduction system.

1	$\Box P$	$A(\rightarrow I)$
2	$\Box$ $\Box P$	1, S4R
3	$\Box\Box P$	2–2, $\Box I$
4	$\Box P \rightarrow \Box\Box P$	1–3, $\rightarrow I$

So too is '$\Diamond\Diamond P \rightarrow \Diamond P$':

1	$\Diamond\Diamond P$	$A(\rightarrow I)$
2	$\sim\Diamond P$	$A(\sim E)$
3	$\Box\sim P$	2, MN
4	$\Diamond$ $\Diamond P$	$A(\Diamond E)$
5	$\Box\sim P$	3, S4R
6	$\sim\Diamond P$	5, MN
7	$\sim P$	$A(\sim E)$
8	$\Diamond P \wedge \sim\Diamond P$	4, 6, $\wedge I$
9	P	7–8, $\sim E$
10	$\Diamond P$	1, 4–9, $\Diamond E$
11	$\Diamond P \wedge \sim\Diamond P$	2, 10, $\wedge I$
12	$\Diamond P$	2–11, $\sim E$
13	$\Diamond\Diamond P \rightarrow \Diamond P$	1–12, $\rightarrow I$

We may also show that $\vdash_{S4D} \Box\Diamond P \rightarrow \Box\Diamond\Box\Diamond P$:

1	$\Box\Diamond P$	$A(\rightarrow I)$
2	$\Box$ $\Box\Diamond P$	1, S4R
3	$\Diamond\Box\Diamond P$	2, $\Diamond I$
4	$\Box\Diamond\Box\Diamond P$	2–3, $\Box I$
5	$\Box\Diamond P \rightarrow \Box\Diamond\Box\Diamond P$	1–4, $\rightarrow I$

Here's an S4-derivation to show that $\vdash_{S4D} \Box(\Box P \to Q) \to \Box(\Box P \to \Box Q)$:

1	$\mid\ \Box(\Box P \to Q)$	$A(\to I)$
2	$\mid\ \Box\mid\ \Box(\Box P \to Q)$	1, S4R
3	$\mid\ \mid\ \mid\ \Box P$	$A(\to I)$
4	$\mid\ \mid\ \mid\ \Box\mid\ \Box(\Box P \to Q)$	2, S4R
5	$\mid\ \mid\ \mid\ \mid\ \Box P \to Q$	4, $\Box E$
6	$\mid\ \mid\ \mid\ \mid\ \Box P$	3, S4R
7	$\mid\ \mid\ \mid\ \mid\ Q$	5, 6, $\to E$
8	$\mid\ \mid\ \mid\ \Box Q$	4–7, $\Box I$
9	$\mid\ \mid\ \Box P \to \Box Q$	3–8, $\to I$
10	$\mid\ \Box(\Box P \to \Box Q)$	2–9, $\Box I$
11	$\Box(\Box P \to Q) \to \Box(\Box P \to \Box Q)$	1–10, $\to I$

And, finally, here's an S4-derivation to show that $\{P \to \Box Q\} \vdash_{S4D} P \to \Box(R \to \Box Q)$:

1	$\mid\ P \to \Box Q$	
2	$\mid\ \mid\ P$	$A(\to I)$
3	$\mid\ \mid\ \Box Q$	1, 2, $\to E$
4	$\mid\ \mid\ \Box\mid\ \Box Q$	3, S4R
5	$\mid\ \mid\ \mid\ \mid\ R$	$A(\to I)$
6	$\mid\ \mid\ \mid\ \mid\ \Box Q$	4, R
7	$\mid\ \mid\ \mid\ R \to \Box Q$	5–6, $\to I$
8	$\mid\ \mid\ \Box(R \to \Box Q)$	4–7, $\Box I$
9	$\mid\ P \to \Box(R \to \Box Q)$	2–8, $\to I$

8. The System S5

Suppose we add to our axiomatic system T the following axiom

$$\text{S5} : \Diamond P \to \Box \Diamond P$$

or, equivalently,

$$\text{S5}' : \Diamond \Box P \to \Box P$$

This gives us the axiomatic system S5. S5 will have the following axioms:

$$\vdash_{S5} \phi, \text{ for all theorems of } PL, \ulcorner\phi\urcorner \quad (PL)$$
$$\vdash_{S5} \Box(P \to Q) \to (\Box P \to \Box Q) \quad (K)$$
$$\vdash_{S5} \Box P \to P \quad (T)$$
$$\vdash_{S5} \Diamond P \to \Box\Diamond P \quad (\text{S5})$$

And the following rules of inference:

$$\text{all } PL\text{-valid inferences} \quad (PLR)$$
$$\text{from } \vdash_{S5} \phi[\alpha_1, \alpha_2, \ldots, \alpha_N], \text{ infer } \vdash_{S5} \phi[\psi_1/\alpha_1, \psi_2/\alpha_2, \ldots, \psi_N/\alpha_N] \quad (US)$$
$$\text{from } \vdash_{S5} \phi, \text{ infer } \vdash_{S5} \Box\phi \quad (N)$$

Since all the axioms and rules of inference of K are axioms and rules of inference of S5, we retain all the derived rules from §3.

