Course Materials for *Introduction to Logic*

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Draft of Fall, 2019

Contents

Part I

Basic Concepts of Logic

What is Logic?

1. Logic is the study of *arguments*
2. An *argument* is any collection of reasons to think that some claim is true.
   (a) The claim being argued *for* is called the 'conclusion' of the argument.
   (b) The reasons to think that the conclusion is true are call the 'premises' of the argument.
3. A sample argument:

   > We must give up some privacy in the name of security. For if the homeland is not secure, terrorist attacks order of magnitude larger than 9/11 will find their way to our shores. And no amount of privacy is worth enduring an attack like this.

   (a) In order to make it clear what the conclusion is and what the premises are, we will write the claims in an argument in a vertical stack, with the premises on the top and the conclusion at the bottom, prefaced with the symbol '∴', which means '*therefore*'.

   Premise
   Premise
   ∴ Conclusion

   (b) Some other arguments:

   Bernie would beat Warren in a one-on-one race
   Biden would beat Bernie in a one-on-one race
   ∴ Biden would beat Warren in a one-on-one race

   Something can only harm you if you are aware of it
   No one is aware of their own death
   ∴ No one is harmed by their own death

   It is possible for me to survive the death of my body
   ∴ Me and my body are two different things

   Every contingent being is caused
   Nothing is caused by itself
   ∴ A necessary being exists

   If moral theory is studies empirically, then examples of conduct will be considered
   If examples of conduct are considered, then principles for selecting examples will be used
   If principles for selecting examples are used, then moral theory is not studied empirically
   ∴ Moral theory is not studied empirically

4. The goal of logic is to give a theory of when arguments are good, when they are bad, and in what ways they are good and bad.
   (a) One important good-making feature of an argument is that its premises are *true*. However, this isn't enough to make an argument good. Consider the following, terrible argument:

   Paris is the capital of France
   Sunday is the day before Monday
   ∴ Climate change is a Chinese hoax

   The premises of this argument are all true—but that's not enough to make this a good argument.

   (b) Another good-making feature of an argument is this: the conclusion *necessarily follows from* the premises. That is: necessarily, if the premises are true, then the conclusion is true, too. An argument with this feature is called *valid*. An argument which lacks this feature is called *invalid*.

> An argument is *valid* if and only if it impossible for its premises to be true while its conclusion is false.
>
> An argument is *invalid* if and only it is possible for its premises to be true while its conclusion is false.

(c) If a valid argument additionally has all true premises, then we will say that it is *sound*

> An argument is *sound* if and only if it is both valid and all of its premises are true.

- So, if an argument is sound, then its conclusion is true. That's a reason to care about soundness. And since soundness decomposes into truth and validity, it's a reason to care about validity.

5. Unfortunately, logic alone cannot teach us which premises are true and which are false. However, it *can* teach us which argument are valid. And this is something worth knowing—it's not trivial.

   (a) Working out which arguments are valid and which are invalid is, in general, a difficult and subtle matter.

   (b) The study of logic will put you in a better position to think this question through.

   (c) It will also teach you a general theory which will allow you to think quickly and intuitively about which arguments are valid and which are not.

What will we learn in this class?

6. In this class, we will learn some logical theories which form the backbone of all other logical theories.

7. We will begin with *sentence logic*. From there, we will move on to *predicate logic*.

   (a) If you learn these theories, you will be in a position to learn about *modal logic*, *tense logic*, *conditional logics*, *set theory*, *higher-order logic*, and so on.

   (b) Familiarity with *sentence logic* and *predicate logic* is important to understand topics and debates in all areas of philosophy. It has important applications in computer science and mathematics.

   (c) But learning sentence logic and predicate logic won't only put you in a position to study these other topics. The skills we acquire here will, moreover, be helpful with reasoning about arguments in a wide variety of contexts, about a wide variety of subject matters—even if the arguments are too complex for either sentence logic or predicate logic to handle.

8. These theories will allow us to show that two of our sample arguments *are* valid. It will also provide us with a formal system in which we can rigorously *prove* that they are valid.

Arguments

1. An *argument* is a collection of reasons for believing some claim

2. We typically use arguments to attempt to *persuade* one another

   (a) The claim we're trying to presuade each other to believe—the thing that the argument is arguing *for*—is called the conclusion of the argument

   (b) The reasons which are presented in the conclusion's favor are called the *premises* of the argument.

3. For our purposes in this class, we'll adopt a slightly more general and more formal definition of an argument.

> An *argument* is a collection of statements, at most one of which is designated as the conclusion, the others of which are designated as the premises

   (a) On this definition, the following will count as an argument, even though the premises don't intuitively give you *any* reason to accept its conclusion:

   Bacon isn't meat

   Samuel Huntington is spry

   Summer will never come

   ∴ Elmer Fudd isn't fictional

   (b) A *statement* is a sentence which is capable of being true or false.

   (c) A test: given some sentence, '$\mathscr{A}$', if 'It is true that $\mathscr{A}$' makes sense, then '$\mathscr{A}$' is a statement. If 'It is true that $\mathscr{A}$' does not make sense, then '$\mathscr{A}$' is not a statement.

   Statements: 'I ate my car keys', 'Nobody knows the trouble I've seen', 'Chocolate is tasty'

   Non-statements: 'Try jiggling the handle', 'Who ate the car keys?', 'Ouch!'

Validity

4. One important way for an argument to be good: the truth of all of its premises guarantees the truth of its conclusion. If the premises are all true, then the conclusion must be true as well.

> An argument is *valid* if and only if it is impossible for its premises to all be true while its conclusion is simultaneously false.
>
> An argument is *invalid* if and only if it is possible for its premises to all be true while its conclusion is simultaneously false.

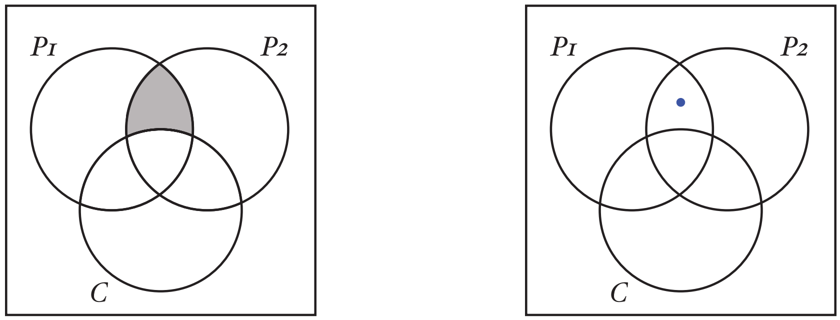
   (a) ***A valid arguments can have false premises***

   (b) ***A valid arguments can have a false conclusion***

   (c) When it comes to validity, it doesn't matter whether the premises and conclusion are actually true or false. The only thing that matters is whether it's *possible* for the premises to all be true while the conclusion is false.

   (d) If the premises of a valid argument are all true, then we say that the argument is *sound*.

> An argument is *sound* if and only if it is valid and it has all true premises.



(a) Valid (b) Invalid

Figure 1: All the points in the circle *P1* are possibilities in which the premise P1 is true. All points outside of the circle *P1* are possibilities in which the premise P1 is false. All the points in the circle *P2* are possibilities in which P2 is true. All points outside *P2* are possibilities in which P2 is false. All points in *C* are possibilities in which the conclusion C is true; all those outside of *C* are possibilities in which C is false. For the argument P1, P2, ∴ C to be valid is for there to be no possibilities in the grey region—no possibilities in which P1 and P2 are both true, while C is false. For the argument to be invalid is for there to be a possibility like this.

(e) Consider an argument from the premises P1 and P2 to the conclusion C: P1, P2, ∴ C. In the Venn diagram in figure 1, the circle labeled *P1* contains all the possibilities in which P1 is true. The circle labeled *P2* contains all the possibilities in which P2 is true. The circle labeled *C* contains all the possibilities in which C is true. Then, what it is for the argument to be valid is for the area in grey to be *empty*—it is for there to be no possibilities in which P1 and P2 are both true, but C is false. For it to be invalid is for there to be some possibility like this.

(f) So, if the argument P1, P2, ∴, C is invalid, then you should be able to point to a possibility in which P1 and P2 are true but C is false. This is a way to demonstrate that an argument is invalid.

5. Special cases of validity:

   (a) If it is impossible for an argument's premises to all be true, then it is automatically impossible for all of the argument's premises to be true while its conclusion is false—no matter what its conclusion is.

   (b) So, if it's impossible for an argument's premises to all be true, then it will automatically be valid—no matter what its conclusion is.

   (c) Also, if it is impossible for an argument's conclusion to be false, then it is automatically impossible for all of the argument's premises to be true while its conclusion is false—no matter what its premises are.

   (d) So, if it's impossible for an argument's conclusion to be false, then it will automatically be valid—no matter what its premises are.

Joint Possibility and Joint Impossibility

1. Suppose I make the following claims:
    (a) Whenever it rains, I go shopping
    (b) Last Tuesday, I didn't go shopping
    (c) It rained last Tuesday

    Then, you can know that I've said something false. Suppose everything I said was true. Then, it must have rained last Tuesday. And, since whenever it rains, I go shopping, I went shopping last Tuesday. But I *also* say that I *didn't* go shopping last Tuesday. So I've said something false.

2. These three claims are not *jointly possible*. It's not possible for them to all be true together at once.

> Statements are *jointly possible* if and only if it is possible for them to all be true.
>
> Statements are *jointly impossible* if and only if it is impossible for them to all be true.

    (a) To figure out whether a collection of statements is jointly possible or not, here's a test: try to imagine a scenario in which all of the statements are true at once. If you succeed, then you know that the statements are jointly possible. If you fail after trying very hard, then you may guess that they are jointly impossible.

Validity and Joint Impossibility

3. We can also characterize the notion of validity in terms of joint possibility

4. To understand how, first notice that a statement, '$\mathscr{A}$', is false if and only if 'it is not the case that $\mathscr{A}$' is true.

5. So, we may replace our definition of validity with the following:

    An argument *P1*, *P2* ∴ *C* is valid if and only if it is impossible for '*P1*' and '*P2*' to be true while 'not-*C*' is true.

    or:

    An argument *P1*, *P2* ∴ *C* is valid if and only if it is impossible for '*P1*', '*P2*', and 'not-*C*' to all be true together.

    or:

    argument *P1*, *P2* ∴ *C* is valid if and only if '*P1*', '*P2*', and 'not-*C*' are jointly impossible

6. This means that, whenever we say that an argument is valid, we could just as well have said that a certain collection of statements—the premises, together with 'not-' the conclusion—are jointly impossible.

7. In these terms, we may re-express the special cases of validity in terms that might make them more easily intelligible:
    (a) If not-*C* is impossible, then *P1*, *P2*, and not-*C* are jointly impossible.
    (b) If *P1* and *P2* are jointly impossible, then *P1*, *P2*, and not-*C* are jointly impossible.

Necessary Truths, Necessary Falsehoods, and Contingencies

8. Consider the following statements:

   (a) Either Trump will win in 2020 or Trump will not win in 2020

   (b) If it snows here tomorrow, then it precipitates here tomorrow

   (c) I'm not taller than myself

   When we try to imagine any of these statements being false, we come up short. They seem to be *necessarily* true. Call a statement like this a 'necessary truth'.

   > A statement is a *necessary truth* iff it is impossible for that statement to be false.

9. Consider the following statements:

   (a) Trump will win in 2020 and he will not win in 2020

   (b) It will snow here tomorrow, but it won't precipitate here tomorrow

   (c) I am taller than myself.

   When we try to imagine any of these statements being true, we come up short. They seem to be *necessarily* false. Call a statement like this a 'necessary falsehood'.

   > A statement is a *necessary falsehood* iff it is impossible for that statement to be true.

10. Consider the following statements:

    (a) Trump will win in 2020

    (b) It will snow here tomorrow

    (c) I am taller than Travis

    We can imagine each of these statements being true, and we can imagine them being false. They are neither necessarily true nor necessarily false. Call a statement like this a 'contingency'

    > A statement is a *contingency* iff it is possible for the statement to be true and it is possible for the statement to be false.

Part 1: Validity

For each of the arguments below, if the argument is valid, then choose (a). If the argument is invalid, then say which of the possibilities described in the other answer choices show that it is invalid (there may be more than one).

Trump won't win if the economy is weak
The economy is not weak
∴ Trump will win

1. (a) The argument is valid.
   (b) The economy will be weak, because of Trump's trade policies; so he won't win.
   (c) Trump will win, even if the economy is weak; and the economy isn't weak.
   (d) Trump won't win, whether or not the economy is weak. And the economy isn't weak.
   (e) Trump will win if the economy is weak, but if the economy isn't weak, then he'll lose. And the economy isn't weak.

Either you and your sister stop arguing or I turn this car around
You and your sister don't stop arguing
∴ I turn this car around

2. (a) The argument is valid.
   (b) You and your sister stop arguing, and I turn this car around.
   (c) You and your sister don't stop arguing, and I turn this car around.
   (d) You and your sister stop arguing, and I don't turn this car around.
   (e) You and your sister don't stop arguing, and I don't turn this car around.

Nobody knows the trouble I've seen
Karen knows the trouble I've seen
∴ Giraffes have long necks

3. (a) The argument is valid.
   (b) Giraffes have long necks, Bill knows the trouble I've seen, and Karen does not know the trouble I've seen.
   (c) Giraffes do not have long necks, and everyone knows the trouble I've seen.
   (d) I don't know the trouble I've seen, nor does Karen know the trouble I've seen. And giraffes have long legs and short necks.
   (e) Giraffes know the trouble I've seen, but Karen doesn't. Karen's neck is long, but giraffe's necks are short.

Whenever it rains, Gerald goes to the movies

If Gerald went to the movies on Monday, then he saw *The Lion King*

It didn't rain on Monday

∴ Gerald hasn't seen *The Lion King*

4. (a) The argument is valid.
   (b) Gerald only goes to the movies when it doesn't rain. It didn't rain on Monday, and he went to the movies on Monday, but he saw *Once Upon a Time in Hollywood*.... He hasn't seen *The Lion King*.
   (c) Gerald goes to the movies every day—whether it rains or shines. On Monday, it didn't rain, and Gerald went to the movies. He saw *The Lion King*.
   (d) Gerald goes to the movies when it rains—if it doesn't rain, he stays home. On Sunday, it rained, and Gerald saw *The Lion King*. He loved it, and planned to go back if it rained again on Monday; but, on Monday, it didn't rain.
   (e) Gerald never goes to the movies, and hasn't seen *The Lion King*.

I never order meat if there's something without meat on the menu

At *Panera*, there's something without meat on the menu

At *Panera*, I ordered meat

∴ I'm a vegan

5. (a) The argument is valid
   (b) I sometimes order meat, even when there's something without meat on the menu—but only when I'm really craving it.
   (c) I never order meat if there's something without meat on the menu, but at Panera, there's only meat options. So, when I'm at Panera, I order meat. But, since I sometimes eat meat, I'm not a vegan.
   (d) I never order meat—ever. So at Panera, I don't order meat. I'm a vegan.
   (e) Who are you to judge? You eat meat, like, every day. It's not a big deal if I have some at Panera every now and again—mind your own business.

If Sam found the golden egg, she won $100

Sam didn't win $100

∴ Sam didn't find the golden egg

6. (a) The argument is valid.
   (b) There was a prize of $100 attached to finding the golden egg, but Sam never found the golden egg, so she didn't win the prize.
   (c) If Sam found the golden egg, she only won $50. But she didn't find it, so she didn't win $100.
   (d) There was a prize of $100 attached to finding the golden egg. Sam found the golden egg and won $100.
   (e) Sam didn't even want the $100 anyhow. She never bothered to look. That's why she didn't find the golden egg.

Part 2: Joint Possibility and Joint Impossibility

7. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. In jointly impossible, say why they are jointly impossible.

   In the Smithsonian, there's a square circle which is is both completely red all over completely green all over, and weighs more than itself

   Sunday is the Lord's day

   Bacon isn't meat

8. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. If jointly impossible, say why they are jointly impossible.

   Mr. Smithers never leaves work before 5:00p.m.

   On Monday, Homer left work after Mr. Burns did.

   Mr. Burns never leaves work before Mr. Smithers does.

   On Monday, Homer left work at 4:50p.m.

   On Monday, Homer, Mr. Burns, and Mr. Smithers were all at work.

9. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. If jointly impossible, say why they are jointly impossible.

   Either the Butler, the Gardener, or the Maid did it.

   If the Maid did it, then the Butler knows that the Maid did it.

   Whoever did it killed anyone (else) who knows that they did it.

   The Gardener and the Butler are alive.

   If the Butler knows who did it, then he told the Gardener.

10. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. If jointly impossible, say why they are jointly impossible.

    Tammy only speaks the truth

    Franny only speaks falsehoods

    Franny says "Tammy only speaks the truth"

11. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. If jointly impossible, say why they are jointly impossible.

    One of Tammy and Franny always speaks the truth—the other only ever speaks falsehoods.

    Either Tammy has the amulet or Franny does (and they don't both have it).

    Tammy says "I don't have the amulet".

    Franny says "I have the amulet".

    Tammy has the amulet.

12. Are the following statements jointly possible or jointly impossible? If jointly possible, describe a possibility in which they are all true. If jointly impossible, say why they are jointly impossible.

    Exactly one of Franny, Sammy, and Tammy has the amulet—the others don't have it.

    Exactly one of Franny, Sammy, and Tammy speaks the truth—the others only speak falsehoods.

    Franny says "I have the amulet".

    Sammy says "I don't have the amulet".

    Tammy says "Franny doesn't have the amulet".