Above, I said that adding the axiom '$\Diamond\Box P \to \Box P$' to T was equivalent to adding '$\Diamond P \to \Box\Diamond P$' to T. That's because we can derive '$\Diamond\Box P \to \Box P$' as a theorem in S5:

1.	$\vdash_{S5} \Diamond P \to \Box\Diamond P$	(S5)
2.	$\vdash_{S5} \Diamond{\sim}P \to \Box\Diamond{\sim}P$	1 (US)
3.	$\vdash_{S5} {\sim}\Box P \to \Box{\sim}\Box P$	2 (MN)
4.	$\vdash_{S5} {\sim}\Box P \to {\sim}\Diamond\Box P$	3 (MN)
5.	$\vdash_{S5} \Diamond\Box P \to \Box P$	4 (PLR)

And, if we take '$\Diamond\Box P \to \Box P$' as an axiom, then we may derive '$\Diamond P \to \Box\Diamond P$' as a theorem:

1.	$\vdash_{S5'} \Diamond\Box P \to \Box P$	(S5′)
2.	$\vdash_{S5'} \Diamond\Box{\sim}P \to \Box{\sim}P$	1 (US)
3.	$\vdash_{S5'} \Diamond{\sim}\Diamond P \to {\sim}\Diamond P$	2 (MN)
4.	$\vdash_{S5'} {\sim}\Box\Diamond P \to {\sim}\Diamond P$	3 (MN)
5.	$\vdash_{S5'} \Diamond P \to \Box\Diamond P$	4 (PLR)

Note that we didn't include either the B axiom or the S4 axiom in the system S5. However, both of these axioms are theorems of S5. Here is a proof showing that the B axiom is a theorem in S5:

1.	$\vdash_{S5} \Box P \to P$	(T)
2.	$\vdash_{S5} \Box{\sim}P \to {\sim}P$	1 (US)
3.	$\vdash_{S5} {\sim}\Diamond P \to {\sim}P$	2 (MN)
4.	$\vdash_{S5} P \to \Diamond P$	3 (PLR)
5.	$\vdash_{S5} \Diamond P \to \Box\Diamond P$	(S5)
6.	$\vdash_{S5} P \to \Box\Diamond P$	4, 5 (PLR)

If we extend the above proof, we may show that the S4 axiom is a theorem in S5:

7.	$\vdash_{S5} \Box P \to \Diamond\Box P$	4 (*US*)
8.	$\vdash_{S5} \Diamond\Box P \to \Box P$	(S5′)
9.	$\vdash_{S5} \Box P \leftrightarrow \Diamond\Box P$	7, 8 (*PLR*)
10.	$\vdash_{S5} \Box P \to \Box\Diamond\Box P$	6 (*US*)
11.	$\vdash_{S5} \Box P \to \Box\Box P$	9, 10 (*SE*)

(On line 8, I appealed to the fact that (S5′) is a theorem in S5, since we already established this above.)

So both (S4) and (B) are redundant axioms for the system S5. In fact, adding the (S5) axiom to the system T is *equivalent* to adding *both* the (S4) axiom *and* the (B) axiom to T. We have already shown that adding (S5) to T brings along both (S4) and (B) as theorems. We will now show that adding to T both the axiom (S4) and the axiom (B) brings along (S5) as a theorem:

1.	$\vdash_{TBS4} \Diamond\Diamond P \to \Diamond P$	(S4)
2.	$\vdash_{TBS4} P \to \Diamond P$	(T')
3.	$\vdash_{TBS4} \Diamond P \to \Diamond\Diamond P$	2 (*US*)
4.	$\vdash_{TBS4} \Diamond P \leftrightarrow \Diamond\Diamond P$	1, 3 (*PLR*)
5.	$\vdash_{TBS4} P \to \Box\Diamond P$	(B)
6.	$\vdash_{TBS4} \Diamond P \to \Box\Diamond\Diamond P$	5 (*US*)
7.	$\vdash_{TBS4} \Diamond P \to \Box\Diamond P$	4, 6 (*SE*)

(I used (T') on line 2 because we have already shown that (T') follows from (T).) This establishes that everything which can be proved in the system S5 may be proved in the system *TB*S4 (the system which results from adding (B) and (S4) to the system T). That's because—as we've just shown—all the axioms of S5 are either axioms or theorems of *TB*S4, and all the rules of inference of S5 are rules of inference of *TB*S4. And we have already shown above that everything which can be proved in the system *TB*S4 may be proved in S5. That's because we showed above that all of the axioms of *TB*S4 are either axioms or theorems of S5, and all the rules of inference of *TB*S4 are rules of inference of S5. So what we've shown is that S5 and *TB*S4 are *equivalent*, in the following sense: for any set of wffs Γ and any wff $\ulcorner\phi\urcorner$,

$$\Gamma \vdash_{TBS4} \phi \quad \text{if and only if} \quad \Gamma \vdash_{S5} \phi$$

In fact, once we have the axioms (B) and (S4), we no longer need the full strength of axiom (T). We could get by with just (D), (B) and (S4). The system which we get by adding to K the axioms (D), (B) and (S4)—call that system '*DB*S4'—is also equivalent to S5. We have already shown that (B) and (S4) are theorems of S5; Since (T) entails (D), (D) will also be a theorem of S5. We will now show that (T'), which is, recall, equivalent to (T), is a theorem of *DB*S4. Since (S5) follows from (T), (B), and (S4), this shows that (S5) is a theorem of *DB*S4.

1. $\vdash_{DBS4} P \to \Box\Diamond P$ (B)
2. $\vdash_{DBS4} \Box P \to \Diamond P$ (D)
3. $\vdash_{DBS4} \Box\Diamond P \to \Diamond\Diamond P$ 2 (US)
4. $\vdash_{DBS4} P \to \Diamond\Diamond P$ 1, 3 (PLR)
5. $\vdash_{DBS4} \Diamond\Diamond P \to \Diamond P$ (S4′)
6. $\vdash_{DBS4} P \to \Diamond P$ 4, 5 (PLR)

(In the proof above, I used (S4′) instead of (S4), since we have already shown that they are equivalent.) Therefore, *DB*S4 is equivalent to S5. That is: for any set of wffs Γ and any wff ⌜ϕ⌝

$$\Gamma \vdash_{DBS4} \phi \quad \text{if and only if} \quad \Gamma \vdash_{S5} \phi$$

With the axiomatic system S5, the addition of extra modal operators beyond the first *makes no difference*, no matter what those modal operators are. That is: if you start with a modalized wff, then no matter how many additional modal operators you tack on, you will always get something that is equivalent to the modalized wff you started out with. Let's use '△' as a metavariable which ranges over modal operators—that is, ⌜△⌝ can refer to either '□' or '◇'. Then, for any wff of PML ⌜ϕ⌝,

$$\vdash_{S5} \triangle\triangle\cdots\triangle\Box\phi \leftrightarrow \Box\phi$$

and

$$\vdash_{S5} \triangle\triangle\cdots\triangle\Diamond\phi \leftrightarrow \Diamond\phi$$

To show this, we will first show that, for any wff ⌜ϕ⌝, the following are theorems of S5:

$$\begin{aligned} &\vdash_{S5} \Box\Diamond\phi \leftrightarrow \Diamond\phi \\ &\vdash_{S5} \Box\Box\phi \leftrightarrow \Box\phi \\ &\vdash_{S5} \Diamond\Diamond\phi \leftrightarrow \Diamond\phi \\ &\vdash_{S5} \Diamond\Box\phi \leftrightarrow \Box\phi \end{aligned} \qquad (OC)$$

(I've named these theorem schemata '(OC)' for 'operator collapse'.) The following proof schemata establish these theorems, for every ⌜ϕ⌝ (the second and third are just the ones we saw from the system S4 above):

1. $\vdash_{S5} \Diamond P \to \Box\Diamond P$ (S5)
2. $\vdash_{S5} \Diamond\phi \to \Box\Diamond\phi$ 1 (US)
3. $\vdash_{S5} \Box P \to P$ (T)
4. $\vdash_{S5} \Box\Diamond\phi \to \Diamond\phi$ 3 (US)
5. $\vdash_{S5} \Box\Diamond\phi \leftrightarrow \Diamond\phi$ 2, 4 (PLR)

1. $\vdash_{S5} \Box P \to \Box\Box P$ (S4)
2. $\vdash_{S5} \Box\phi \to \Box\Box\phi$ 1 (US)
3. $\vdash_{S5} \Box P \to P$ (T)
4. $\vdash_{S5} \Box\Box\phi \to \Box\phi$ 3 (US)
5. $\vdash_{S5} \Box\Box\phi \leftrightarrow \Box\phi$ 2, 4 (PLR)

1.	$\vdash_{S5} \Diamond\Diamond P \to \Diamond P$	(S4′)
2.	$\vdash_{S5} \Diamond\Diamond\phi \to \Diamond\phi$	1 (US)
3.	$\vdash_{S5} P \to \Diamond P$	(T')
4.	$\vdash_{S5} \Diamond\phi \to \Diamond\Diamond\phi$	3 (US)
5.	$\vdash_{S5} \Diamond\Diamond\phi \leftrightarrow \Diamond\phi$	2, 4 (PLR)

1.	$\vdash_{S5} \Diamond\Box P \to \Box P$	(S5′)
2.	$\vdash_{S5} \Diamond\Box\phi \to \Box\phi$	1 (US)
3.	$\vdash_{S5} P \to \Diamond P$	(T')
4.	$\vdash_{S5} \Box\phi \to \Diamond\Box\phi$	3 (US)
5.	$\vdash_{S5} \Diamond\Box\phi \leftrightarrow \Box\phi$	2, 4 (PLR)