1. (d)
2. (a)
3. (a)
4. (c) and (d)
5. (a)
6. (a)
7. They are jointly impossible, because it is impossible for there to be a square circle; it is impossible for something to be both completely red and completely green; and it is impossible for something to weigh more than itself.
8. They are jointly impossible. If Homer left work at 4:50 on Monday, and Homer left work after Mr. Burns did, then Mr. Burns must have left work before 4:50 on Monday. But if Mr. Burns never leaves work before Mr. Smithers, then that means that Mr. Smithers must have left work before 4:50 on Monday. But the first statement tells us that this never happens. It's not possible for Mr. Smithers to both leave work before 5:00pm on Monday and to never leave work before 5:00pm. So it's not possible for all of these statements to be true at once.
9. They are jointly possible. Suppose that the Gardener did it, but that nobody other than the victim knows that the Gardener did it. It could still be true that, if the Maid did it, then the Butler knows that she did it—but, since the Maid didn't do it, the Butler doesn't know that she did it. And it could still be true that the Gardener killed anyone (else) who knows that he did it. Likewise, if the Butler knows who did it, then he would have told the Gardener. But the Butler doesn't know who did it.
10. They are jointly impossible. If Franny only speaks falsehoods, and she says "Tammy only speaks the truth", then the first statement, "Tammy only speaks the truth", must be false. So if the second two statements are true, then the first is false. So they can't all be true together.
11. They are jointly impossible. The final statement tells us that Tammy has the amulet. Then, when Tammy says "I don't have the amulet", she says something false. So Tammy must be the one who speaks only falsehoods. So Franny must speak only the truth. So, when Franny says "I have the amulet", she must be speaking truly. So Franny has the amulet. But the final statement said that it was *Tammy* who has the amulet. So these statements can't all be true at once.
12. They are jointly possible. Suppose that Tammy speaks the truth and Sammy has the amulet. Then, Franny speaks falsely when she says "I have the amulet". Sammy speaks falsely when he says "I don't have the amulet". And Tammy speaks truly when she says "Franny doesn't have the amulet". So all the claims are true together.

Argument Forms

1. Consider these arguments:

	John went to the store unless it rained		Tabitha will be late unless she hurries		The test is Friday unless I'm mistaken
	It didn't rain		Tabitha won't hurry		I'm not mistaken
∴	John went to the store	∴	Tabitha will be late	∴	The test is Friday

Each of these arguments is valid; and they appear to be valid for the same reason. Notice: they all share the following *form*:

$\mathscr{A}$ unless $\mathscr{B}$
It is not the case that $\mathscr{B}$
∴ $\mathscr{A}$

2. Moreover, it seems like recognizing the form of these arguments is all that it takes to see that they are valid. Indeed, it seems that we can tell that any argument with any of the following forms *must* be valid, no matter which sentences we plug in for '$\mathscr{A}$' and '$\mathscr{B}$':

	If $\mathscr{A}$, then $\mathscr{B}$		Either $\mathscr{A}$ or $\mathscr{B}$		Both $\mathscr{A}$ and $\mathscr{B}$
	$\mathscr{A}$		It is not the case that $\mathscr{A}$	∴	$\mathscr{A}$
∴	$\mathscr{B}$	∴	$\mathscr{B}$		

3. Let's think more carefully about argument forms.

   (a) A *variable* is just a place-holder for which you can substitute some kind of expression. For instance: we could use 'x' as a variable for which you can substitute a *number*, as in: '$f(x) = x^2$'. Or we could use 'x' and 'y' as variables for which you can substitute *names*, as in: 'x loves y'. And, in this class, we will use calligraphic letters like '$\mathscr{A}$' and '$\mathscr{B}$' as variables for which you can substitute *statements*, as in: 'If $\mathscr{A}$, then $\mathscr{B}$.'

   (b) Some things can take the place of a variable, and other things cannot. So, when we use a variable, we should be clear about which kinds of things can take the place of that variable, and which cannot. For the calligraphic letters '$\mathscr{A}$', '$\mathscr{B}$', and '$\mathscr{C}$', the only things that can take their place is *statements*. We say that the things which can take the place of a variable are in its *range*—or, we say that the variable *ranges over* those things. So, as we're using it in this class, '$\mathscr{A}$' ranges over statements.

   (c) A *statement form* is a string of words containing variables such that, if you replace the variable with something in its range, then you get a statement. For instance, 'It is both the case that $\mathscr{A}$ and $\mathscr{B}$' is a statement form, since, if we replace '$\mathscr{A}$' with 'Eli is hungry' and '$\mathscr{B}$' with 'Bob is sad', we get the statement 'It is both the case that Eli is hungry and Bob is sad', which is a statement. For another: if 'x' and 'y' are variables ranging over names, then 'x loves y' is a statement form, since, if we replace 'x' with 'Sabeen', and 'y' with 'Matthew', then we get the statement 'Sabeen loves Matthew'. However, if 'x' and 'y' range over names, then 'It is both the case that x and y' is *not* a statement form, since 'It is both the case that Bob and Mary' is not a statement.

   (d) An *argument form* is just a collection of statement forms, at most one of which is labeled as the conclusion, and the rest of which are labeled as the premises. For instance, each of the following are argument forms:

	x loves y		Either $\mathscr{A}$ or $\mathscr{B}$		x is the brother of y
∴	y loves x	∴	Both $\mathscr{A}$ and $\mathscr{B}$	∴	y is the brother of x

There's something wrong with each of these argument forms. They are each *invalid*. We've defined validity for *arguments*—but what does it mean to call an argument *form* valid or invalid?

Validity of Argument Forms

4. To define the notion of validity for argument forms, let's first define the notion of a *substitution instance*.

   (a) Take a statement form, and uniformly replace its variables with anything in the range of those variables. What you get is a *substitution instance* of that statement form. For instance: 'If Eli is hungry, then Barcelona is in France' is a substitution instance of 'If $\mathscr{A}$, then $\mathscr{B}$'.

   (b) Similarly, take an argument form, and uniformly replace its variables with anything in the range of those variables. What you get is a *substitution instance* of that argument form. For instance, if 'P' and 'Q' are variables ranging over *kinds*, and 'S' is a variable ranging over *people*, then the argument on the left is a substitution instance of the argument form in the center.

All people are mortal	All Ps are Q	All people are mortal
Socrates is a person	S is a P	Socrates is a person
∴ Socrates is mortal	∴ S is Q	∴ Aristotle is mortal

   Notice that we have to replace each occurrence of the variable with the *same* thing. The argument on the right above is *not* a substitution instance of the argument form in the center. For, in the second premise of that argument, we replaced 'S' with 'Socrates'—but, in the conclusion, we replaced 'S' with 'Aristotle'.

   (c) Now, an argument *form* is valid if and only if there's no substitution instance of it with all true premises and a false conclusion. If there *is* a substitution instance of the argument form with all true premises and a false conclusion, then the argument form is invalid.

> An argument form is *valid* if and only if there is no substitution instance of it with all true premises and a false conclusion.

> An argument form is *invalid* if and only if there is some substitution instance of it with all true premises and a false conclusion.

5. To prove that an argument form is invalid, then, it is enough to provide a single substitution instance which has all true premises and a false conclusion.

Validity of Arguments and Validity of Argument Forms

6. The validity of argument *forms* is very different from the validity of arguments. What is the connection between them?

   (a) For now, I will assert—but not attempt to persuade you of—the following bold and provocative and completely non-obvious claim: *if an argument has a valid form, then it is a valid argument*.

   (b) *Note*: distinguish this true claim from the following, false and pernicious claim: *if an argument has an* invalid *form, then it is an* invalid *argument*. **This is not true.** Any two-premise argument has the following invalid form: $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$. But not every two-premise argument is invalid.

7. If every argument with a valid form is valid, then this means that, if we can prove that an argument has a valid form, we can prove that it is valid. But how can we prove that an argument *form* is valid? Doesn't this require us to consider every possible substitution instance? And aren't there infinitely many possible substitution instances of an argument form?

A. Which of the following are substitution instances of the statement form

if *x* loves *y*, then *y* loves *x*

where *x* and *y* range over names?

1. If Bob loves Mary, then John loves Suzy.
2. If Bob loves Bob, then Bob loves Bob.
3. If Janice loves Jeremy, then Janice loves Jeremy.
4. If Bob loves Mary, then nobody loves anybody else.
5. If Robin loves Zelda, then Zelda loves Robin.

B. Which of the arguments below have the following form?

If $\mathscr{A}$, then both $\mathscr{B}$ and $\mathscr{C}$
It is not the case that both $\mathscr{A}$ and $\mathscr{B}$
∴ It is not the case that $\mathscr{C}$.

If today is Sunday, then both tomorrow is Monday and yesterday is Saturday.
It is not the case that both today is Sunday and yesterday is Saturday.
∴ It is not the case that yesterday is Saturday.

If Rand Paul is a Senator, then both Paul Ryan is a Senator and Marsha Brady is a Senator.
It is not the case that both Paul Ryan is a Senator and Marsha Brady is a Senator.
∴ It is not the case that Rand Paul is a Senator.

If I will sleep in, then both I will miss my appointment and I will not have time to study.
It is not the case that both I will sleep in and I will miss my appointment.
∴ It is not the case that I will not have time to study.

If I live in Manhattan, then both I live in New York City and I live in New York State.
It is not the case that both I live in Manhattan and I live in New York City.
∴ It is not the case that I live in New York State.

C. Do any of the following show you that this argument form is invalid? (If so, which?) Do any show you that this argument form is valid? (If so, which?)

All Fs are G.
Some Gs are H.
$\therefore$ Some Fs are H.

1. All senators are citizens.
   Some citizens are unemployed.
   ∴ Some senators are unemployed.

2. No penguins are carnivores.
   Some carnivores are mammals.
   ∴ No penguins are mammals.

3. All rectangles are polygons.
   Some polygons are equilateral.[1]
   ∴ Some rectangles are equilateral.

4. All cats are animals.
   Some dogs are animals.
   ∴ Some cats are dogs.

[1] A polygon is equilateral if and only if all of its sides are of the same length.

Part II

Sentence Logic

1. We want to look at the *form* of an argument as a way of proving that the arguments is valid. Unfortunately, the English language is very messy and complicated. So thinking about the form of English sentences requires a lot of thought about the *meaning* of those sentences. That's important work—but it's work for another class (a class in *semantics* or the philosophy of language). In this class, we're going to take a different approach. We will introduce an artificial *formal* language, which we'll call 'SL'—for '*sentence logic*'. This formal language will be far less messy and less complicated than English, and it will allow us to think rigorously about the validity of some common argument forms.

2. The Language SL is going to allow us to focus on the following statement forms:

   ▹ It is not the case that $\mathscr{A}$

   ▹ Both $\mathscr{A}$ and $\mathscr{B}$

   ▹ Either $\mathscr{A}$ or $\mathscr{B}$

   ▹ If $\mathscr{A}$, then $\mathscr{B}$

   ▹ $\mathscr{A}$ if and only if $\mathscr{B}$

3. English statements without any of these forms will be called *atomic* statements. They will be translated into SL using *statement letters*—capital italic letters, $A, B, C, \ldots, Z$.

   (a) In order to help us translate into SL, we will provide a *symbolization key* which tells us which English sentences each (relevant) statement letter is standing for. For instance:

   N : Nobody knows the trouble I've seen
   A : Ants ate my car keys
   S : Santa Claus exists

   ▹ *N.B.*: the script '$\mathscr{A}$' and '$\mathscr{B}$' are very different from the statement letters 'A' and 'B'. '$\mathscr{A}$' and '$\mathscr{B}$' are *variables*. They don't represent any particular statements. 'A' and 'B' are not variables. They are used in SL to represent one and only one statement at a time. The symbolization key tells us which statements the statement letters represent.

4. The logical forms above will be translated using the following symbols:

English sentence	**SL sentence**	**Name**
It is not the case that $\mathscr{A}$	$\neg\mathscr{A}$	negation
Both $\mathscr{A}$ and $\mathscr{B}$	$(\mathscr{A} \wedge \mathscr{B})$	conjunction
Either $\mathscr{A}$ or $\mathscr{B}$	$(\mathscr{A} \vee \mathscr{B})$	disjunction
If $\mathscr{A}$, then $\mathscr{B}$	$(\mathscr{A} \rightarrow \mathscr{B})$	conditional
$\mathscr{A}$ if and only if $\mathscr{B}$	$(\mathscr{A} \leftrightarrow \mathscr{B})$	biconditional

5. In SL, a sentence of the form '$\neg\mathscr{A}$' is a *negation*. '$\neg\mathscr{A}$' means 'It is not the case that $\mathscr{A}$'.

6. In SL, a sentence of the form '$(\mathscr{A} \wedge \mathscr{B})$' is a *conjunction*. In this sentence, both '$\mathscr{A}$' and '$\mathscr{B}$' are called *conjuncts*. '$(\mathscr{A} \wedge \mathscr{B})$' means 'Both $\mathscr{A}$ and $\mathscr{B}$'.

   (a) Using the symbolization key A : Abelard loves Heloise; and H : Heloise loves Abelard, we may translate 'Abelard loves Heloise and Heloise doesn't love Abelard' as '$(A \wedge \neg H)$'.

   (b) SL doesn't distinguish between the meaning of 'and' and 'but'. This is one of the ways that SL is less messy than English. All of the following are translated into SL as conjunctions:

$$\left.\begin{array}{c}\mathscr{A}\text{ and }\mathscr{B}\\ \mathscr{A}\text{, but }\mathscr{B}\\ \mathscr{A}\text{; however, }\mathscr{B}\\ \mathscr{A}\text{, though }\mathscr{B}\\ \mathscr{A}\text{ as well as }\mathscr{B}\end{array}\right\}\mapsto (\mathscr{A}\wedge\mathscr{B})$$

$$\text{Not both }\mathscr{A}\text{ and }\mathscr{B}\ \}\mapsto \neg(\mathscr{A}\wedge\mathscr{B})$$

7. In SL, a sentence of the form '$(\mathscr{A}\vee\mathscr{B})$' is a *disjunction*. In this sentence, both '$\mathscr{A}$' and '$\mathscr{B}$' are called *disjuncts*. '$(\mathscr{A}\vee\mathscr{B})$' means 'Either $\mathscr{A}$ or $\mathscr{B}$'.
   (a) We must distinguish two different meanings 'or' might have.
   (b) If 'or' is *exclusive*, then 'Either $\mathscr{A}$ or $\mathscr{B}$' is *false* when both '$\mathscr{A}$' and '$\mathscr{B}$' are true.
      i. E.g., "Either you clean your room or you're grounded"
   (c) If 'or' is *inclusive*, then 'Either $\mathscr{A}$ or $\mathscr{B}$' is *true* when both '$\mathscr{A}$' and '$\mathscr{B}$' are true.
      i. E.g., "Either Adam or Betsy could lift that"
   (d) In SL, '$(\mathscr{A}\vee\mathscr{B})$' translates the *inclusive 'or'*. In fact, throughout this class, we will *always* understand 'or' as being inclusive.
   (e) A translation guide:

$$\left.\begin{array}{c}\text{Either }\mathscr{A}\text{ or }\mathscr{B}\\ \mathscr{A}\text{ unless }\mathscr{B}\end{array}\right\}\mapsto (\mathscr{A}\vee\mathscr{B})$$

$$\text{Neither }\mathscr{A}\text{ nor }\mathscr{B}\ \}\mapsto \neg(\mathscr{A}\vee\mathscr{B})$$

8. In SL, a sentence of the form '$(\mathscr{A}\rightarrow\mathscr{B})$' is a *conditional*. In this sentence, '$\mathscr{A}$' is called the *antecedent*, and '$\mathscr{B}$' is called the *consequent*. '$(\mathscr{A}\rightarrow\mathscr{B})$' means 'If $\mathscr{A}$, then $\mathscr{B}$'.
   (a) A translation guide:

$$\left.\begin{array}{c}\text{If }\mathscr{A}\text{, then }\mathscr{B}\\ \mathscr{A}\text{ only if }\mathscr{B}\\ \mathscr{B}\text{ if }\mathscr{A}\end{array}\right\}\mapsto (\mathscr{A}\rightarrow\mathscr{B})$$

9. In SL, a sentence of the form '$(\mathscr{A}\leftrightarrow\mathscr{B})$' is a *biconditional*. In this sentence, '$\mathscr{A}$' is called the *left-hand-side*, and '$\mathscr{B}$' is called the *right-hand-side*. '$(\mathscr{A}\leftrightarrow\mathscr{B})$' means '$\mathscr{A}$ if and only if $\mathscr{B}$'. (We'll see later on that it means the same thing as '$((\mathscr{A}\rightarrow\mathscr{B})\wedge(\mathscr{B}\rightarrow\mathscr{A}))$'.
   (a) A translation guide:

$$\left.\begin{array}{c}\mathscr{A}\text{ if and only if }\mathscr{B}\\ \mathscr{A}\text{ when and only when }\mathscr{B}\end{array}\right\}\mapsto (\mathscr{A}\leftrightarrow\mathscr{B})$$

10. A tip: to translate a sentence of English into SL, first find another sentence of English which is synonymous with the first, and which uses *only* the canonical logical forms introduced above: 'it is not the case that', 'both...and...', 'either...or...', 'if..., then...', and '...if and only if...'. Then, translate the sentence into SL using the translation guides provided above. For instance:
   ▹ 'I won't go if John does' ↦ 'If John goes, then I won't go' ↦ 'If John goes, then it is not the case that I go' ↦ '(John goes → it is not the case that I go)' ↦ '(John goes → ¬ I go)' ↦ '$(J\rightarrow\neg I)$'
   ▹ (Here, I've used the statement letter 'J' for 'John goes' and 'I' for 'I go'.)

11. Beware! If we implement this procedure carelessly, we may end up mis-translating a sentence. For instance:

- ‘I hate getting what I want and I hate not getting what I want’ ↦ ‘Both I hate getting what I want and it is not the case that I hate getting what I want’ ↦ ‘(I hate getting what I want ∧ it is not the case that I hate getting what I want)’ ↦ ‘(I hate getting what I want ∧¬ I hate getting what I want)’ ↦ ‘$(H \land \neg H)$’.