To see how this allows us to show that any wff of the form $\ulcorner\triangle\triangle\cdots\triangle\Diamond\phi\urcorner$ is equivalent to one of the form $\ulcorner\Diamond\phi\urcorner$ (and similarly for $\ulcorner\triangle\triangle\cdots\triangle\Box\phi\urcorner$ and $\ulcorner\Box\phi\urcorner$), let's walk through an example (hopefully this will make it clear how to provide a more rigorous proof by mathematical induction, but I won't bother to provide it here): Using instances of the theorem schemata (OC), we may show that '$\Box\Box\Diamond\Diamond\Box\Diamond P$' is equivalent to '$\Diamond P$':

1.	$\vdash_{S5} \Box\Diamond P \leftrightarrow \Diamond P$	(OC)
2.	$\vdash_{S5} \Diamond\Box\Diamond P \leftrightarrow \Box\Diamond P$	(OC)
3.	$\vdash_{S5} \Diamond\Box\Diamond P \leftrightarrow \Diamond P$	1, 2 (SE)
4.	$\vdash_{S5} \Diamond\Diamond\Box\Diamond P \leftrightarrow \Diamond\Box\Diamond P$	(OC)
5.	$\vdash_{S5} \Diamond\Diamond\Box\Diamond P \leftrightarrow \Diamond P$	3, 4 (SE)
6.	$\vdash_{S5} \Box\Diamond\Diamond\Box\Diamond P \leftrightarrow \Diamond\Diamond\Box\Diamond P$	(OC)
7.	$\vdash_{S5} \Box\Diamond\Diamond\Box\Diamond P \leftrightarrow \Diamond P$	5, 6 (SE)
8.	$\vdash_{S5} \Box\Box\Diamond\Diamond\Box\Diamond P \leftrightarrow \Box\Diamond\Diamond\Box\Diamond P$	(OC)
9.	$\vdash_{S5} \Box\Box\Diamond\Diamond\Box\Diamond P \leftrightarrow \Diamond P$	7, 8 (SE)

8.1. **Semantics for S5.** If we require that the accessibility relation R be *reflexive* and *euclidean*—that is, that every world sees itself and all the worlds that it sees see each other—then we get a semantics for the system S5.

An *S5-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive* and *euclidean* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$. Recall,

REFLEXIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is REFLEXIVE iff, for all $a \in \mathbf{A}$, Raa.

$$\forall a \ Raa$$

EUCLIDEAN:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is EUCLIDEAN iff, for all $a, b, c \in \mathbf{A}$, if Rab and Rac, then Rbc.

$$\forall a \, \forall b \, \forall c \ \left((Rab \wedge Rac) \to Rbc\right)$$

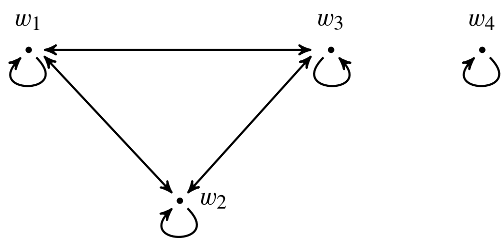


FIGURE 7. An *S5-frame* consisting of the worlds $w_1, w_2,$, w_3, and w_4, and an accessibility relation R such that Rw_1w_1, Rw_1w_2, Rw_1w_3, Rw_2w_1, Rw_2w_2, Rw_2w_3, Rw_3w_1, Rw_3w_2, Rw_3w_3, and Rw_4w_4.

A sample S5-frame is shown in figure 7.

The (S5) axiom imposes the constraint that the accessibility relation be euclidean; the (T) axiom imposes the constraint that it be reflexive. Above, we saw that adding (S5) and (T) to the system K is equivalent to adding (S4), (B), and (T) to K. From the perspective of the semantics, the reason for this is that any binary relation which is reflexive and euclidean is automatically symmetric and transitive as well. Here is a predicate logic derivation establishing that, if a relation R is reflexive and euclidean, then it will be symmetric as well:

1	$\forall x\, Rxx$	
2	$\forall x\, \forall y\, \forall z\, ((Rxy \wedge Rxz) \rightarrow Ryz)$	
3	Rab	$A(\rightarrow I)$
4	Raa	$1\ \forall E[x \rightarrow a]$
5	$(Rab \wedge Raa) \rightarrow Rba$	$2\ \forall E[x \rightarrow a, y \rightarrow b, z \rightarrow a]$
6	$Rab \wedge Raa$	$3, 4\ \wedge I$
7	Rba	$5, 6 \rightarrow E$
8	$Rab \rightarrow Rba$	$3–7, \rightarrow I$
9	$\forall x\, \forall y\, (Rxy \rightarrow Ryx)$	$8\ \forall I[b \rightarrow y, a \rightarrow x]$

And here is a predicate logic derivation establishing that, if a relation is euclidean and symmetric, then it will be transitive as well.

1	$\forall x\, \forall y\, (Rxy \to Ryx)$	
2	$\forall x\, \forall y\, \forall z\, ((Rxy \wedge Rxz) \to Ryz)$	
3	$\quad Rab \wedge Rbc$	$A(\to I)$
4	$\quad Rab \to Rba$	$1\ \forall E[x \to a, y \to b]$
5	$\quad Rab$	$3\ \wedge E$
6	$\quad Rba$	$5, 6 \to E$
7	$\quad (Rba \wedge Rbc) \to Rac$	$2\ \forall E[x \to b, y \to a, z \to c]$
8	$\quad Rbc$	$3\ \wedge E$
9	$\quad Rba \wedge Rbc$	$6, 8\ \wedge I$
10	$\quad Rac$	$7, 9 \to E$
11	$(Rab \wedge Rbc) \to Rac$	$3\text{–}10 \to I$
12	$\forall x\, \forall y\, \forall z((Rxy \wedge Ryz) \to Rxz)$	$11\ \forall I[c \to z, b \to y, a \to x]$

Finally, we will show that, if a binary relation is symmetric and transitive, then it is euclidean, too:

1	$\forall x\, \forall y\, (Rxy \to Ryx)$	
2	$\forall x\, \forall y\, \forall z\, ((Rxy \wedge Ryz) \to Rxz)$	
3	$\quad Rab \wedge Rac$	$A(\to I)$
4	$\quad Rab$	$3\ \wedge E$
5	$\quad Rab \to Rba$	$1\ \forall E[x \to a, y \to b]$
6	$\quad Rba$	$4, 5 \to E$
7	$\quad Rac$	$3\ \wedge E$
8	$\quad Rba \wedge Rac$	$6, 7\ \wedge I$
9	$\quad (Rba \wedge Rac) \to Rbc$	$2\ \forall E[x \to b, y \to a, z \to c]$
10	$\quad Rbc$	$8, 9 \to E$
11	$(Rab \wedge Rac) \to Rbc$	$3\text{–}10 \to I$
12	$\forall x\, \forall y\, \forall z\, ((Rxy \wedge Rxz) \to Ryz)$	$11\ \forall I[c \to z, b \to y, a \to x]$

This shows us that a binary relation is reflexive and euclidean if and only if it is reflexive, symmetric, and transitive. So another, equivalent, definition of an S5-frame is this: An *S5-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *reflexive*, *symmetric*, and *transitive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$.

REFLEXIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is REFLEXIVE iff, for all $a \in \mathbf{A}$, Raa.

$$\forall a \ Raa$$

SYMMETRIC:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is SYMMETRIC iff, for all $a, b \in \mathbf{A}$, if Rab, then Rba.

$$\forall a \, \forall b \ \ (Rab \rightarrow Rba)$$

TRANSITIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is TRANSITIVE iff, for all $a, b, c \in \mathbf{A}$, if Rab and Rbc, then Rac.

$$\forall a \, \forall b \, \forall c \ \ ((Rab \wedge Rbc) \rightarrow Rac)$$

We also saw that the (T) axiom—which corresponds to the reflexivity of the accessibility relation—was not needed in the axiomatic system S5. That is: we saw that, once we had (B) and (S4), we only needed the weaker (D) axiom in order to get an axiomatic system with the full strength of S5. From the standpoint of the semantics, the reason for this is that any binary relation which is serial, symmetric, and transitive is automatically reflexive (and therefore, euclidean). Here's a predicate logic derivation showing that any serial, symmetric, and transitive binary relation is reflexive:

1	$\forall x \, \exists y \, Rxy$	
2	$\forall x \, \forall y \, (Rxy \rightarrow Ryx)$	
3	$\forall x \, \forall y \, \forall z \, ((Rxy \wedge Ryz) \rightarrow Rxz)$	
4	$\exists y Ray$	1 $\forall E[x \rightarrow a]$
5	Rab	4 $\exists E[y \rightarrow b]$
6	$Rab \rightarrow Rba$	2 $\forall E[x \rightarrow a, y \rightarrow b]$
7	Rba	5, 6 $\rightarrow E$
8	$(Rab \wedge Rba) \rightarrow Raa$	3 $\forall E[x \rightarrow a, y \rightarrow b, z \rightarrow a]$
9	$Rab \wedge Rba$	5, 7 $\wedge I$
10	Raa	8, 9 $\rightarrow E$
11	$\forall x \, Rxx$	10 $\forall I[a \rightarrow x]$

And since we have already shown that a reflexive, symmetric, and transitive relation is euclidean (and since every reflexive relation is serial), this means that another, equivalent, definition of an S5-frame is this: An *S5-frame* is a pair $< \mathcal{W}, R >$ of a set of worlds $\mathcal{W}$ and a *serial*, *symmetric*, and *transitive* binary relation $R \subseteq \mathcal{W} \times \mathcal{W}$.

SERIAL:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is SERIAL iff, for all $a \in \mathbf{A}$, there is some $b \in \mathbf{A}$ such that Rab.

$$\forall a \, \exists b \;\; Rab$$

SYMMETRIC:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is SYMMETRIC iff, for all $a, b \in \mathbf{A}$, if Rab, then Rba.

$$\forall a \, \forall b \;\; (Rab \rightarrow Rba)$$

TRANSITIVE:
A binary relation $R \subseteq \mathbf{A} \times \mathbf{A}$ is TRANSITIVE iff, for all $a, b, c \in \mathbf{A}$, if Rab and Rbc, then Rac.

$$\forall a \, \forall b \, \forall c \;\; ((Rab \wedge Rbc) \rightarrow Rac)$$

An *S5-model* is a triple $< \mathcal{W}, R, \mathscr{I} >$ consisting of an S5-frame $< \mathcal{W}, R >$ and an interpretation function $\mathscr{I}$, from pairs of atomic wffs of PML and worlds $w \in \mathcal{W}$ to $\{1, 0\}$.