(a) But the sentence we started with was true, while the sentence we ended up with is a necessary falsehood. Something went wrong—one of the stages in the process didn’t mean the same thing as the sentence which came before it.

1. Our goal today is to rigorously define our formal language SL. In general, to specify a language, we need to provide:
   (a) A vocabulary for the language;
   (b) A grammar for the language; and
   (c) A way to interpret the meaning of every grammatical sentence of the language

   The first two tasks are the tasks of specifying a *syntax* for the language. The final task is the task of specifying a *semantics* for the language.

Syntax for SL

2. The vocabulary of SL includes the following symbols:
   (a) An infinite number of statement letters (uppercase letters, with subscripts, if desired):
   $$A, B, C, \ldots, Y, Z, A_1, B_1, C_1, \ldots, Z_1, A_2, \ldots$$
   (b) Our five logical operators:
   $$\neg, \wedge, \vee, \rightarrow, \leftrightarrow$$
   (c) Parentheses:
   $$(\ ,\ )$$

   Nothing else is included in the vocabulary of SL.

3. Any sequence of the symbols from the vocabulary of SL is an *expression*. However, not all expressions are grammatical sentences. We define a *sentence* of SL with the following rules:

Rules for Sentences

SL)	Any statement letter is, by itself, a sentence.
$\neg$)	If '$\mathscr{A}$' is a sentence, then '$\neg\mathscr{A}$' is a sentence.
$\wedge$)	If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \wedge \mathscr{B})$' is a sentence.
$\vee$)	If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \vee \mathscr{B})$' is a sentence.
$\rightarrow$)	If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \rightarrow \mathscr{B})$' is a sentence.
$\leftrightarrow$)	If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \leftrightarrow \mathscr{B})$' is a sentence.
–)	Nothing else is a sentence.

4. We build sentences of SL up by progressively appealing to these rules, one after the other. With the rules, we can define some other important syntactic notions.
   (a) A non-atomic sentence's *main operator* is just the operator associated with the last rule which would have to be appealed to, were we building the sentence up by appealing to the rules in this way.
   (b) '$\mathscr{B}$' is a *subsentence* of '$\mathscr{A}$' iff, in the course of building up '$\mathscr{A}$' by applying the rules for sentences, we would first have to build up the sentence '$\mathscr{B}$'.
   (c) The *scope* of a logical operator (in a sentence) is the sub-sentence for which that operator is the main operator.

5. A convention: we allow ourselves to drop the outermost parenthases, and to use square brackets, '[,]', to improve readability.

Semantics for SL

6. I will assume that what it is to understand the meaning of a sentence is just to understand what it takes for that sentence to be true and what it takes for that sentence to be false.

   (a) So: to specify a *semantics* for SL, I will say when the sentences of SL are true and when they are false.

7. Every statement letter stands for a sentence of English. If that sentence of English is true, then the statement letter is true. If that sentence of English is false, then the statement letter is false.

8. If '$\mathscr{A}$' is true, then '$\neg\mathscr{A}$ is false. If '$\mathscr{A}$' is false, then '$\neg\mathscr{A}$' is true. We can summarize with this table (called a *truth-table*):

The Meaning of '¬'

$\mathscr{A}$	$\neg\mathscr{A}$
T	F
F	T

9. We can similarly give the meaning of the other operators by providing their characteristic truth-tables. They are:

The Meaning of '∧'

$\mathscr{A}$	B	$\mathscr{A} \wedge \mathscr{B}$
T	T	T
T	F	F
F	T	F
F	F	F

The Meaning of '∨'

$\mathscr{A}$	B	$\mathscr{A} \vee \mathscr{B}$
T	T	T
T	F	T
F	T	T
F	F	F

The Meaning of '→'

$\mathscr{A}$	B	$\mathscr{A} \rightarrow \mathscr{B}$
T	T	T
T	F	F
F	T	T
F	F	T

The Meaning of '↔'

$\mathscr{A}$	B	$\mathscr{A} \leftrightarrow \mathscr{B}$
T	T	T
T	F	F
F	T	F
F	F	T

10. We can work out the truth-value (true or false) of complicated sentences by working out the truth-values of their sub-sentences. This allows us to determine the meaning of any arbitrary sentence of SL. For instance:

P	Q	$\neg$	P	$\wedge$	Q
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	F	F

To understand this *truth-table*, consider the first row: it tells us that, if both 'P' and 'Q' are true, then the sub-sentence '$\neg P$' is false, and the sentence '$\neg P \wedge Q$' is false. The third row tells us that, if 'P' is false and 'Q' is true, then the sub-sentence '$\neg P$' is true, and the sentence '$\neg P \wedge Q$' is true.

A. SENTENCES. Which of the following are sentences of SL? (For this exercise, suspend our normal conventions for parentheses.) If it is not a sentence of SL, write out a new expression, as similar to the given expression as you can manage, which *is* a sentence of SL.

1. $(P \wedge Q) \rightarrow R)$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

2. $((P \vee Q \vee R) \rightarrow S)$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

3. $(P \leftrightarrow (S \leftrightarrow R))$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

4. $(p \rightarrow (q \rightarrow r))$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

5. $((\neg A) \rightarrow A)$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

6. $\neg\neg\neg\neg\neg((\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B))$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

7. $(\mathscr{A} \leftrightarrow \neg\mathscr{A})$

   Is it a sentence? ________ If 'no', then *this* is a sentence: ____

B. MAIN OPERATORS. What is the main operator of the following sentences of SL? What is (are) the subsentence(s) on which the main operator is operating?

1. $\neg A \leftrightarrow (B \rightarrow C)$

   Main Operator: ____ Subsentence(s): ____________ (and ____________)

2. $(C \leftrightarrow A) \vee B$

   Main Operator: ____ Subsentence(s): ____________ (and ____________)

3. $A \rightarrow (B \rightarrow C)$

   Main Operator: ____ Subsentence(s): ____________ (and ____________)

4. $(A \wedge B) \vee C$

   Main Operator: ____ Subsentence(s): ____________ (and ____________)

A. TRUTH-VALUE OF SL SENTENCES. Suppose that A is false, B is true, and C is false. Then, are the following sentences of SL true or false?

1. ____ $\neg A \leftrightarrow (B \rightarrow C)$
2. ____ $\neg(A \leftrightarrow (B \rightarrow C))$
3. ____ $(C \leftrightarrow A) \vee B$
4. ____ $C \leftrightarrow (A \vee B)$
5. ____ $A \rightarrow (B \rightarrow C)$
6. ____ $(A \rightarrow B) \rightarrow C$

B. TRUTH-TABLES. Write out the full truth-table for the following sentences of SL.

1. $(A \rightarrow Z) \vee (Z \rightarrow A)$

2. $(\neg P \wedge (Q \rightarrow P)) \rightarrow Q$

C. TRANSLATIONS. Let B = 'Bob is hungry', L = 'Lucy is impatient', and C = 'Carl wears purple'. Then, translate the following sentences of SL into English.

1. $B \wedge \neg C$ ____
2. $\neg L \rightarrow (\neg B \vee \neg C)$ ____
3. $\neg L \rightarrow \neg(B \wedge C)$ ____

D. TRANSLATIONS. Translate the following English sentences into SL, using the following symbolization key: A = 'Abelard loved Heloise', B = 'Abelard loved philosophy', C = 'Heloise loved philosophy', and D = 'Heloise loved Abelard'.

1. Abelard either loved Heloise or philosophy.
2. If Abelard didn't love philosophy, then he didn't love Heloise, either.
3. If Heloise didn't love philosophy, then Abelard didn't love her.
4. Abelard loved Heloise only if she loved either philosophy or him.
5. If Abelard loved Heloise, then she loved neither philosophy nor him.

Part 1: Invalidity of Argument Forms

For each of the argument forms below, provide a substitution instance which proves that the argument form is invalid (that is: provide a substitution instance with all true premises and a false conclusion). Note: your substitution instances should only involve statements which your recitation leader should know to be true or false.

1. Some Ps are R
   All Qs are R
   ∴ Some Ps are Q

2. Either $\mathscr{A}$ or $\mathscr{B}$
   If $\mathscr{B}$, then $\mathscr{C}$
   ∴ $\mathscr{C}$

3. x is taller than y
   z is taller than y
   ∴ x is taller than z

4. All Qs are R
   No P is Q
   ∴ No P is R

Part 2: Sentences of SL

5. Which of the following expressions are sentences of SL? For each letter, write 'sentence' if it is a sentence of SL, and write 'not a sentence' if it is not a sentence of SL. (Note: for this exercise, suspend our informal convention of dropping the outermost parenthases).

   (a) $(\neg S)$
   (b) $(\neg S) \rightarrow (\neg T)$
   (c) $(\neg A \leftrightarrow \neg\neg\neg X)$
   (d) $\neg(A \vee B \vee C)$
   (e) $(A \wedge (B \vee C) \rightarrow (D \wedge E))$
   (f) $\neg(a \leftrightarrow \neg(b \leftrightarrow (c \leftrightarrow d)))$
   (g) $(X \vee \neg\neg\neg(Y \rightarrow (\neg Z \vee (W \wedge (X \wedge Y)))))$
   (h) $((\mathscr{A} \vee \neg\mathscr{B}) \rightarrow (\mathscr{A} \leftrightarrow \mathscr{B}))$
   (i) $(((A \vee A) \vee A) \vee (A \vee (A \vee A)))$
   (j) $\neg(\neg(C \vee D) \leftrightarrow (\neg A \wedge B))$

Part 3: Translation into SL

6. Using the following symbolization key, translate the English sentences below into SL.

A :	Albert jogs
B :	Bob swims
C :	Carol jogs
H :	Albert is healthy
L :	Bob is lazy
M :	Carol is a marathon runner

(a) Bob swims, unless he's lazy.

(b) If Bob is not lazy, then he swims.

(c) Carol is a marathon runner if and only if she jogs.

(d) Albert isn't healthy and he doesn't jog.

(e) Albert is healthy, but he doesn't jog.

(f) Carol isn't a marathon runner if she doesn't jog.

(g) Carol is a marathon runner only if Albert and her both jog.

(h) If Carol is a marathon runner and Albert is healthy, then Carol and Albert both jog.

(i) Neither Albert nor Carol jog, though Bob isn't lazy and swims.

Part 4: Translation into English

7. Using the following symbolization key, translate each of the following sentences of SL into idiomatic English.

A :	Abelard loves Heloise
H :	Heloise loves Abelard
P :	Heloise loves Philosophy
Q :	Abelard loves Philosophy
M :	Abelard is a monk
N :	Heloise is a nun

(a) $A \rightarrow P$

(b) $H \leftrightarrow Q$

(c) $M \vee H$

(d) $(P \wedge Q) \rightarrow (A \wedge H)$

(e) $(M \vee N) \rightarrow (P \wedge Q)$

(f) $P \rightarrow \neg Q$

(g) $\neg(P \vee Q)$

(h) $\neg P \wedge \neg Q$

(i) $\neg(P \wedge Q)$

(j) $\neg P \vee \neg Q$

(k) $(\neg P \wedge \neg Q) \rightarrow \neg(A \vee H)$

Part 1: Invalidity of Argument Forms

For each of the argument forms below, provide a substitution instance which proves that the argument form is invalid (that is: provide a substitution instance with all true premises and a false conclusion). Note: your substitution instances should only involve statements which your recitation leader should know to be true or false.

1. Some Ps are R
   All Qs are R
   ∴ Some Ps are Q

   Some Republican Senators are Senators. [true]
   All Democratic Senators are Senators. [true]
   ∴ Some Republican Senators are Democratic Senators. [false]

2. Either $\mathscr{A}$ or $\mathscr{B}$
   If $\mathscr{B}$, then $\mathscr{C}$
   ∴ $\mathscr{C}$

   Either Trump won or Clinton won. [true]
   If Clinton won, then a Democrat won. [true]
   ∴ A Democrat won. [false]

3. x is taller than y
   z is taller than y
   ∴ x is taller than z

   Trump is taller than a mouse. [true]
   The Empire State Building is taller than a mouse. [true]
   ∴ Trump is taller than the Empire State Building. [false]

4. All Qs are R
   No P is Q
   ∴ No P is R

   All Democratic Senators are Senators. [true]
   No Republican Senator is a Democratic Senator. [true]
   ∴ No Republican Senator is a Senator. [false]

Part 2: Sentences of SL

5. Which of the following expressions are sentences of SL? For each letter, write 'sentence' if it is a sentence of SL, and write 'not a sentence' if it is not a sentence of SL. (Note: for this exercise, suspend our informal convention of dropping the outermost parenthases).

   (a) $(\neg S)$
   not a sentence ('$\neg S$' is a sentence.)

   (b) $(\neg S) \rightarrow (\neg T)$
   not a sentence ('$(\neg S \rightarrow \neg T)$' is a sentence.)

   (c) $(\neg A \leftrightarrow \neg\neg\neg X)$
   sentence

   (d) $\neg(A \vee B \vee C)$
   not a sentence ('$\neg(A \vee (B \vee C))$' is a sentence.)

   (e) $(A \wedge (B \vee C) \rightarrow (D \wedge E))$
   not a sentence ('$((A \wedge (B \vee C)) \rightarrow (D \wedge E))$' is a sentence.)

   (f) $\neg(a \leftrightarrow \neg(b \leftrightarrow (c \leftrightarrow d)))$
   not a sentence ('$\neg(A \leftrightarrow \neg(B \leftrightarrow (C \leftrightarrow D)))$' is a sentence.)

   (g) $(X \vee \neg\neg\neg(Y \rightarrow (\neg Z \vee (W \wedge (X \wedge Y)))))$
   sentence

   (h) $((\mathscr{A} \vee \neg\mathscr{B}) \rightarrow (\mathscr{A} \leftrightarrow \mathscr{B}))$
   not a sentence ('$((A \vee \neg B) \rightarrow (A \leftrightarrow B))$' is a sentence.)

   (i) $(((A \vee A) \vee A) \vee (A \vee (A \vee A)))$
   sentence

   (j) $\neg(\neg(C \vee D) \leftrightarrow (\neg A \wedge B))$
   sentence

Part 3: Translation into SL

6. Using the following symbolization key, translate the English sentences below into SL.

A :	Albert jogs
B :	Bob swims
C :	Carol jogs
H :	Albert is healthy
L :	Bob is lazy
M :	Carol is a marathon runner

   (a) Bob swims, unless he's lazy.

$$B \vee L$$

   (b) If Bob is not lazy, then he swims.

$$\neg L \rightarrow B$$

   (c) Carol is a marathon runner if and only if she jogs.

$$M \leftrightarrow C$$

   (d) Albert isn't healthy and he doesn't jog.

$$\neg H \wedge \neg A$$

(e) Albert is healthy, but he doesn't jog.

$$H \wedge \neg A$$

(f) Carol isn't a marathon runner if she doesn't jog.

$$\neg C \rightarrow \neg M$$

(g) Carol is a marathon runner only if Albert and her both jog.

$$M \rightarrow (A \wedge C)$$

(h) If Carol is a marathon runner and Albert is healthy, then Carol and Albert both jog.

$$(M \wedge H) \rightarrow (C \wedge A)$$

(i) Neither Albert nor Carol jog, though Bob isn't lazy and swims.

$$\neg(A \vee C) \wedge (\neg L \wedge B)$$

Part 4: Translation into English

7. Using the following symbolization key, translate each of the following sentences of SL into idiomatic English.

A :	Abelard loves Heloise
H :	Heloise loves Abelard
P :	Heloise loves Philosophy
Q :	Abelard loves Philosophy
M :	Abelard is a monk
N :	Heloise is a nun

(a) $A \rightarrow P$

Abelard loves Heloise only if she loves Philosophy.

(b) $H \leftrightarrow Q$

Heloise loves Abelard if and only if he loves Philosophy.

(c) $M \vee H$

Heloise loves Abelard unless he's a monk.

(d) $(P \wedge Q) \rightarrow (A \wedge H)$

If Abelard and Heloise both love Philosophy, then they love each other.

(e) $(M \vee N) \rightarrow (P \wedge Q)$

If either Abelard is a monk or Heloise is a nun, then they both love Philosophy

(f) $P \rightarrow \neg Q$

If Heloise loves Philosophy, then Abelard doesn't love Philosophy

(g) $\neg(P \vee Q)$

Neither Abelard nor Heloise loves Philosophy.

(h) $\neg P \wedge \neg Q$

Heloise doesn't love Philosophy, and neither does Abelard.

(i) $\neg(P \wedge Q)$

Not both Heloise and Abelard love Philosophy.

(j) $\neg P \vee \neg Q$

Either Heloise doesn't love Philosophy or Abelard doesn't.

(k) $(\neg P \wedge \neg Q) \rightarrow \neg(A \vee H)$

If both Abelard and Heloise don't love Philosophy, then neither of them loves the other.

1. Recall the meanings of the five logical operators '¬', '∧', '∨', '→', and '↔'.

$\mathscr{A}$	$\neg\mathscr{A}$
T	F
F	T

$\mathscr{A}$	$\mathscr{B}$	$\mathscr{A} \wedge \mathscr{B}$	$\mathscr{A} \vee \mathscr{B}$	$\mathscr{A} \to \mathscr{B}$	$\mathscr{A} \leftrightarrow \mathscr{B}$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

2. We can work out the truth-value (true or false) of complicated sentences by working out the truth-values of their sub-sentences. This allows us to determine the meaning of any arbitrary sentence of SL. For instance:

P	Q	$\neg$	P	$\wedge$	Q
T	T	F	T	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	F	F	F

(a) To understand this *truth-table*, consider the first row: it tells us that, if both 'P' and 'Q' are true, then the sub-sentence '$\neg P$' is false, and the sentence '$\neg P \wedge Q$' is false. The third row tells us that, if 'P' is false and 'Q' is true, then the sub-sentence '$\neg P$' is true, and the sentence '$\neg P \wedge Q$' is true.

(b) More generally, in every row, the truth-value of a sub-sentence is written beneath the *main operator* associated with that sub-sentence. To emphasize that '∧' is the main operator of the entire sentence, I've placed a box around that column of the truth-table. (You should do this on your problem sets and tests.)

3. In SL, the truth-value of non-atomic sentences is a function of the truth-values of the atomic sentences appearing therein.

▹ That's because the operators of SL are *truth-functional*. Not every operator is truth-functional in this way. Consider 'because'.

Because the logical operators of SL are truth-functional, then only thing we need to know in order to say whether a sentence of SL are true or false is what we will call a *valuation*.

> A VALUATION is an assignment of truth-values (either true or false) to the statement letters of SL.

▹ Each row of a truth-table represents a valuation. And the rows of the truth-table represent *all possible* valuations (for the statements letters appearing in the sentences of interest).

Entailment

4. If every valuation which makes all of $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ true also makes $\mathscr{C}$ true, then we will say that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *entail* $\mathscr{C}$.

> $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ ENTAIL $\mathscr{C}$ iff there is no valuation on which $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are all true and $\mathscr{C}$ is false.

▹ Notation: if $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ entail $\mathscr{C}$, then we will write '$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \models \mathscr{C}$'.

5. If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ entail $\mathscr{C}$, then the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is valid.

   (a) But just because $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *doesn't* entail $\mathscr{C}$, this doesn't tell us that the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is *invalid*

   (b) In general, every possible assignment of truth-values is represented in some valuation. But not every valuation represents some possible assignment of truth-values. There are some *bogus* valuations.

      ▹ For instance, let $S :=$ Sally is taller than John and let $J :=$ John is taller than Sally. Then, there will be a valuation which makes both S and J true—but there is no possibility in which S and J are both true.

   (c) In general, then, if we know something about *every* valuation, then we know something about every possibility.

   (d) But just because we know something about *some* valuation, this doesn't tell us anything about any possibility.

   (e) When we learn that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *doesn't* entail $\mathscr{C}$, we learn that there is *some* valuation which makes all of $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ true and which makes $\mathscr{C}$ false. But this doesn't tell us that there is a *possibility* like this. This valuation might be a *bogus* valuation.

   (f) On the other hand, if we learn that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *does* entail $\mathscr{C}$, we learn that *every* valuation either makes one of $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ false or else it makes $\mathscr{C}$ true. So we can infer that every possibility either makes one of $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ false, or else it makes $\mathscr{C}$ true. So we can infer that the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is valid.

Satisfiability

6. If there is a valuation which makes a collection of sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ true, then we will say that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are *satisfiable*. Otherwise, we will say that they are *unsatisfiable*.

> $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are SATSIFIABLE iff there is some valuation on which $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are all true.
>
> $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are UNSATSIFIABLE iff there is no valuation on which $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are all true.

   (a) If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are unsatisfiable, then they are jointly impossible.

      ▹ If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are unsatisfiable, then we know that *every* valuation makes one of them false.

   (b) However, just because $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are satisfiable, this doesn't mean that they are jointly possible.

      ▹ Learning that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are satisfiable only tells us that *there is some* valuation which makes them all true.

Tautologies and Contradictions

7. A sentence is a *tautology* iff it is true in every valuation. A sentence is a *contradiction* iff it is false in every valuation.

> $\mathscr{A}$ is a TAUTOLOGY iff $\mathscr{A}$ is true on every valuation.
>
> $\mathscr{A}$ is a CONTRADICTION iff $\mathscr{A}$ is false on every valuation.

   (a) If a sentence is a tautology, then it is a necessary truth. If it is a contradiction, then it is a necessary falsehood.

      ▹ If $\mathscr{A}$ is a tautology/contradiction, then it is true/false on *every* valuation.

   (b) However, just because a sentence is neither a tautology nor a contradiction, this doesn't mean that it's not a necessary truth or a necessary falsehood.

      ▹ Learning that $\mathscr{A}$ is *neither* a tautology *nor* a contradiction tells us simply that *there are some* valuations which make it true and some which make it false.

A. ENTAILMENT. Write out truth-tables to determine whether the following claims are true or false

1. $P \models P \vee Q$

2. $A \models B \rightarrow A$

3. $X \rightarrow Y \models Y \rightarrow X$

4. $S \leftrightarrow T$, $\neg S \models \neg T$

B. SATISFIABILITY. Write out truth-tables to determine whether the following collections of sentences of SL are satisfiable or unsatisfiable.

1. $P \rightarrow Q,\ \neg P \rightarrow Q,\ \neg Q$

2. $\neg(X \wedge Y),\ \neg(X \vee Y)$

C. TAUTOLOGIES AND CONTRADICTIONS. Write out truth-tables to determine whether the following sentences of SL are tautologies, contradictions, or neither tautologies nor contradictions.

1. $(P \rightarrow P) \rightarrow P$

2. $P \rightarrow \neg P$

3. $\neg(P \rightarrow P)$

Remember: when you write out your truth-tables, the statement letters on the left-hand-side should be in alphabetical order, and you should indicate in some way which column is under the main operator of every sentence.

Part 1: Entailment

Write out truth-tables to determine whether the following claims are true or false.

1. $P \to Q, Q \to P \models P \leftrightarrow Q$
2. $A \wedge \neg A \models Y$
3. $Y \models A \vee \neg A$
4. $P \vee Q,\ Q \to P \models P$
5. $P \to Q,\ Q \to R \models \neg R \to \neg P$

Part 2: Satisfiability

Write out truth-tables to determine whether the following collections of sentences of SL are satisfiable or unsatisfiable.

6. $(J \to J) \to H,\ \neg J,\ \neg H$
7. $(A \to B) \leftrightarrow (\neg B \vee A),\ A$
8. $A \to B,\ B \to C,\ C \to \neg A$

Part 3: Tautologies and Contradictions

Write out truth-tables to determine whether the following sentences of SL are tautologies, contradictions, or neither tautologies nor contradictions.

10. $\neg B \to [(B \vee D) \to D]$
11. $(M \leftrightarrow N) \wedge (M \leftrightarrow \neg N)$

Part 4: True/False

Are the following claims true or false?

12. If a sentence of SL is a contradiction, then it is a necessary falsehood.
13. If an argument is valid, then, when translated into SL, its premises will entail its conclusion.
14. If $\mathscr{P}$ doesn't entail $\mathscr{C}$, then the argument $\mathscr{P} \therefore \mathscr{C}$ is invalid.
15. If a collection of sentences is unsatisfiable, then they are jointly impossible.

Part 1: Entailment

Write out truth-tables to determine whether the following claims are true or false.

1. $P \to Q, Q \to P \models P \leftrightarrow Q$

P	Q	$P \to Q$			$Q \to P$			$P \leftrightarrow Q$		
T	T	T	**T**	T	T	**T**	T	T	**T**	T
T	F	T	**F**	F	F	**T**	T	T	**F**	F
F	T	F	**T**	T	T	**F**	F	F	**F**	T
F	F	F	**T**	F	F	**T**	F	F	**T**	F

The claim is true. There's no row in which '$P \to Q$' and '$Q \to P$' are both true and in which '$P \leftrightarrow Q$' is false, so '$P \to Q$' and '$Q \to P$' entail '$P \leftrightarrow Q$'.

2. $A \wedge \neg A \models Y$

A	Y	$A \wedge \neg A$				Y
T	T	T	**F**	F	T	**T**
T	F	T	**F**	F	T	**F**
F	T	F	**F**	T	F	**T**
F	F	F	**F**	T	F	**F**

The claim is true. There's no row in which '$A \wedge \neg A$' is true and in which 'Y' is false (since there's no row in which '$A \wedge \neg A$' is true), so '$A \wedge \neg A$' entails 'Y'.

3. $Y \models A \vee \neg A$

A	Y	Y	$A \vee \neg A$			
T	T	**T**	T	**T**	F	T
T	F	**F**	T	**T**	F	T
F	T	**T**	F	**T**	T	F
F	F	**F**	F	**T**	T	F

The claim is true. There's no row in which 'Y' is true and in which '$A \vee \neg A$' is false (since there's no row in which '$A \vee \neg A$' is true), so 'Y' entails '$A \vee \neg A$'.

4. $P \vee Q,\ Q \to P \models P$

P	Q	$P \vee Q$			$Q \to P$			P
T	T	T	**T**	T	T	**T**	T	**T**
T	F	T	**T**	F	F	**T**	T	**T**
F	T	F	**T**	T	T	**F**	F	**F**
F	F	F	**F**	F	F	**T**	F	**F**

The claim is true. There's no row in which '$P \vee Q$' and '$Q \to P$' are both true and in which 'P' is false, so '$P \vee Q$' and '$Q \to P$' entail 'P'.

5. $P \to Q,\ Q \to R \models \neg R \to \neg P$

P	Q	R	P	$\rightarrow$	Q	Q	$\rightarrow$	R	$\sim$	R	$\rightarrow$	$\sim$	P
T	T	T	T	T	T	T	T	T	F	T	T	F	T
T	T	F	T	T	T	T	F	F	T	F	F	F	T
T	F	T	T	F	F	F	T	T	F	T	T	F	T
T	F	F	T	F	F	F	T	F	T	F	F	F	T
F	T	T	F	T	T	T	T	T	F	T	T	T	F
F	T	F	F	T	T	T	F	F	T	F	T	T	F
F	F	T	F	T	F	F	T	T	F	T	T	T	F
F	F	F	F	T	F	F	T	F	T	F	T	T	F

The claim is true. There's no row in which '$P \rightarrow Q$' and '$Q \rightarrow R$' are both true and in which '$\neg R \rightarrow \neg P$' is false, so '$P \rightarrow Q$' and '$Q \rightarrow R$' entail '$\neg R \rightarrow \neg P$'.

Part 2: Satisfiability

Write out truth-tables to determine whether the following collections of sentences of SL are satisfiable or unsatisfiable.

6. $(J \rightarrow J) \rightarrow H,\ \neg J,\ \neg H$

H	J	$(J$	$\rightarrow$	$J)$	$\rightarrow$	H	$\neg$	J	$\sim$	H
T	T	T	T	T	T	T	F	T	F	T
T	F	F	T	F	T	T	T	F	F	T
F	T	T	T	T	F	F	F	T	T	F
F	F	F	T	F	F	F	T	F	T	F

There's no row in which all of the sentences are true, so the sentences are unsatisfiable.

7. $(A \rightarrow B) \leftrightarrow (\neg B \vee A),\ A$

A	B	$(A$	$\rightarrow$	$B)$	$\leftrightarrow$	$(\neg$	B	$\vee$	$A)$	A
T	T	T	T	T	T	F	T	T	T	T
T	F	T	F	F	F	T	F	T	T	T
F	T	F	T	T	F	F	T	F	F	F
F	F	F	T	F	T	T	F	T	F	F

Both sentences are true in the first row, so the sentences are satisfiable.

8. $A \rightarrow B,\ B \rightarrow C,\ C \rightarrow \neg A$

A	B	C	A	$\rightarrow$	B	B	$\rightarrow$	C	C	$\rightarrow$	$\neg$	A
T	T	T	T	T	T	T	T	T	T	F	F	T
T	T	F	T	T	T	T	F	F	F	T	F	T
T	F	T	T	F	F	F	T	T	T	F	F	T
T	F	F	T	F	F	F	T	F	F	T	F	T
F	T	T	F	T	T	T	T	T	T	T	T	F
F	T	F	F	T	T	T	F	F	F	T	T	F
F	F	T	F	T	F	F	T	T	T	T	T	F
F	F	F	F	T	F	F	T	F	F	T	T	F

All sentences are true in the 5th, 7th, and 8th rows. So the sentences are satisfiable.

Part 3: Tautologies and Contradictions

Write out truth-tables to determine whether the following sentences of SL are tautologies, contradictions, or neither tautologies nor contradictions.

10. $\neg B \rightarrow [(B \vee D) \rightarrow D]$

B	D	$\neg$	B	$\rightarrow$	[(B	$\vee$	D)	$\rightarrow$	D]
T	T	F	T	**T**	T	T	T	T	T
T	F	F	T	**T**	T	T	F	F	F
F	T	T	F	**T**	F	T	T	T	T
F	F	T	F	**T**	F	F	F	T	F

The sentence is true in every row, so it is a tautology.

11. $(M \leftrightarrow N) \wedge (M \leftrightarrow \neg N)$

M	N	(M	$\leftrightarrow$	N)	$\wedge$	(M	$\leftrightarrow$	$\neg$	N)
T	T	T	T	T	**F**	T	F	F	T
T	F	T	F	F	**F**	T	T	T	F
F	T	F	F	T	**F**	F	T	F	T
F	F	F	T	F	**F**	F	F	T	F

The sentence is false in every row, so it is a contradiction.

Part 4: True/False

Are the following claims true or false?

12. If a sentence of SL is a contradiction, then it is a necessary falsehood.

    True. If a sentence is false on every valuation, then it must be false in every possibility.

13. If an argument is valid, then, when translated into SL, its premises will entail its conclusion.

    False. Some valid arguments have premises that don't entail their conclusions.

14. If $\mathscr{P}$ doesn't entail $\mathscr{C}$, then the argument $\mathscr{P}\ \therefore\ \mathscr{C}$ is invalid.

    False. Some valid arguments have premises that don't entail their conclusions.

15. If a collection of sentences is unsatisfiable, then they are jointly impossible.

    True. If every valuation makes one of the sentences false, then every possibility must make one of the sentences false.

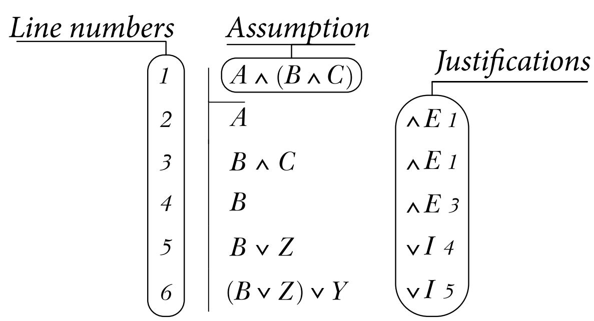
1. With truth-tables, we are able to prove whether an argument's premises entail its conclusion or not. But this can be prohibitively difficult. Consider the argument of SL:

$$(P \leftrightarrow Q) \rightarrow R,\ R \leftrightarrow S,\ S \leftrightarrow T,\ T \leftrightarrow U,\ U \leftrightarrow V,\ \neg V \ \therefore\ (P \wedge \neg Q) \vee (\neg P \wedge Q)$$

This argument's premises entail its conclusion, but verifying this with truth-tables would require a truth table with $2^7 = 128$ rows. In the next few classes, we'll learn a method of constructing proofs which will allow us to establish more easily that some sentences of SL entail some other sentence of SL.

2. Some preliminary orientation: a natural deduction proof will contain:
   - a certain number of lines, each one numbered.
   - at the top of the proof, some *assumptions*
   - on each line of the proof beneath the assumptions, a sentence of SL, along with a *justification*, explaining why we are allowed to write that sentence down on that line.

   Here's a sample SL natural deduction proof with these elements clearly labeled:



3. In order for a natural deduction proof to be *legal*,
   - the symbols appearing on each line must be sentences of SL (or a special symbol, ⊥, which we'll meet later).
   - each line which is not an assumption must *follow from* the lines cited in the justification, according to the rule cited in the justification.
   - the lines cited in the justification must *precede* the lines on which the justification is written.
     - Only lines which are *accessible* may be cited; and only the preceding lines are *accessible*. (We'll give a more careful definition of accessibility later on.)

4. Notation: if there is a legal SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its final line,[2] then I will write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \ \vdash \mathscr{C}$$

5. For each logical operator of SL, there will be one *introduction rule* for that operator, and one *elimination rule* for that operator. For instance, for conjunction, we will have:

[2] I'll get more careful about this in a later class.

Conjunction Introduction ($\wedge I$)

	$\mathscr{A}$
	$\mathscr{B}$
▹	$\mathscr{A} \wedge \mathscr{B}$

Conjunction Elimination ($\wedge E$)

	$\mathscr{A} \wedge \mathscr{B}$
▹	$\mathscr{A}$
▹	$\mathscr{B}$

(a) *Conjunction Introduction* says: if you have accessible a line on which '$\mathscr{A}$' is written, and you have accessible a line on which '$\mathscr{B}$' is written, then you may write down '$\mathscr{A} \wedge \mathscr{B}$'. (It doesn't matter whether the line on which '$\mathscr{A}$' is written precedes the line on which '$\mathscr{B}$' is written or not.)

- ▹ When you do so, you should write out, in the justification, the rule that you're using, '$\wedge I$', and cite the line on which '$\mathscr{A}$' appears as well as the line on which '$\mathscr{B}$' appears.

(b) *Conjunction Elimination* says: if you have accessible a line on which '$\mathscr{A} \wedge \mathscr{B}$' is written, then may write down '$\mathscr{A}$'. Also: if you have accessible a line on which '$\mathscr{A} \wedge \mathscr{B}$' is written, then you may write down '$\mathscr{B}$'.

- ▹ When you do so, you should write out, in the justification, that rule that you're using, '$\wedge E$', and cite the line on which '$\mathscr{A} \wedge \mathscr{B}$' appears.

(c) Note: these rules *cannot* be applied to sub-sentences. In order for $\wedge E$ to be applied, you must apply it to a sentence whose *main operator* is the conjunction.