Given an S5-model $< \mathcal{W}, R, \mathscr{I} >$, we construct a *valuation* function $V_{\mathcal{M}}$ in the usual way. That is: for every world $w \in \mathcal{W}$, every atomic wff $\ulcorner \alpha \urcorner$, and every pair of wffs $\ulcorner \phi \urcorner$, $\ulcorner \psi \urcorner$,

(1) $V_{\mathcal{M}}(\alpha, w) = 1$ iff $\mathscr{I}(\alpha, w) = 1$.
(2) $V_{\mathcal{M}}(\sim\phi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$.
(3) $V_{\mathcal{M}}(\phi \rightarrow \psi, w) = 1$ iff $V_{\mathcal{M}}(\phi, w) = 0$ or $V_{\mathcal{M}}(\psi, w) = 1$.
(4) $V_{\mathcal{M}}(\Box\phi, w) = 1$ iff, for every w', if Rww', then $V_{\mathcal{M}}(\phi, w') = 1$.

8.2. **S5-Consequence.** We will say that $\ulcorner \phi \urcorner$ is an S5-consequence of a set of wffs Γ, or that the argument from Γ to $\ulcorner \phi \urcorner$ is S5-valid,

$$\Gamma \models_{S5} \phi$$

iff there is no S5-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$, such that $V_{\mathcal{M}}(\gamma, w) = 1$ for every $\gamma \in \Gamma$, yet $V_{\mathcal{M}}(\phi, w) = 0$. Or, equivalently: iff for every world in every S5-model at which all the premises in Γ are true, $\ulcorner \phi \urcorner$ is true as well.

And we will say that a wff $\ulcorner \phi \urcorner$ is an *S5-tautology*, or *S5-valid*,

$$\models_{S5} \phi$$

if and only if $\ulcorner \phi \urcorner$ is true at every world in every S5 model.

Interesting and Unexpected and Fantastic Fact: for every set of wffs Γ and every wff $\ulcorner \phi \urcorner$,

$$\Gamma \vdash_{S5} \phi \quad \text{if and only if} \quad \Gamma \models_{S5} \phi$$

8.3. **Establishing Validity in S5.** If we wish to show that an argument from the premises in Γ to the conclusion $\ulcorner \phi \urcorner$ is S5-valid, then we may provide a semantic proof. Suppose, for instance, that we wish to show that

$$\models_{S5} \Diamond\Box P \to \Box P$$

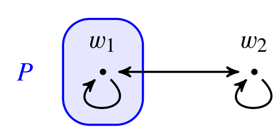
1.	Suppose that there is an S5-model $< \mathcal{W}, R, \mathscr{I} >$, with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Box P \to \Box P, w) = 0$.	*Assumption*
2.	Then, $V_{\mathcal{M}}(\Diamond\Box P \to \Box P, w) = 0$	1
3.	So $V_{\mathcal{M}}(\Diamond\Box P, w) = 1$ and $V_{\mathcal{M}}(\Box P, w) = 0$.	2, *def.* $\to$
4.	So $V_{\mathcal{M}}(\Diamond\Box P, w) = 1$.	3
5.	So there is some w' such that Rww' and $V_{\mathcal{M}}(\Box P, w) = 1$—call it '$x$'.	4, *def.* $\Diamond$
6.	So Rwx and $V_{\mathcal{M}}(\Box P, x) = 1$.	5
7.	So $V_{\mathcal{M}}(\Box P, x) = 1$.	6
8.	So, for all w', if Rxw', then $V_{\mathcal{M}}(P, w') = 1$.	7, *def.* $\Box$
9.	And $V_{\mathcal{M}}(\Box P, w) = 0$.	3
10.	So it is not the case that $V_{\mathcal{M}}(\Box P, w) = 1$.	9, *bivalence*
11.	So it is not the case that, for all w', if Rww', then $V_{\mathcal{M}}(P, w') = 1$.	10, *def.* $\Box$
12.	So there is some w' such that Rww' and $V_{\mathcal{M}}(P, w') \neq 1$—call it '$y$'.	11, *QL*
13.	So Rwy and $V_{\mathcal{M}}(P, y) \neq 1$.	12
14.	So Rwy	13
15.	And Rwx	6
16.	So Rwx and Rwy	14, 15 $\wedge I$
17.	If Rwx and Rwy, then Rxy	*def. S5-model*
18.	So Rxy	16, 17 *MP*
19.	And if Rxy, then $V_{\mathcal{M}}(P, y) = 1$.	8, *QL*
20.	So $V_{\mathcal{M}}(P, y) = 1$.	18, 19 *MP*
21.	But $V_{\mathcal{M}}(P, y) \neq 1$.	13
22.	Our assumption that there is an S5-model $< \mathcal{W}, R, \mathscr{I} >$ with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Box P \to \Box P, w) = 0$ has led to a contradiction.	20, 21
23.	So that assumption is false, and there is no S5-model $< \mathcal{W}, R, \mathscr{I} >$ with some $w \in \mathcal{W}$ such that $V_{\mathcal{M}}(\Diamond\Box P \to \Box P, w) = 0$.	22