6. For today, we'll want to get familiar with the following rules, too (there will be more rules on Wednesday):

Disjunction Introduction ($\vee I$)

	$\mathscr{A}$
▹	$\mathscr{A} \vee \mathscr{B}$
▹	$\mathscr{B} \vee \mathscr{A}$

Conditional Elimination ($\rightarrow E$)

	$\mathscr{A} \rightarrow \mathscr{B}$
	$\mathscr{A}$
▹	$\mathscr{B}$

Biconditional Elimination ($\leftrightarrow E$)

	$\mathscr{A} \leftrightarrow \mathscr{B}$
	$\mathscr{A}$
▹	$\mathscr{B}$

	$\mathscr{A} \leftrightarrow \mathscr{B}$
	$\mathscr{B}$
▹	$\mathscr{A}$

7. Here's a legal proof using all five rules:

1	$A \wedge B$	
2	$(A \vee C) \rightarrow (B \leftrightarrow D)$	
3	A	$\wedge E$ 1
4	$A \vee C$	$\vee I$ 3
5	$B \leftrightarrow D$	$\rightarrow E$ 2, 4
6	B	$\wedge E$ 1
7	D	$\leftrightarrow E$ 5, 6
8	$B \wedge D$	$\wedge I$ 6, 7

Complete the following natural deduction proofs. (For this part, you need only use the rules $\wedge E$, $\wedge I$, $\vee I$, $\rightarrow E$, and $\leftrightarrow E$).

1 | P Prove: $(P \vee Q) \vee R$
2 |

1 | $X \wedge Y$
2 | $Y \leftrightarrow (X \rightarrow Z)$ Prove: $(Z \wedge Y) \vee (Y \rightarrow X)$
3 |

1 | H
2 | $H \leftrightarrow \neg J$ Prove: $(\neg J \vee X) \wedge H$
3 |

1 | P
2 | $P \leftrightarrow (P \leftrightarrow (P \leftrightarrow Q))$ Prove: Q
3 |

1 | X
2 | $X \to (X \to (Y \land (Y \to A)))$ Prove: A
3 |

1 | $\neg\neg A$
2 | $\neg\neg A \to B$ Prove: $(Q \leftrightarrow P) \lor (\neg\neg A \land B)$
3 |

1 | $A \land ((B \lor C) \to D)$
2 | B Prove: D
3 |

1 | O
2 | $\neg X \lor (Y \leftrightarrow Z)$
3 | $(\neg X \lor (Y \leftrightarrow Z)) \to (O \to X)$ Prove: X
4 |

1. A new rule, and an illustrative natural deduction proof which utilizes it:

Conditional Introduction ($\rightarrow I$)

n.	$\mathscr{A}$	
	$\vdots$	
k.	$\mathscr{B}$	
▹	$\mathscr{A} \rightarrow \mathscr{B}$	$\rightarrow I$ n–k

1	$P \wedge (P \rightarrow Q)$	
2	P	$\wedge E$ 1
3	$P \rightarrow Q$	$\wedge E$ 1
4	Q	$\rightarrow E$ 2, 3
5	$(P \wedge (P \rightarrow Q)) \rightarrow Q$	$\rightarrow I$ 1–4

(a) In the proof on the right-hand-side, call the vertical line descending from line 1 down to line 4 a *scope line*. And say that the sentences of SL which appear next to that scope line lie *within the scope of the assumption* $P \wedge (P \rightarrow Q)$.

(b) In a natural deduction proof, anything which lies within the scope of some assumptions is *entailed by* those assumptions.

(c) Notice that the sentence on the final line, '$(P \wedge (P \rightarrow Q)) \rightarrow Q$', lies outside the scope of *any* assumptions. So it is entailed by *no* assumptions. We have proven it without relying upon any assumptions at all. I'll express this by writing:

$$\vdash (P \wedge (P \rightarrow Q)) \rightarrow Q$$

(d) Proving a sentence from *no* assumptions shows that it is entailed by anything—that is, it shows that it is a tautology. So the proof above shows that '$(P \wedge (P \rightarrow Q)) \rightarrow Q$' is a tautology.

2. Another new rule, and an illustrative proof:

Assumption

▹	$\mathscr{A}$	

1	$P \rightarrow Q$	
2	$Q \rightarrow R$	
3	P	Ass. ($\rightarrow I$)
4	Q	$\rightarrow E$ 1, 3
5	R	$\rightarrow E$ 2, 4
6	$P \rightarrow R$	$\rightarrow I$ 3–5

(a) The new assumption and the vertical scope line descending from it is called a *subproof*.

(b) Notice that, in the rule $\rightarrow I$, we don't cite any particular line—instead, we cite *the entire subproof* running from line n through line k.

(c) The rule *Assumption* tells you that you are allowed to start a subproof *whenever you wish*. You are also allowed to end a subproof *whenever you wish*.

3. Once we have multiple subproofs, we need to think more carefully about which we can cite and which we cannot. The following proof, for instance, is *not* legal (good thing, too, since $A \rightarrow B$ doesn't entail B):

1	$A \rightarrow B$	
2	A	Ass.
3	B	$\rightarrow E$ 1, 2 ← **MISTAKE!** (line 2 is not accessible at line 3)

(a) Here are the rules for when a sentence or a subproof is accessible to be cited in a justification:

> A *sentence* is accessible at your line iff it comes before your line and does not lie inside of a completed subproof.
>
> A *subproof* is accessible at your line iff it comes before your line and does not lie inside of a completed subproof.

4. For illustration, for the following proof I'll say which lines and which subproofs are accessible at each line.

1	$C \wedge Z$	
2	$\quad A$	Ass. $(\rightarrow I)$
3	$\quad\quad B$	Ass. $(\rightarrow I)$
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \rightarrow C$	$\rightarrow I$ 3–4
6	$A \rightarrow (B \rightarrow C)$	$\rightarrow I$ 2–5
7	$\quad B \rightarrow A$	Ass. $(\rightarrow I)$
8	$\quad C$	$\wedge E$ 1
9	$(B \rightarrow A) \rightarrow C$	$\rightarrow I$ 7–8

Line	Accessible Lines/ Subproofs	Inaccessible Lines/ Subproofs (above you)
2	1	
3	1, 2	
4	1, 2, 3	
5	1, 2, 3–4	3, 4
6	1, 2–5	2, 3, 4, 5, 3–4
7	1, 2–5, 6	2, 3, 4, 5, 3–4
8	1, 2–5, 6, 7	2, 3, 4, 5, 3–4
9	1, 2–5, 6, 7–8	2, 3, 4, 5, 7, 8, 3–4

5. More rules:

Biconditional Introduction $(\leftrightarrow I)$

n.	$\quad \mathscr{A}$	
	$\quad \vdots$	
k.	$\quad \mathscr{B}$	
m.	$\quad \mathscr{B}$	
	$\quad \vdots$	
l.	$\quad \mathscr{A}$	
▹	$\mathscr{A} \leftrightarrow \mathscr{B}$	$\leftrightarrow I$ n–k, m–l

1	$A \wedge B$	
2	$\quad A$	Ass. $(\leftrightarrow I)$
3	$\quad B$	$\wedge E$ 1
4	$\quad B$	Ass. $(\leftrightarrow I)$
5	$\quad A$	$\wedge E$ 1
6	$A \leftrightarrow B$	$\leftrightarrow I$ 2–3, 4–5

Disjunction Elimination $(\vee E)$

n.	$\mathscr{A} \vee \mathscr{B}$	
i.	$\quad \mathscr{A}$	
	$\quad \vdots$	
j.	$\quad \mathscr{C}$	
k.	$\quad \mathscr{B}$	
	$\quad \vdots$	
l.	$\quad \mathscr{C}$	
▹	$\mathscr{C}$	$\vee E$ n, i–j, k–l

1	$A \vee B$	
2	$A \rightarrow C$	
3	$B \rightarrow D$	
4	$\quad A$	Ass. $(\vee E)$
5	$\quad C$	$\rightarrow E$ 2, 4
6	$\quad C \vee D$	$\vee I$ 5
7	$\quad B$	Ass. $(\vee E)$
8	$\quad D$	$\rightarrow E$ 3, 7
9	$\quad C \vee D$	$\vee I$ 8
10	$C \vee D$	$\vee E$ 1, 4–6, 7–9

6. The final four rules of our proof system have to do with negation and an additional symbol which we'll introduce—'$\bot$', which is pronounced '*Contradiction!*'. You get to write this down whenever you have two sentences, one of which is the negation of the other.

Contradiction Introduction ($\bot I$)

	$\mathscr{A}$	
	$\neg\mathscr{A}$	
▹	$\bot$	

Negation Introduction ($\neg I$)

n.	$\mid$ $\mathscr{A}$	
	$\mid$ $\vdots$	
k.	$\mid$ $\bot$	
▹	$\neg\mathscr{A}$	$\neg I$ n–k

Negation Elimination ($\neg E$)

n.	$\mid$ $\neg\mathscr{A}$	
	$\mid$ $\vdots$	
k.	$\mid$ $\bot$	
▹	$\mathscr{A}$	$\neg E$ n–k

1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\mid$ $\neg Y \wedge X$	Ass. ($\neg I$)
4	$\mid$ $\neg Y$	$\wedge E$ 3
5	$\mid$ $\bot$	$\bot I$ 2, 4
6	$\neg(\neg Y \wedge X)$	$\neg I$ 3–5

1	P	
2	$\mid$ $\neg P$	Ass. ($\neg I$)
3	$\mid$ $\bot$	$\bot I$ 1, 2
4	$\neg\neg P$	$\neg I$ 2–3

1	$\neg\neg A$	
2	$\mid$ $\neg A$	Ass. ($\neg E$)
3	$\mid$ $\bot$	$\bot I$ 1, 2
4	A	$\neg E$ 2–3

7. The final rule tells us that *everything follows from a contradiction*. If you have a contradiction, then you can get anything you want.

Contradiction Elimination ($\bot E$)

	$\bot$	
▹	$\mathscr{A}$	

1	$P \rightarrow \neg P$	
2	$\mid$ P	Ass. ($\rightarrow I$)
3	$\mid$ $\neg P$	$\rightarrow E$ 1, 2
4	$\mid$ $\bot$	$\bot I$ 2, 3
5	$\mid$ Q	$\bot E$ 4
6	$P \rightarrow Q$	$\rightarrow I$ 2–5

8. More sample proofs:

1	$\mid$ G	Ass. ($\leftrightarrow I$)
2	$\mid$ $\mid$ $\neg G$	Ass. ($\neg I$)
3	$\mid$ $\mid$ $\bot$	$\bot I$ 1, 2
4	$\mid$ $\neg\neg G$	$\neg I$ 2–3
5	$\mid$ $\neg\neg G$	Ass. ($\leftrightarrow I$)
6	$\mid$ $\mid$ $\neg G$	Ass. ($\neg E$)
7	$\mid$ $\mid$ $\bot$	$\bot I$ 5, 6
8	$\mid$ G	$\neg E$ 6–7
9	$G \leftrightarrow \neg\neg G$	$\leftrightarrow I$ 1–4, 5–8

1	$\mid$ $\neg(A \vee \neg A)$	Ass. ($\neg E$)
2	$\mid$ $\mid$ A	Ass. ($\neg I$)
3	$\mid$ $\mid$ $A \vee \neg A$	$\vee I$ 2
4	$\mid$ $\mid$ $\bot$	$\bot I$ 1, 3
5	$\mid$ $\neg A$	$\neg I$ 2–4
6	$\mid$ $A \vee \neg A$	$\vee I$ 5
7	$\mid$ $\bot$	$\bot I$ 1, 6
8	$A \vee \neg A$	$\neg E$ 1–7

Complete the following natural deduction proofs.

1 | Q Prove: $P \to Q$
2 |

1 | $A \to B$ Prove: $A \to (A \wedge B)$
2 |

1 | $P \to R$
2 | $\neg R$ Prove: $\neg P$
3 |

1 | $P \vee Q$
2 | $P \to Q$ Prove: Q
3 |

1 | $M \leftrightarrow P$

2 | $\neg P$ Prove: $\neg M$

3 |

1 | X

2 | $X \rightarrow Y$ Prove: $X \leftrightarrow Y$

3 |

1 | $\neg A \wedge B$

2 | $B \rightarrow A$ Prove: X

3 |

1 | $P \leftrightarrow Q$

2 | $Q \rightarrow (P \rightarrow W)$ Prove: $P \rightarrow W$

3 |

Assumption

▷ | | $\mathcal{A}$

Conjunction Introduction ($\wedge I$)

| $\mathcal{A}$
| $\mathcal{B}$
▷ | $\mathcal{A} \wedge \mathcal{B}$

Conjunction Elimination ($\wedge E$)

| $\mathcal{A} \wedge \mathcal{B}$
▷ | $\mathcal{A}$
▷ | $\mathcal{B}$

Disjunction Introduction ($\vee I$)

| $\mathcal{A}$
▷ | $\mathcal{A} \vee \mathcal{B}$
▷ | $\mathcal{B} \vee \mathcal{A}$

Disjunction Elimination ($\vee E$)

| $\mathcal{A} \vee \mathcal{B}$
| | $\mathcal{A}$
| | $\vdots$
| | $\mathcal{C}$
| | $\mathcal{B}$
| | $\vdots$
| | $\mathcal{C}$
▷ | $\mathcal{C}$

Conditional Introduction ($\rightarrow I$)

| | $\mathcal{A}$
| | $\vdots$
| | $\mathcal{B}$
▷ | $\mathcal{A} \rightarrow \mathcal{B}$

Conditional Elimination ($\rightarrow E$)

| $\mathcal{A} \rightarrow \mathcal{B}$
| $\mathcal{A}$
▷ | $\mathcal{B}$

Biconditional Introduction ($\leftrightarrow I$)

| | $\mathcal{A}$
| | $\vdots$
| | $\mathcal{B}$
| | $\mathcal{B}$
| | $\vdots$
| | $\mathcal{A}$
▷ | $\mathcal{A} \leftrightarrow \mathcal{B}$

Biconditional Elimination ($\leftrightarrow E$)

| $\mathcal{A} \leftrightarrow \mathcal{B}$
| $\mathcal{A}$
▷ | $\mathcal{B}$

| $\mathcal{A} \leftrightarrow \mathcal{B}$
| $\mathcal{B}$
▷ | $\mathcal{A}$

Negation Introduction ($\neg I$)

| | $\mathcal{A}$
| | $\vdots$
| | $\bot$
▷ | $\neg\mathcal{A}$

Negation Elimination ($\neg E$)

| | $\neg\mathcal{A}$
| | $\vdots$
| | $\bot$
▷ | $\mathcal{A}$

Contradiction Introduction ($\bot I$)

| $\mathcal{A}$
| $\neg\mathcal{A}$
▷ | $\bot$

Contradiction Elimination ($\bot E$)

| $\bot$
▷ | $\mathcal{A}$

Part A

The following two 'proofs' are *incorrect*. Explain the mistakes they make.

1	$(\neg L \wedge A) \vee L$	
2	$\quad \neg L \wedge A$	
3	$\quad \neg L$	$\wedge E$ 3
4	$\quad A$	$\wedge E$ 1
5	$\quad L$	
6	$\quad \bot$	$\bot I$ 3, 5
7	$\quad A$	$\bot E$ 6
8	A	$\vee E$ 1, 2–4, 5–7

1	$A \wedge (B \wedge C)$	
2	$(B \vee C) \rightarrow D$	
3	B	$\wedge E$ 1
4	$B \vee C$	$\vee I$ 3
5	D	$\rightarrow E$ 4, 2

Part B

The following three proofs are missing their justifications (rule and line numbers). Add them to turn them into *boda fide* proofs.

1	$P \wedge S$
2	$S \rightarrow R$
3	P
4	S
5	R
6	$R \vee E$

[3] All of these exercises come from *Forall x: An Introduction to Formal Logic*, by P. D. Magnus and Tim Button.

1	$A \to D$
2	$\quad A \wedge B$ [assumption]
3	$\quad A$
4	$\quad D$
5	$\quad D \vee E$
6	$(A \wedge B) \to (D \vee E)$

1	$\neg L \to (J \vee L)$
2	$\neg L$
3	$J \vee L$
4	$\quad J$ [assumption]
5	$\quad J \wedge J$
6	$\quad J$
7	$\quad L$ [assumption]
8	$\quad \bot$
9	$\quad J$
10	J

Part C

Give a proof for each of the following arguments. **Use the proof checker at jdmitrigallow.com/proofs to make sure that your proofs are correct.**

1. $J \to \neg J \therefore \neg J$
2. $Q \to (Q \wedge \neg Q) \therefore \neg Q$
3. $A \to (B \to C) \therefore (A \wedge B) \to C$
4. $K \wedge L \therefore K \leftrightarrow L$
5. $(C \wedge D) \vee E \therefore E \vee D$
6. $A \leftrightarrow B, B \leftrightarrow C \therefore A \leftrightarrow C$
7. $\neg F \to G, F \to H \therefore G \vee H$
8. $(Z \wedge K) \vee (K \wedge M), K \to D \therefore D$
9. $P \wedge (Q \vee R), P \to \neg R \therefore Q \vee E$
10. $S \leftrightarrow T \therefore S \leftrightarrow (T \vee S)$
11. $\neg(P \to Q) \therefore \neg Q$
12. $\neg(P \to Q) \therefore P$

A.

1	$(\neg L \wedge A) \vee L$	
2	$\quad \neg L \wedge A$	
3	$\quad \neg L$	$\wedge E$ 3 ⟵ Mistake! Should be '$\wedge E$ 2'
4	$\quad A$	$\wedge E$ 1 ⟵ Mistake! Should be '$\wedge E$ 2'
5	$\quad L$	
6	$\quad \bot$	$\bot I$ 3, 5 ⟵ Mistake! Line 3 is not accessible
7	$\quad A$	$\bot E$ 6
8	A	$\vee E$ 1, 2–4, 5–7

1	$A \wedge (B \wedge C)$	
2	$(B \vee C) \rightarrow D$	
3	B	$\wedge E$ 1 ⟵ Mistake! 'B' is not a conjunct of '$A \wedge (B \wedge C)$'
4	$B \vee C$	$\vee I$ 3
5	D	$\rightarrow E$ 4, 2

B. $P \wedge S, S \rightarrow R \therefore R \vee E$

1	$P \wedge S$	
2	$S \rightarrow R$	
3	P	$\wedge E$ 1
4	S	$\wedge E$ 1
5	R	$\rightarrow E$ 2, 4
6	$R \vee E$	$\vee I$ 5

$A \rightarrow D \therefore (A \wedge B) \rightarrow (D \vee E)$

1	$A \rightarrow D$	
2	$\quad A \wedge B$	A ($\rightarrow I$)
3	$\quad A$	$\wedge E$ 2
4	$\quad D$	$\rightarrow E$ 1, 3
5	$\quad D \vee E$	$\vee I$ 4
6	$(A \wedge B) \rightarrow (D \vee E)$	$\rightarrow I$ 2–5

$\neg L \to (J \vee L), \neg L \therefore J$

1	$\neg L \to (J \vee L)$	
2	$\neg L$	
3	$J \vee L$	$\to E$ 1, 2
4	$\quad J$	A ($\vee E$)
5	$\quad J \wedge J$	$\wedge I$ 4, 4
6	$\quad J$	$\wedge E$ 5
7	$\quad L$	A ($\vee E$)
8	$\quad \bot$	$\bot I$ 2, 7
9	$\quad J$	$\bot E$ 8
10	J	$\vee E$ 3, 4–6, 7–9

C. 1. $J \to \neg J \ \vdash \neg J$

1	$J \to \neg J$	
2	$\quad J$	A ($\neg I$)
3	$\quad \neg J$	$\to E$ 1, 2
4	$\quad \bot$	$\bot I$ 2, 3
5	$\neg J$	$\neg I$ 2–4

2. $Q \to (Q \wedge \neg Q) \ \vdash \neg Q$

1	$Q \to (Q \wedge \neg Q)$	
2	$\quad Q$	A($\neg I$)
3	$\quad Q \wedge \neg Q$	$\to E$ 1, 2
4	$\quad \neg Q$	$\wedge E$ 3
5	$\quad \bot$	$\bot I$ 2, 4
6	$\neg Q$	$\neg I$ 2–5

3. $A \to (B \to C) \ \vdash (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$\quad A \wedge B$	A($\to I$)
3	$\quad A$	$\wedge E$ 2
4	$\quad B \to C$	$\to E$ 1, 3
5	$\quad B$	$\wedge E$ 2
6	$\quad C$	$\to E$ 4, 5
7	$(A \wedge B) \to C$	$\to I$ 2–6

4. $K \wedge L \vdash K \leftrightarrow L$

1	$K \wedge L$	
2	$\quad K$	A($\leftrightarrow I$)
3	$\quad L$	$\wedge E$ 1
4	$\quad L$	A($\leftrightarrow I$)
5	$\quad K$	$\wedge E$ 1
6	$K \leftrightarrow L$	$\leftrightarrow I$ 2–3, 4–5

5. $(C \wedge D) \vee E \vdash E \vee D$

1	$(C \wedge D) \vee E$	
2	$\quad C \wedge D$	A ($\vee E$)
3	$\quad D$	$\wedge E$ 2
4	$\quad E \vee D$	$\vee I$ 3
5	$\quad E$	A ($\vee E$)
6	$\quad E \vee D$	$\vee I$ 5
7	$E \vee D$	$\vee E$ 1, 2–4, 5–6

6. $A \leftrightarrow B, B \leftrightarrow C \vdash A \leftrightarrow C$

1	$A \leftrightarrow B$	
2	$B \leftrightarrow C$	
3	$\quad A$	A ($\leftrightarrow I$)
4	$\quad B$	$\leftrightarrow E$ 1, 3
5	$\quad C$	$\leftrightarrow E$ 2, 4
6	$\quad C$	A ($\leftrightarrow E$)
7	$\quad B$	$\leftrightarrow E$ 2, 6
8	$\quad A$	$\leftrightarrow E$ 1, 7
9	$A \leftrightarrow C$	$\leftrightarrow I$ 3–5, 6–8

7. $\neg F \to G, F \to H \;\vdash\; G \lor H$

1	$\neg F \to G$	
2	$F \to H$	
3	$\quad \neg(G \lor H)$	A ($\neg E$)
4	$\quad\quad F$	A ($\neg I$)
5	$\quad\quad H$	$\to E$ 2, 4
6	$\quad\quad G \lor H$	$\lor I$ 5
7	$\quad\quad \bot$	$\bot I$ 3, 6
8	$\quad \neg F$	$\neg I$ 4–7
9	$\quad G$	$\to E$ 1, 8
10	$\quad G \lor H$	$\lor I$ 9
11	$\quad \bot$	$\bot I$ 3, 10
12	$G \lor H$	$\neg E$ 3–11

8. $(Z \land K) \lor (K \land M), K \to D \;\vdash\; D$

1	$(Z \land K) \lor (K \land M)$	
2	$K \to D$	
3	$\quad Z \land K$	A ($\lor E$)
4	$\quad K$	$\land E$ 3
5	$\quad D$	$\to E$ 2, 4
6	$\quad K \land M$	A ($\lor E$)
7	$\quad K$	$\land E$ 6
8	$\quad D$	$\to E$ 2, 7
9	D	$\lor E$ 1, 3–5, 6–8

9. $P \wedge (Q \vee R), P \rightarrow \neg R \ \vdash\ Q \vee E$

1	$P \wedge (Q \vee R)$	
2	$\underline{P \rightarrow \neg R}$	
3	P	$\wedge E$ 1
4	$\neg R$	$\rightarrow E$ 2, 3
5	$Q \vee R$	$\wedge E$ 1
6	$\quad \mid \underline{Q}$	A ($\vee E$)
7	$\quad \mid Q \vee E$	$\vee I$ 6
8	$\quad \mid \underline{R}$	A ($\vee E$)
9	$\quad \mid \bot$	$\bot I$ 4, 8
10	$\quad \mid Q \vee E$	$\bot E$ 9
11	$Q \vee E$	$\vee E$ 5, 6–7, 8–10

10. $S \leftrightarrow T \ \vdash\ S \leftrightarrow (T \vee S)$

1	$\underline{S \leftrightarrow T}$	
2	$\quad \mid \underline{S}$	A ($\leftrightarrow I$)
3	$\quad \mid T \vee S$	$\vee I$ 2
4	$\quad \mid \underline{T \vee S}$	A ($\leftrightarrow I$)
5	$\quad \mid \quad \mid \underline{T}$	A ($\vee E$)
6	$\quad \mid \quad \mid S$	$\leftrightarrow E$ 1, 5
7	$\quad \mid \quad \mid \underline{S}$	A ($\leftrightarrow I$)
8	$\quad \mid \quad \mid S$	R 7
9	$\quad \mid S$	$\vee E$ 4, 5–6, 7–8
10	$S \leftrightarrow (T \vee S)$	$\leftrightarrow I$ 2–3, 4–9

11. $\neg(P \to Q) \vdash \neg Q$

1	$\neg(P \to Q)$	
2	$\quad Q$	A ($\neg I$)
3	$\quad\quad P$	A ($\to I$)
4	$\quad\quad Q$	R 2
5	$\quad P \to Q$	$\to I$ 3–4
6	$\quad \bot$	$\bot I$ 1, 5
7	$\neg Q$	$\neg I$ 2–6

12. $\neg(P \to Q) \vdash P$

1	$\neg(P \to Q)$	
2	$\quad \neg P$	A ($\neg E$)
3	$\quad\quad P$	A ($\to I$)
4	$\quad\quad \bot$	$\bot I$ 2, 3
5	$\quad\quad Q$	$\bot E$ 4
6	$\quad P \to Q$	$\to I$ 3–5
7	$\quad \bot$	$\bot I$ 1, 6
8	P	$\neg E$ 2–7

Proof Strategies

1. Constructing a natural deduction proof requires some measure of *creativity*. There is no algorithmic procedure I can teach—no step-by-step instructions I can give—which will always lead you to construct a legal proof. However, I can give you some tips to follow. If you practice with these tips—and I mean *really practice*, not just do the proofs on the problem set—then you will learn how to prove things in the natural deduction system. The first, and most important tip, is this:

> **Tip #0:** Try to form a 'big picture' strategy for completing the proof.

2. Our rules are helpfully named. Each logical operator has an introduction rule and an associated elimination rule. My first two tips are to use the main operators of the sentences you have available, and the sentences you're trying to derive, to guide your choice of strategy.

> **Tip #1:** Try to use the introduction rule for the main operator of the sentence you want to write down.

- ▹ For instance, if you want to prove $\neg P$ from $P \rightarrow \neg P$, it makes sense to try to use the *introduction* rule for negation. So it makes sense to adopt the following 'big picture' strategy:

$P \rightarrow \neg P$			
	P	Ass. ($\neg I$)	Goal: '$\bot$'
	$\vdots$		
	$\bot$		
$\neg P$		$\neg I$	

  (At the beginning, you may not see how to fill in the ellipses—but you don't have to worry about that now; you're just forming a 'big picture' strategy.)

> **Tip #2:** Try to use the elimination rule for the main operator of a sentence you have accessible.

- ▹ For instance: if you are trying to prove Q from $P \vee Q$ and $P \rightarrow Q$, then it makes sense to use the elimination rule for '$\vee$' and adopt the following 'big picture' strategy:

$P \vee Q$			
$P \rightarrow Q$			
	P	Ass. ($\vee E$)	Goal: 'Q'
	$\vdots$		
	Q		
	Q	Ass. ($\vee E$)	Goal: 'Q'
	$\vdots$		
	Q		
Q		$\vee E$	

3. Notice that, when I was constructing the 'big picture' strategies above, I wrote in a 'goal' next to each new sub-proof. That's my third tip: as you go along and form new sub-proofs, you should also explicitly form new sub-goals for what you want to accomplish within that subproof. This will help you structure your thinking about the proof.

> **Tip #3:** As you form new sub-proofs, explicitly write down your new sub-goals within that sub-proof.

4. My next tip is to never forget the power of a contradiction. Once you have a contradiction, you can use the rule '$\bot E$' to get *literally anything you want*.

> **Tip #4:** Keep in mind: once you have a contradiction, $\bot$, you can get *anything you want*.

5. Next tip: think about the *meaning* of the sentences you're writing down. If your strategy requires you to derive $\mathscr{B}$ from $\mathscr{A}$, think about whether or not $\mathscr{A}$ *entails* $\mathscr{B}$.

> **Tip #5:** When you're forming a 'big picture' strategy, think about whether the things that strategy requires you to derive *are actually entailed* by your assumptions. If they are not, abandon that strategy.

▹ For instance, suppose you're trying to prove $\neg A \vee \neg B$ from $\neg(A \wedge B)$. And suppose you adopt the following 'big picture' strategy:

$\neg(A \wedge B)$		
$\quad A$	Ass. ($\neg I$)	Goal: '$\bot$'
$\quad \vdots$		
$\quad \bot$		
$\neg A$	$\neg I$	
$\neg A \vee \neg B$	$\vee I$	

Stop and think about that for a second—the proof system won't allow you to prove anything that isn't entailed by your assumptions. But this strategy requires us to prove $\neg A$ from $\neg(A \wedge B)$. But $\neg(A \wedge B)$ just tells us that it's not the case that *both* A and B are true. So for all the assumption tells us, it could be that A is true and B is false. So $\neg A$ isn't entailed by $\neg(A \wedge B)$. So we won't be able to prove it. So this strategy is doomed. We should abandon it.

6. What do we do, then? How *can* we prove $\neg A \vee \neg B$ from $\neg(A \wedge B)$? The next tip says: if you're stuck, and you can't find any other 'big picture' strategy: try negation elimination.

> **Tip #6**: If you see no other 'big picture' strategies, then try a negation elimination strategy, where you assume the negation of the thing you want to prove, and attempt to derive a contradiction.

▹ If anything works, negation elimination will.

7. One final tip: you will often end up stuck, unsure about how to proceed. In those circumstances, the absolute worst thing to do is to stare at a blank page, expecting that something will jump out at you. If you've no ideas, just do *something*—*anything*. Make a bold assumption, see what follows from it. Oftentimes, it's only after embarking on a chain of reasoning that you can begin to see where it's taking you. So if you're not sure how to get started: don't worry about it, and just *get started*.

> **Tip #7:** If you don't know how to proceed, and you don't have any ideas, ***just do something***.

1. A sentence of SL, $\mathscr{A}$, is a *theorem* iff it is possible to prove $\mathscr{A}$ from no assumptions.

> $\mathscr{A}$ is a THEOREM iff there is a legal SL proof which has $\mathscr{A}$ on its final line, and $\mathscr{A}$ appears outside of the scope of any assumption.

- ▹ If there is a proof like this, then we will write:

$$\vdash \mathscr{A}$$

2. For instance, this proof shows that $A \rightarrow (B \rightarrow A)$ is a theorem:

1	A	Ass. ($\rightarrow I$)
2	$\quad B$	Ass. ($\rightarrow I$)
3	$\quad A \wedge A$	$\wedge I$ 1
4	$\quad A$	$\wedge E$ 3
5	$B \rightarrow A$	$\rightarrow I$ 2–4
6	$A \rightarrow (B \rightarrow A)$	$\rightarrow I$ 1–5

3. Two sentences, $\mathscr{A}$ and $\mathscr{B}$, are *provably equivalent* iff it is possible to prove $\mathscr{B}$ from $\mathscr{A}$ and it is possible to prove $\mathscr{A}$ from $\mathscr{B}$.

> $\mathscr{A}$ and $\mathscr{B}$ are *provably equivalent* iff 1) there is a legal proof with $\mathscr{A}$ as its only assumption and $\mathscr{B}$ written on its main scope line, and 2) there is a legal proof with $\mathscr{B}$ as its only assumption and $\mathscr{A}$ written on its main scope line.

- ▹ If $\mathscr{A}$ and $\mathscr{B}$ are provably equivalent, then we write:

$$\mathscr{A} \dashv\vdash \mathscr{B}$$

4. For instance, this pair of proofs show that $A \leftrightarrow B$ and $(A \vee B) \rightarrow (A \wedge B)$ are provably equivalent.

1	$A \leftrightarrow B$	
2	$\quad A \vee B$	Ass. ($\rightarrow I$)
3	$\quad\quad A$	Ass. ($\vee E$)
4	$\quad\quad B$	$\leftrightarrow E$ 1, 3
5	$\quad\quad A \wedge B$	$\wedge I$ 3, 4
6	$\quad\quad B$	Ass. ($\vee E$)
7	$\quad\quad A$	$\leftrightarrow E$ 1, 6
8	$\quad\quad A \wedge B$	$\wedge I$ 6, 7
9	$\quad A \wedge B$	$\vee E$ 2, 3–5, 6–8
10	$(A \vee B) \rightarrow (A \wedge B)$	$\rightarrow I$ 2–9

1	$(A \vee B) \rightarrow (A \wedge B)$	
2	$\quad A$	Ass. ($\leftrightarrow I$)
3	$\quad A \vee B$	$\vee I$ 2
4	$\quad A \wedge B$	$\rightarrow E$ 1, 3
5	$\quad B$	$\wedge E$ 4
6	$\quad B$	Ass. ($\leftrightarrow I$)
7	$\quad A \vee B$	$\vee I$ 6
8	$\quad A \wedge B$	$\rightarrow E$ 1, 7
9	$\quad A$	$\wedge E$ 8
10	$A \leftrightarrow B$	$\leftrightarrow I$ 2–5, 6–9

5. A collection of sentences, $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are *provably inconsistent* iff it is possible to prove $\bot$ from them.

> $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are PROVABLY INCONSISTENT iff there is a legal proof which has $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\bot$ written down on its main scope line.

   (a) If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are provably inconsistent, then we write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \ \vdash \ \bot$$

6. For instance, the following proof shows that $A \leftrightarrow B$ and $B \leftrightarrow \neg A$ are provably inconsistent:

1	$A \leftrightarrow B$	
2	$B \leftrightarrow \neg A$	
3	$\quad A$	Ass. $(\neg I)$
4	$\quad B$	$\leftrightarrow E$ 1, 3
5	$\quad \neg A$	$\leftrightarrow E$ 2, 4
6	$\quad \bot$	$\bot I$ 3, 5
7	$\neg A$	$\neg I$ 3–6
8	B	$\leftrightarrow E$ 2, 7
9	A	$\leftrightarrow E$ 1, 8
10	$\bot$	$\bot I$ 7, 9

***Reiteration* (R)**

	$\mathscr{A}$
▷	$\mathscr{A}$

***Modus Tollens* (MT)**

	$\mathscr{A} \to \mathscr{B}$
	$\neg\mathscr{B}$
▷	$\neg\mathscr{A}$

***Disjunctive Syllogism* (DS)**

	$\mathscr{A} \lor \mathscr{B}$
	$\neg\mathscr{A}$
▷	$\mathscr{B}$

	$\mathscr{A} \lor \mathscr{B}$
	$\neg\mathscr{B}$
▷	$\mathscr{A}$

***Law of Excluded Middle* (*LEM*)**

	$\mathscr{A}$ (assumption)
	⋮
	$\mathscr{B}$
	$\neg\mathscr{A}$ (assumption)
	⋮
	$\mathscr{B}$
▷	$\mathscr{B}$

***Double Negation Elimination* (DNE)**

	$\neg\neg\mathscr{A}$
▷	$\mathscr{A}$

***DeMorgan's Rules* (DeM)**

$\neg(\mathscr{A} \land \mathscr{B})$ ◁▷ $\neg\mathscr{A} \lor \neg\mathscr{B}$

$\neg(\mathscr{A} \lor \mathscr{B})$ ◁▷ $\neg\mathscr{A} \land \neg\mathscr{B}$

Part A

The following proofs are missing their justifications (rule and line numbers). Add them wherever they are required.