8.4. **Establishing Invalidity in S5.** In order to establish that an argument from the premises in Γ to the conclusion $\ulcorner \phi \urcorner$ is invalid in S5, that is, that $\Gamma \not\models_{S5} \phi$, it is enough to provide an S5-model $< \mathcal{W}, R, \mathscr{I} >$ in which all of the premises in Γ are true at some world in $\mathcal{W}$, yet $\ulcorner \phi \urcorner$ is false at that world. For instance, suppose that we wish to show that

$$\{P\} \not\models_{S5} \Diamond\Box P$$

We may do so with the following S5-model:

$$\begin{aligned} \mathcal{W} &= \{w_1, w_2\} \\ R &= \{< w_1, w_1 >, < w_1, w_2 >, \\ &\quad < w_2, w_2 >, < w_2, w_1 >\} \\ \mathscr{I}(P, w_1) &= 1 \\ \mathscr{I}(P, w_2) &= 0 \end{aligned}$$



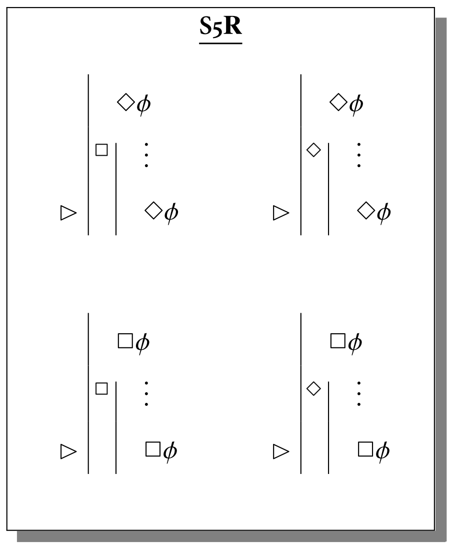
'P' is true at w_1 in this model. However, '$\Box P$' is false at both w_1 and w_2. '$\Box P$ is false at w_1 because w_1 sees w_2 and 'P' is false at w_2; and '$\Box P$' is false at w_2 because w_2 sees itself, and 'P' is false at w_2. Therefore, there is no world that w_1 sees at which '$\Box P$' is true. So '$\Diamond\Box P$' is false at w_1. So the premise of this argument is true at w_1, yet its conclusion is false at w_1. So the argument is S5-invalid.

The same model shows that

$$\{\Diamond P\} \not\models_{S5} \Diamond\Box P$$

For '$\Diamond P$' is true at w_1, since w_1 sees itself. However, '$\Diamond\Box P$' is still false at w_1, since '$\Box P$' is false at both w_1 and w_2.

8.5. **Natural Deduction for S5.** To get a natural deduction system for S5, we will take our natural deduction system for T and add to it the rule S5R:



This rule says: if you have a wff of the form $\ulcorner\Box\phi\urcorner$ outside of the scope of a strict subproof, then you may reiterate $\ulcorner\Box\phi\urcorner$ within that strict subproof (whether it is a box or a diamond strict subproof); and, if you have a wff of the form $\ulcorner\Diamond\phi\urcorner$ outside of the scope of a strict subproof, then you may reiterate $\ulcorner\Diamond\phi\urcorner$ within the scope of that strict subproof (whether it is a box or a diamond strict subproof).

Here is an S5-derivation of the S5 axiom:

1	$\Diamond P$	$A(\rightarrow I)$
2	$\Box$ $\Diamond P$	1, S5R
3	$\Box\Diamond P$	2–2, $\Box I$
4	$\Diamond P \rightarrow \Box\Diamond P$	1–3, $\rightarrow I$

And here is an S5-derivation of (S5′), ‘$\Diamond\Box P \rightarrow \Box P$’:

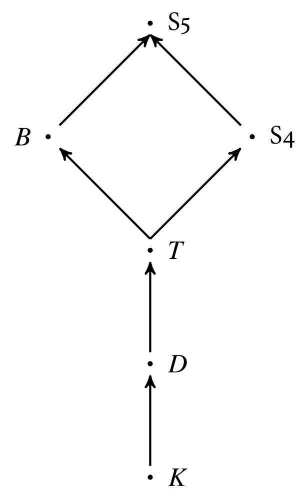
1	$\mid\ \Diamond\Box P$	$A(\rightarrow I)$
2	$\mid\ \mid\ \sim\Box P$	$A(\sim E)$
3	$\mid\ \mid\ \Diamond\sim P$	2, MN
4	$\mid\ \mid\ \Box\mid\ \Diamond\sim P$	3, S5R
5	$\mid\ \mid\ \ \ \mid\ \sim\Box P$	4, MN
6	$\mid\ \mid\ \Box\sim\Box P$	4–5, $\Box I$
7	$\mid\ \mid\ \sim\Diamond\Box P$	6, MN
8	$\mid\ \mid\ \Diamond\Box P \wedge \sim\Diamond\Box P$	1, 7, $\wedge I$
9	$\mid\ \Box P$	2–8, $\sim E$
10	$\Diamond\Box P \rightarrow \Box P$	1–9, $\rightarrow I$