1	$W \to \neg B$
2	$A \wedge W$
3	$B \vee (J \wedge K)$
4	W
5	$\neg B$
6	$J \wedge K$
7	K

1	$L \leftrightarrow \neg O$
2	$L \vee \neg O$
3	$\quad \neg L$
4	$\quad \neg O$
5	$\quad L$
6	$\quad \bot$
7	$\neg\neg L$
8	L

1	$Z \to (C \wedge \neg N)$
2	$\neg Z \to (N \wedge \neg C)$
3	$\quad \neg(N \vee C)$
4	$\quad \neg N \wedge \neg C$
5	$\quad \neg N$
6	$\quad \neg C$
7	$\quad\quad Z$
8	$\quad\quad C \wedge \neg N$
9	$\quad\quad C$
10	$\quad\quad \bot$
11	$\quad \neg Z$
12	$\quad N \wedge \neg C$
13	$\quad N$
14	$\quad \bot$
15	$\neg\neg(N \vee C)$
16	$N \vee C$

Part B

Give a proof for each of these arguments.

1. $E \vee F, F \vee G, \neg F \therefore E \wedge G$

2. $M \vee (N \to M) \therefore \neg M \to \neg N$
3. $(M \vee N) \wedge (O \vee P), N \to P, \neg P \therefore M \wedge O$
4. $(X \wedge Y) \vee (X \wedge Z), \neg(X \wedge D), D \vee M \therefore M$

Part C

Show that each of the following sentences is a theorem:

1. $O \to O$
2. $N \vee \neg N$
3. $J \leftrightarrow [J \vee (L \wedge \neg L)]$
4. $[(A \to B) \to A] \to A$

Part D

Provide proofs to show each of the following:

1. $C \to (E \wedge G), \neg C \to G \vdash G$
2. $M \wedge (\neg N \to \neg M) \vdash (N \wedge M) \vee \neg M$
3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \to M) \vdash Y \to Z$
4. $(W \vee X) \vee (Y \vee Z), X \to Y, \neg Z \vdash W \vee Y$

Part E

Show that each of the following pairs of sentences are provably equivalent:

1. $R \leftrightarrow E, E \leftrightarrow R$
2. $G, \neg\neg\neg\neg G$
3. $T \to S, \neg S \to \neg T$
4. $U \to I, \neg(U \wedge \neg I)$
5. $\neg(C \to D), C \wedge \neg D$

Part 1

A.

1	$W \to \neg B$	
2	$A \wedge W$	
3	$B \vee (J \wedge K)$	
4	W	$\wedge E$ 2
5	$\neg B$	$\to E$ 1, 4
6	$J \wedge K$	DS 3, 5
7	K	$\wedge E$ 6

1	$L \leftrightarrow \neg O$	
2	$L \vee \neg O$	
3	$\quad \neg L$	Ass. ($\neg I$)
4	$\quad \neg O$	DS 2, 3
5	$\quad L$	$\leftrightarrow E$ 1, 4
6	$\quad \bot$	$\bot I$ 3, 5
7	$\neg\neg L$	$\neg I$ 3–6
8	L	DNE 7

1	$Z \to (C \wedge \neg N)$	
2	$\neg Z \to (N \wedge \neg C)$	
3	$\quad \neg(N \vee C)$	Ass. ($\neg I$)
4	$\quad \neg N \wedge \neg C$	DeM 3
5	$\quad \neg N$	$\wedge E$ 4
6	$\quad \neg C$	$\wedge E$ 4
7	$\quad\quad Z$	Ass. ($\neg I$)
8	$\quad\quad C \wedge \neg N$	$\to E$ 1, 7
9	$\quad\quad C$	$\wedge E$ 8
10	$\quad\quad \bot$	$\bot I$ 6, 9
11	$\quad \neg Z$	$\neg I$ 7–10
12	$\quad N \wedge \neg C$	$\to E$ 2, 11
13	$\quad N$	$\wedge E$ 12
14	$\quad \bot$	$\bot I$ 5, 13
15	$\neg\neg(N \vee C)$	$\neg I$ 3–14
16	$N \vee C$	DNE 15

B. 1. $E \vee F, F \vee G, \neg F \therefore E \wedge G$

1	$E \vee F$	
2	$F \vee G$	
3	$\neg F$	
4	E	DS 1, 3
5	G	DS 2, 3
6	$E \wedge G$	$\wedge I$ 4, 5

2. $M \vee (N \to M) \therefore \neg M \to \neg N$

1	$M \vee (N \to M)$	
2	$\quad \neg M$	Ass ($\to I$)
3	$\quad N \to M$	DS 1, 2
4	$\quad\quad N$	Ass ($\neg I$)
5	$\quad\quad M$	$\to E$ 3, 4
6	$\quad\quad \bot$	$\bot I$ 2, 5
7	$\quad \neg N$	$\neg I$ 4–6
8	$\neg M \to \neg N$	$\to I$ 2–7

3. $(M \vee N) \wedge (O \vee P), N \to P, \neg P \therefore M \wedge O$

1	$(M \vee N) \wedge (O \vee P)$	
2	$N \to P$	
3	$\neg P$	
4	$\neg N$	MT 2, 3
5	$M \vee N$	$\wedge E$ 1
6	M	DS 4, 5
7	$O \vee P$	$\wedge E$ 1
8	O	DS 3, 7
9	$M \wedge O$	$\wedge I$ 6, 8

4. $(X \wedge Y) \vee (X \wedge Z), \neg(X \wedge D), D \vee M \therefore M$

1	$(X \wedge Y) \vee (X \wedge Z)$	
2	$\neg(X \wedge D)$	
3	$D \vee M$	
4	$\quad \neg M$	Ass ($\neg E$)
5	$\quad D$	DS 3, 4
6	$\quad\quad \neg D$	Ass ($\neg I$)
7	$\quad\quad \bot$	$\bot I$ 5, 6
8	$\quad \neg\neg D$	$\neg I$ 6–7
9	$\quad \neg X \vee \neg D$	DeM 2
10	$\quad \neg X$	DS 8, 9
11	$\quad \neg X \vee \neg Y$	$\vee I$ 10
12	$\quad \neg(X \wedge Y)$	DeM 11
13	$\quad X \wedge Z$	DS 1, 12
14	$\quad X$	$\wedge E$ 13
15	$\quad \bot$	$\bot I$ 10, 14
16	M	$\neg E$ 4–15

Part 2

A. 1. $\vdash O \rightarrow O$

1	$\quad O$	Ass ($\rightarrow I$)
2	$\quad O$	R 1
3	$O \rightarrow O$	$\rightarrow I$ 1–2

2. $\vdash N \vee \neg N$

1	$\quad N$	Ass (LEM)
2	$\quad N \vee \neg N$	$\vee I$ 1
3	$\quad \neg N$	Ass (LEM)
4	$\quad N \vee \neg N$	$\vee I$ 3
5	$N \vee \neg N$	LEM 1–2, 3–4

3. $\vdash J \leftrightarrow [J \vee (L \wedge \neg L)]$

1	$\quad J$	Ass ($\leftrightarrow I$)
2	$\quad J \vee (L \wedge \neg L)$	$\vee I$ 1
3	$\quad J \vee (L \wedge \neg L)$	Ass ($\leftrightarrow I$)
4	$\quad\quad L \wedge \neg L$	Ass ($\neg I$)
5	$\quad\quad L$	$\wedge E$ 4
6	$\quad\quad \neg L$	$\wedge E$ 4
7	$\quad\quad \bot$	$\bot I$ 5, 6
8	$\quad \neg(L \wedge \neg L)$	$\neg I$ 4–7
9	$\quad J$	DS 3, 8
10	$J \leftrightarrow [J \vee (L \wedge \neg L)]$	$\leftrightarrow I$ 1–2, 3–9

4. $\vdash ((A \to B) \to A) \to A$

1	$\quad (A \to B) \to A$	Ass ($\to I$)
2	$\quad\quad \neg A$	Ass ($\neg E$)
3	$\quad\quad\quad A$	Ass ($\to I$)
4	$\quad\quad\quad \bot$	$\bot I$ 2, 3
5	$\quad\quad\quad B$	$\bot E$ 4
6	$\quad\quad A \to B$	$\to I$ 3–5
7	$\quad\quad A$	$\to E$ 1, 6
8	$\quad\quad \bot$	$\bot I$ 2, 7
9	$\quad A$	$\neg E$ 2–8
10	$((A \to B) \to A) \to A$	$\to I$ 1–9

B. 1. $C \to (E \wedge G), \neg C \to G \ \vdash G$

1	$\quad C \to (E \wedge G)$	
2	$\quad \neg C \to G$	
3	$\quad\quad C$	Ass (LEM)
4	$\quad\quad E \wedge G$	$\to E$ 1, 3
5	$\quad\quad G$	$\wedge E$ 4
6	$\quad\quad \neg C$	Ass (LEM)
7	$\quad\quad G$	$\to E$ 2, 6
8	$\quad G$	LEM 3–5, 6–7

2. $M \wedge (\neg N \rightarrow \neg M) \;\vdash\; (N \wedge M) \vee \neg M$

1	$M \wedge (\neg N \rightarrow \neg M)$	
2	$\quad \neg((N \wedge M) \vee \neg M)$	Ass ($\neg E$)
3	$\quad \neg(N \wedge M) \wedge \neg\neg M$	DeM 2
4	$\quad \neg(N \wedge M)$	$\wedge E$ 3
5	$\quad \neg N \vee \neg M$	DeM 4
6	$\quad \neg\neg M$	$\wedge E$ 3
7	$\quad \neg N$	DS 5, 6
8	$\quad \neg N \rightarrow \neg M$	$\wedge E$ 1
9	$\quad \neg M$	$\rightarrow E$ 7, 8
10	$\quad M$	$\wedge E$ 1
11	$\quad \bot$	$\bot I$ 9, 10
12	$(N \wedge M) \vee \neg M$	$\neg E$ 2–11

3. $(Z \wedge K) \leftrightarrow (Y \wedge M), D \wedge (D \rightarrow M) \;\vdash\; Y \rightarrow Z$

1	$(Z \wedge K) \leftrightarrow (Y \wedge M)$	
2	$D \wedge (D \rightarrow M)$	
3	D	$\wedge E$ 2
4	$D \rightarrow M$	$\wedge E$ 2
5	M	$\rightarrow E$ 3, 4
6	$\quad Y$	Ass ($\rightarrow I$)
7	$\quad Y \wedge M$	$\wedge I$ 5, 6
8	$\quad Z \wedge K$	$\leftrightarrow E$ 1, 7
9	$\quad Z$	$\wedge E$ 8
10	$Y \rightarrow Z$	$\rightarrow I$ 6–9

4. $(W \vee X) \vee (Y \vee Z)$, $X \rightarrow Y$, $\neg Z$ $\vdash$ $W \vee Y$

1	$(W \vee X) \vee (Y \vee Z)$	
2	$X \rightarrow Y$	
3	$\neg Z$	
4	$\quad \neg(W \vee Y)$	Ass. $(\neg E)$
5	$\quad \neg W \wedge \neg Y$	DeM 4
6	$\quad \neg Y$	$\wedge E$ 5
7	$\quad \neg X$	MT
8	$\quad \neg W$	$\wedge E$ 5
9	$\quad \neg W \wedge \neg X$	$\wedge I$ 6, 8
10	$\quad \neg(W \vee X)$	DeM 9
11	$\quad Y \vee Z$	DS 1, 10
12	$\quad Y$	DS 3, 11
13	$\quad \bot$	$\bot I$ 6, 12
14	$W \vee Y$	$\neg E$ 4–13

C. 1. $R \leftrightarrow E$ $\dashv\vdash$ $E \leftrightarrow R$

1	$R \leftrightarrow E$	
2	$\quad E$	Ass. $(\leftrightarrow I)$
3	$\quad R$	$\leftrightarrow E$ 1, 2
4	$\quad R$	Ass. $(\leftrightarrow I)$
5	$\quad E$	$\leftrightarrow E$ 1, 4
6	$E \leftrightarrow R$	$\leftrightarrow I$ 2–3, 4–5

1	$E \leftrightarrow R$	
2	$\quad R$	Ass. $(\leftrightarrow I)$
3	$\quad E$	$\leftrightarrow E$ 1, 2
4	$\quad E$	Ass. $(\leftrightarrow I)$
5	$\quad R$	$\leftrightarrow E$ 1, 4
6	$R \leftrightarrow E$	$\leftrightarrow I$ 2–3, 4–5

2. G $\dashv\vdash$ $\neg\neg\neg\neg G$

1	G	
2	$\quad \neg\neg\neg G$	Ass. $(\neg I)$
3	$\quad\quad \neg G$	Ass. $(\neg I)$
4	$\quad\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\neg G$	$\neg I$ 3–4
6	$\quad \bot$	$\bot I$ 2, 5
7	$\neg\neg\neg\neg G$	$\neg I$ 2–6

1	$\neg\neg\neg\neg G$	
2	$\quad \neg G$	Ass. $(\neg E)$
3	$\quad\quad \neg\neg\neg G$	Ass. $(\neg E)$
4	$\quad\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\neg G$	$\neg E$ 3–4
6	$\quad \bot$	$\bot I$ 2, 5
7	G	$\neg E$ 2–6

3. $T \to S \dashv\vdash \neg S \to \neg T$

1	$T \to S$	
2	$\neg S$	Ass. ($\to I$)
3	$\neg T$	MT 1, 2
4	$\neg S \to \neg T$	$\to I$ 2–3

1	$\neg S \to \neg T$	
2	T	Ass. ($\to I$)
3	$\neg S$	Ass. ($\neg E$)
4	$\neg T$	$\to E$ 1, 3
5	$\bot$	$\bot I$ 2, 4
6	S	$\neg E$ 3–5
7	$T \to S$	$\to I$ 2–6

4. $U \to I \dashv\vdash \neg(U \wedge \neg I)$

1	$U \to I$	
2	$U \wedge \neg I$	Ass. ($\neg I$)
3	U	$\wedge E$ 2
4	I	$\to E$ 1, 3
5	$\neg I$	$\wedge E$ 2
6	$\bot$	$\bot I$ 4, 5
7	$\neg(U \wedge \neg I)$	$\neg I$ 2–6

1	$\neg(U \wedge \neg I)$	
2	U	Ass. ($\to I$)
3	$\neg I$	Ass. ($\neg E$)
4	$U \wedge \neg I$	$\wedge I$ 2, 3
5	$\bot$	$\bot I$ 1, 4
6	I	$\neg E$ 3–5
7	$U \to I$	$\to I$ 2–6

5. $\neg(C \to D) \dashv\vdash C \wedge \neg D$

1	$\neg(C \to D)$	
2	$\neg C$	Ass. ($\neg E$)
3	C	Ass. ($\to I$)
4	$\bot$	$\bot I$ 2, 3
5	D	$\bot E$ 4
6	$C \to D$	$\to I$ 3–5
7	$\bot$	$\bot I$
8	C	$\neg E$ 2–7
9	D	Ass. ($\neg I$)
10	C	Ass. ($\to I$)
11	D	R 9
12	$C \to D$	$\to I$ 10–11
13	$\bot$	$\bot I$ 1, 12
14	$\neg D$	$\neg I$ 9–13
15	$C \wedge \neg D$	$\wedge I$ 8, 14

1	$C \wedge \neg D$	
2	$C \to D$	Ass. ($\neg I$)
3	C	$\wedge E$ 1
4	D	$\to E$ 2, 3
5	$\neg D$	$\wedge E$ 1
6	$\bot$	$\bot I$ 4, 5
7	$\neg(C \to D)$	$\neg I$ 2–6

If you'd like some additional natural deduction problems to practice with, here are some to play around with:

1. $A \to C \ \vdash (A \wedge B) \to C$
2. $A \to B \ \vdash A \to (A \wedge B)$
3. $\neg B \leftrightarrow A \ \vdash A \to \neg B$
4. $A \to \neg B,\ \neg B \to C \ \vdash A \to C$
5. $B \to (A \wedge \neg B) \ \vdash \neg B$
6. $A \leftrightarrow B,\ \neg A \ \vdash \neg B$
7. $A \to (B \wedge C),\ \neg C \ \vdash \neg A$
8. $D \ \vdash A \to [B \to (C \to D)]$
9. $A \leftrightarrow B,\ B \leftrightarrow C \ \vdash A \leftrightarrow C$
10. $A \to (B \to C),\ D \to B \ \vdash A \to (D \to C)$
11. $M \leftrightarrow P,\ \neg P \ \vdash \neg M$
12. $D \ \vdash A \to (B \to D)$
13. $A \to C,\ (\neg A \vee C) \to (D \to B) \ \vdash D \to B$
14. $\neg A \to \neg B,\ A \to C,\ B \vee D,\ D \to E \ \vdash E \vee C$
15. $\neg N,\ (\neg N \to L) \wedge [D \leftrightarrow (\neg N \vee A)] \ \vdash L \wedge D$
16. $\neg A \vee B,\ \neg A \to B,\ B \leftrightarrow C \ \vdash C$
17. $\neg A \leftrightarrow B \ \dashv\vdash A \leftrightarrow \neg B$
18. $P \to Q \ \dashv\vdash \neg P \vee Q$
19. $Q,\ \neg(P \to Q) \ \vdash \bot$
20. $A \leftrightarrow \neg B,\ B \leftrightarrow C,\ A \leftrightarrow C \ \vdash \bot$
21. $\neg(A \to B),\ \neg(B \to C) \ \vdash \bot$
22. $A \to B,\ \neg(B \wedge \neg C) \to A \ \vdash B$
23. $\neg A \to B,\ C \to \neg B,\ \neg(\neg C \wedge \neg A) \ \vdash A$
24. $A \vee (B \wedge C),\ C \to \neg A \ \vdash B \vee \neg C$
25. $(A \to B) \to \neg B \ \vdash \neg B$
26. $(A \vee B) \to C,\ (D \vee E) \to [(F \vee G) \to A] \ \vdash D \to (F \to C)$
27. $(F \vee G) \to (H \wedge I) \ \vdash \neg F \vee H$
28. $A \to \neg(B \vee C),\ (C \vee D) \to A,\ \neg F \to (D \wedge \neg E) \ \vdash B \to F$
29. $(A \wedge B) \leftrightarrow (A \vee B),\ C \wedge (C \leftrightarrow \neg\neg A) \ \vdash B$
30. $F \to (G \vee H),\ \neg(\neg F \vee H),\ \neg G \ \vdash H$
31. $\neg(A \to B) \wedge (C \wedge \neg D),\ (B \vee \neg A) \vee [(C \wedge E) \to D] \ \vdash \neg E$

Your challenge, should you choose to accept it, is to provide natural deduction proofs for the following arguments. You should feel free to use the derived rules. For each natural deduction, if you complete it correctly, you will earn the indicated number of points—these points will be added to your midterm grade (your grade on the midterm will be out of 100 points).