Here’s a derivation showing that $\vdash_{S5D} \Diamond\Box P \rightarrow \Box\Diamond P$

1	$\mid\ \Diamond\Box P$	$A(\rightarrow I)$
2	$\mid\ \Box\mid\ \Diamond\Box P$	1 S5R
3	$\mid\ \ \ \mid\ \Diamond\mid\ \Box P$	$A(\Diamond E)$
4	$\mid\ \ \ \mid\ \ \ \mid\ P$	3 $\Box E$
5	$\mid\ \ \ \mid\ \Diamond P$	2, 3–4, $\Diamond E$
6	$\mid\ \Box\Diamond P$	2–5, $\Box I$
7	$\Diamond\Box P \rightarrow \Box\Diamond P$	1–6, $\rightarrow I$

9. Relationships Between the Systems

We may visualize the relationship between the logical systems we have learned with the following graph.

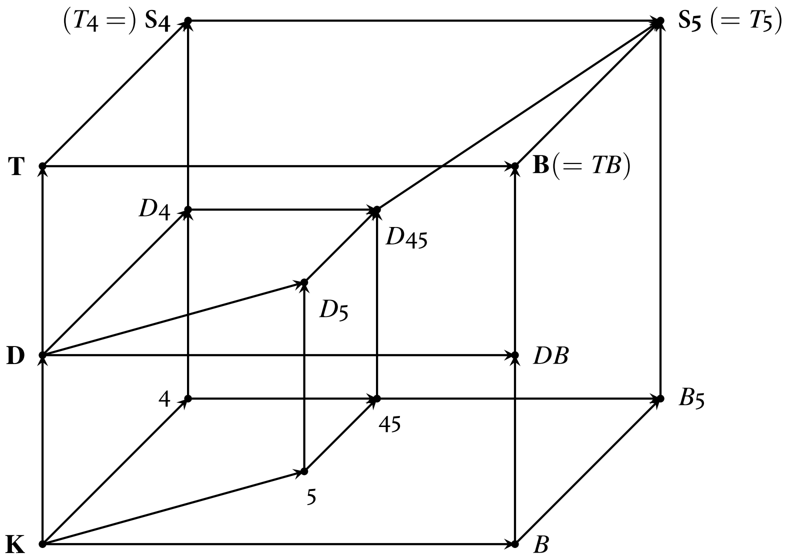


Here, the arrows correspond to relations of *validity preservation*. The graph tells us that, if an argument or a wff of PML is valid in K, then it will be valid in D. If it is valid in D, then it will be valid in T. If it is valid in T, then it will be valid in B, and it will be valid in S4. And, if an argument or a wff of PML is valid in either B or S4, then it will be valid in S5.

Validity-preservation is transitive, so the graph also tells us, for instance, that if an argument or wff of PML is valid in K, then it is valid in B; and that, if it is valid in D, then it is valid in S5. A common metaphorical way of speaking about validity-preservation is in terms of the *strength* of the modal system. If all the validities of the system S are validities of the system S', and S' has more validities besides, then we say that S' is *stronger* than S; and that S is *weaker* than S'.

There are other modal systems out there. Some of these are *weaker* than the system K. That is, there are arguments or wffs that are valid in K which are *not* valid in these modal logics. Such modal logics are known as *non-normal* modal logics. Any modal logic which is at least as strong as K is known as a *normal* modal logic. That is: any modal logic L which is such that all the arguments or wffs which are valid in K are valid in L is a normal modal logic.

Even amongst the normal modal logics, there are a great many which we have not explored here. For a taste, here are some of the possible modal logics which we can get just from mixing and matching the modal axioms we have already seen—namely, (D), (T), (B), (S4), and (S5). Each of the systems shown below has the axioms and rules of system K, plus some combination of the axioms (D), (T), (B), (S4), and (S5). In the diagram, they are ordered in terms of validity preservation, or strength.



The boldfaced logics are the ones we have studied. '*D*4' is the logic that you get if you add (*D*) and (S4) to the system *K*; '*B*5' is the logic that you get if you add (*B*) and (S5) to *K*; and so on. For each of these normal modal logics, you may get a possible worlds semantics for the logic which is sound and complete by imposing constraints on the accessibility relation corresponding to those axioms. So, for instance, to get a semantics for the logic 45, you simply define a 45-frame to be a pair $< \mathcal{W}, R >$ of a set of worlds and a binary relation R which is *transitive* and *euclidean*; and a 45-model is defined in the usual way.