1. $\neg P \vee Q \;\therefore\; P \rightarrow Q$ (1/3 pt.)
2. $P \rightarrow Q \;\therefore\; \neg P \vee Q$ (1/3 pt.)
3. $\neg(J \rightarrow E) \;\therefore\; J$ (1/3 pt.)
4. $Q \leftrightarrow (Q \rightarrow \neg Q) \;\therefore\; A$ (1/3 pt.)
5. $J \;\therefore\; K \rightarrow (J \leftrightarrow K)$ (1/3 pt.)
6. $A \leftrightarrow B \;\therefore\; \neg A \leftrightarrow \neg B$ (1/3 pt.)
7. $A \leftrightarrow (B \leftrightarrow C),\ B \;\therefore\; A \leftrightarrow C$ (1 pt.)
8. $P \rightarrow R \;\therefore\; (P \rightarrow Q) \vee (Q \rightarrow R)$ (1 pt.)
9. $D \leftrightarrow E \;\therefore\; (D \wedge E) \vee (\neg D \wedge \neg E)$ (1 pt.)
10. $(D \wedge E) \vee (\neg D \wedge \neg E) \;\therefore\; D \leftrightarrow E$ (1 pt.)
11. $F \leftrightarrow \neg G \;\therefore\; \neg(F \leftrightarrow G)$ (1 pt.)
12. $\neg(F \leftrightarrow G) \;\therefore\; F \leftrightarrow \neg G$ (2 pts.)
13. $\neg(F \leftrightarrow G) \;\therefore\; (F \wedge \neg G) \vee (\neg F \wedge G)$ (3 pts.)
14. $(P \wedge Q) \rightarrow (R \vee S) \;\therefore\; (Q \rightarrow \neg P) \vee (\neg S \rightarrow R)$ (3 pts.)
15. $(Q \leftrightarrow R) \leftrightarrow (Q \leftrightarrow \neg R) \;\therefore\; A$ (5 pts.)

1. $\neg P \vee Q \ \therefore\ P \to Q$

1	$\neg P \vee Q$	
2	$\quad P$	Ass. (→I)
3	$\quad\quad \neg Q$	Ass. (¬E)
4	$\quad\quad \neg P$	DS 1, 3
5	$\quad\quad \bot$	⊥I 2, 4
6	$\quad Q$	¬E 3–5
7	$P \to Q$	→I 2–6

2. $P \to Q \ \therefore\ \neg P \vee Q$

1	$P \to Q$	
2	$\quad P$	Ass. (LEM)
3	$\quad Q$	→E 1, 2
4	$\quad \neg P \vee Q$	∨I 3
5	$\quad \neg P$	Ass. (LEM)
6	$\quad \neg P \vee Q$	∨I 5
7	$\neg P \vee Q$	LEM 2–4, 5–6

3. $\neg(J \to E) \ \therefore\ J$

1	$\neg(J \to E)$	
2	$\quad \neg J$	Ass. (¬E)
3	$\quad\quad J$	Ass. (→I)
4	$\quad\quad \bot$	⊥I 2, 3
5	$\quad\quad E$	⊥E 4
6	$\quad J \to E$	→I 3–5
7	$\quad \bot$	⊥I 1, 6
8	J	¬E 2–7

4. $Q \leftrightarrow (Q \rightarrow \neg Q) \therefore A$

1	$Q \leftrightarrow (Q \rightarrow \neg Q)$	
2	│ Q	Ass. ($\rightarrow$I)
3	│ $Q \rightarrow \neg Q$	$\leftrightarrow$E 1, 2
4	│ $\neg Q$	$\rightarrow$E 2, 3
5	$Q \rightarrow \neg Q$	$\rightarrow$I 2–4
6	Q	$\leftrightarrow$E 1, 5
7	$\neg Q$	$\rightarrow$E 5, 6
8	$\bot$	$\bot$I 6, 7
9	A	$\bot$E 8

5. $J \therefore K \rightarrow (J \leftrightarrow K)$

1	J	
2	│ K	Ass. ($\rightarrow$I)
3	│ │ J	Ass. ($\leftrightarrow$I)
4	│ │ K	R 2
5	│ │ K	Ass. ($\leftrightarrow$I)
6	│ │ J	R 1
7	│ $J \leftrightarrow K$	$\leftrightarrow$I 3–4, 5–6
8	$K \rightarrow (J \leftrightarrow K)$	$\rightarrow$I 2–7

6. $A \leftrightarrow B \therefore \neg A \leftrightarrow \neg B$

1	$A \leftrightarrow B$	
2	│ $\neg A$	Ass. ($\leftrightarrow$I)
3	│ │ B	Ass. ($\neg$I)
4	│ │ A	$\leftrightarrow$E 1, 3
5	│ │ $\bot$	$\bot$I 2,4
6	│ $\neg B$	$\neg$I 3–5
7	│ $\neg B$	Ass. ($\leftrightarrow$I)
8	│ │ A	Ass. ($\neg$I)
9	│ │ B	$\leftrightarrow$E 1, 8
10	│ │ $\bot$	$\bot$I 7, 9
11	│ $\neg A$	$\neg$I 8–10
12	$\neg A \leftrightarrow \neg B$	$\leftrightarrow$I 2–6, 7–11

7. $A \leftrightarrow (B \leftrightarrow C),\ B \therefore A \leftrightarrow C$

1	$A \leftrightarrow (B \leftrightarrow C)$	
2	B	
3	$\quad A$	Ass. ($\leftrightarrow$I)
4	$\quad B \leftrightarrow C$	$\leftrightarrow$E 1, 3
5	$\quad C$	$\leftrightarrow$E 2, 4
6	$\quad C$	Ass. ($\leftrightarrow$I)
7	$\quad\quad B$	Ass. ($\leftrightarrow$I)
8	$\quad\quad C$	R 6
9	$\quad\quad C$	Ass. ($\leftrightarrow$I)
10	$\quad\quad B$	R 2
11	$\quad B \leftrightarrow C$	$\leftrightarrow$I 7–8, 9–10
12	$\quad A$	$\leftrightarrow$E 1, 11
13	$A \leftrightarrow C$	$\leftrightarrow$I 3–5, 6–12

8. $P \rightarrow R \ \therefore\ (P \rightarrow Q) \vee (Q \rightarrow R)$

1	$P \rightarrow R$	
2	$\quad P$	Ass. (LEM)
3	$\quad\quad Q$	Ass. ($\rightarrow$I)
4	$\quad\quad R$	$\rightarrow$E 1, 2
5	$\quad Q \rightarrow R$	$\rightarrow$I 3–4
6	$\quad (P \rightarrow Q) \vee (Q \rightarrow R)$	$\vee$I 6
7	$\quad \neg P$	Ass. (LEM)
8	$\quad\quad P$	Ass. ($\rightarrow$I)
9	$\quad\quad \bot$	$\bot$I 7, 8
10	$\quad\quad Q$	$\bot$E 9
11	$\quad P \rightarrow Q$	$\rightarrow$I 8–10
12	$\quad (P \rightarrow Q) \vee (Q \rightarrow R)$	$\vee$I 11
13	$(P \rightarrow Q) \vee (Q \rightarrow R)$	LEM 2–6, 7–12

9. $D \leftrightarrow E \ \therefore\ (D \wedge E) \vee (\neg D \wedge \neg E)$

1	$D \leftrightarrow E$	
2	| D	Ass. (LEM)
3	| E	$\leftrightarrow$E 1, 2
4	| $D \wedge E$	$\wedge$I 2, 3
5	| $(D \wedge E) \vee (\neg D \wedge \neg E)$	$\vee$I 4
6	| $\neg D$	Ass. (LEM)
7	| | E	Ass. ($\neg$I)
8	| | D	$\leftrightarrow$E 1, 7
9	| | $\bot$	$\bot$I 6, 8
10	| $\neg E$	$\neg$I 7–9
11	| $\neg D \wedge \neg E$	$\wedge$I 6, 10
12	| $(D \wedge E) \vee (\neg D \wedge \neg E)$	$\vee$I 11
13	$(D \wedge E) \vee (\neg D \wedge \neg E)$	LEM 2–5, 6–12

10. $(D \wedge E) \vee (\neg D \wedge \neg E) \ \therefore\ D \leftrightarrow E$

1	$(D \wedge E) \vee (\neg D \wedge \neg E)$	
2	| D	Ass. ($\leftrightarrow$I)
3	| | $\neg D \wedge \neg E$	Ass. ($\neg$I)
4	| | $\neg D$	$\wedge$E 3
5	| | $\bot$	$\bot$I 2, 4
6	| $\neg(\neg D \wedge \neg E)$	$\neg$I 3–5
7	| $D \wedge E$	DS 1, 6
8	| E	$\wedge$E 7
9	| E	Ass. ($\leftrightarrow$I)
10	| | $\neg D \wedge \neg E$	Ass. ($\neg$I)
11	| | $\neg E$	$\wedge$E 10
12	| | $\bot$	$\bot$I 9, 11
13	| $\neg(\neg D \wedge \neg E)$	$\neg$I 10–12
14	| $D \wedge E$	DS 1, 13
15	| D	$\wedge$E 14
16	$D \leftrightarrow E$	$\leftrightarrow$I 2–8, 9–15

11. $F \leftrightarrow \neg G \ \therefore\ \neg(F \leftrightarrow G)$

1	$F \leftrightarrow \neg G$	
2	$\quad F \leftrightarrow G$	Ass. $(\neg I)$
3	$\quad\quad G$	Ass. $(\neg I)$
4	$\quad\quad F$	$\leftrightarrow E$ 2, 4
5	$\quad\quad \neg G$	$\leftrightarrow E$ 1, 4
6	$\quad\quad \bot$	$\bot I$ 3, 5
7	$\quad \neg G$	$\neg I$ 3–6
8	$\quad F$	$\leftrightarrow E$ 1, 7
9	$\quad G$	$\leftrightarrow E$ 2, 8
10	$\quad \bot$	$\bot I$ 7, 9
11	$\neg(F \leftrightarrow G)$	$\neg I$ 2–10

12. $\neg(F \leftrightarrow G)$ $\therefore$ $F \leftrightarrow \neg G$

1	$\neg(F \leftrightarrow G)$	
2	F	Ass. ($\leftrightarrow$I)
3	G	Ass. ($\neg$I)
4	F	Ass. ($\leftrightarrow$I)
5	G	R 3
6	G	Ass. ($\leftrightarrow$I)
7	F	R 2
8	$F \leftrightarrow G$	$\leftrightarrow$I 4–5, 6–7
9	$\bot$	$\bot$I 1, 8
10	$\neg G$	$\neg$I 3–9
11	$\neg G$	Ass. ($\leftrightarrow$I)
12	$\neg F$	Ass. ($\neg$I)
13	F	Ass. ($\leftrightarrow$I)
14	$\bot$	$\bot$I 12, 13
15	G	$\bot$E 14
16	G	Ass. ($\leftrightarrow$I)
17	$\bot$	$\bot$I 11, 16
18	F	$\bot$E 17
19	$F \leftrightarrow G$	$\leftrightarrow$I 13–15, 16–18
20	$\bot$	$\bot$I 1, 19
21	F	$\neg$E 12–20
22	$F \leftrightarrow \neg G$	$\leftrightarrow$I 2–10, 11–21

13. $\neg(F \leftrightarrow G) \therefore (F \wedge \neg G) \vee (\neg F \wedge G)$

1	$\neg(F \leftrightarrow G)$	
2	$\neg[(F \wedge \neg G) \vee (\neg F \wedge G)]$	Ass. ($\neg$E)
3	$\neg(F \wedge \neg G) \wedge \neg(\neg F \wedge G)$	DeM 2
4	$\neg(F \wedge \neg G)$	$\wedge$E 3
5	$\neg(\neg F \wedge G)$	$\wedge$E 3
6	$\neg F \vee \neg\neg G$	DeM 4
7	$\neg\neg F \vee \neg G$	DeM 5
8	F	Ass. ($\leftrightarrow$I)
9	$\neg F$	Ass. ($\neg$I)
10	$\bot$	$\bot$I 8, 9
11	$\neg\neg F$	$\neg$I 9–10
12	$\neg\neg G$	DS 6, 11
13	G	DNE 12
14	G	Ass. ($\leftrightarrow$I)
15	$\neg G$	Ass. ($\neg$I)
16	$\bot$	$\bot$I 14, 15
17	$\neg\neg G$	$\neg$I 15–16
18	$\neg\neg F$	DS 7, 17
19	F	DNE 18
20	$F \leftrightarrow G$	$\leftrightarrow$I 8–13, 14–19
21	$\bot$	$\bot$I 1, 20
22	$(F \wedge \neg G) \vee (\neg F \wedge G)$	$\neg$E 2–21

14. $(P \wedge Q) \to (R \vee S) \;\therefore\; (Q \to \neg P) \vee (\neg S \to R)$

1	$(P \wedge Q) \to (R \vee S)$	
2	$P \wedge Q$	Ass. (LEM)
3	$R \vee S$	$\to$E 1, 2
4	$\neg S$	Ass. ($\to$I)
5	R	DS 3, 4
6	$\neg S \to R$	$\to$I 4–5
7	$(Q \to \neg P) \vee (\neg S \to R)$	$\vee$I 6
8	$\neg(P \wedge Q)$	Ass. (LEM)
9	$\neg P \vee \neg Q$	DeM 8
10	Q	Ass. ($\to$I)
11	$\neg Q$	Ass. ($\neg$I)
12	$\bot$	$\bot$I 10, 11
13	$\neg\neg Q$	$\neg$I 11–12
14	$\neg P$	DS 9, 13
15	$Q \to \neg P$	$\to$I 10–14
16	$(Q \to \neg P) \vee (\neg S \to R)$	$\vee$I 15
17	$(Q \to \neg P) \vee (\neg S \to R)$	LEM 2–7, 8–16

15. $(Q \leftrightarrow R) \leftrightarrow (Q \leftrightarrow \neg R) \therefore A$

1	$\mid (Q \leftrightarrow R) \leftrightarrow (Q \leftrightarrow \neg R)$	
2	$\mid \mid Q$	Ass. (↔I)
3	$\mid \mid \mid R$	Ass. (¬I)
4	$\mid \mid \mid \mid Q$	Ass. (↔I)
5	$\mid \mid \mid \mid R$	R 3
6	$\mid \mid \mid \mid R$	Ass. (↔I)
7	$\mid \mid \mid \mid Q$	R 2
8	$\mid \mid \mid Q \leftrightarrow R$	↔I 4–5, 6–7
9	$\mid \mid \mid Q \leftrightarrow \neg R$	↔E 1, 8
10	$\mid \mid \mid \neg R$	↔E 2, 9
11	$\mid \mid \mid \bot$	⊥I 3, 10
12	$\mid \mid \neg R$	¬I 3–11
13	$\mid \mid \neg R$	Ass. (↔I)
14	$\mid \mid \mid \neg Q$	Ass. (¬E)
15	$\mid \mid \mid \mid Q$	Ass. (↔I)
16	$\mid \mid \mid \mid \bot$	⊥I 14, 15
17	$\mid \mid \mid \mid R$	⊥E 16
18	$\mid \mid \mid \mid R$	Ass. (↔I)
19	$\mid \mid \mid \mid \bot$	⊥I 13, 18
20	$\mid \mid \mid \mid Q$	⊥E 19
21	$\mid \mid \mid Q \leftrightarrow R$	↔I 15–17, 18–20
22	$\mid \mid \mid Q \leftrightarrow \neg R$	↔E 1, 21
23	$\mid \mid \mid Q$	↔E 13, 22
24	$\mid \mid \mid \bot$	⊥I 14, 23
25	$\mid \mid Q$	¬E 14–24
26	$\mid Q \leftrightarrow \neg R$	↔I 2–12, 13–25
27	$\mid Q \leftrightarrow R$	↔E 1, 26
28	$\mid \mid Q$	Ass. (¬E)
29	$\mid \mid R$	↔E 27, 28
30	$\mid \mid \neg R$	↔E 26, 28
31	$\mid \mid \bot$	⊥I 29, 30
32	$\mid \neg Q$	¬I 28–31

$\vdots$

⋮

33	| | R	Ass. (¬I)
34	| | Q	↔E 27, 33
35	| | $\bot$	⊥I 32, 34
36	| $\neg R$	¬I 33–35
37	| Q	↔E 26, 36
38	| $\bot$	⊥I 32, 37
39	| A	⊥E 38

Midterm

You will have fifty minutes to complete the midterm. There are 5 sections, which means you should budget about 10 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. _______ If an argument is valid, then its conclusion is true.
2. _______ The main operator of '$(K \vee L) \to \neg L$' is '$\neg$'.
3. _______ If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are jointly impossible, then the argument $\mathscr{A}, \mathscr{B} \therefore \neg\mathscr{C}$ is valid.
4. _______ If $\mathscr{A}$ is a contradiction, then $X \models \mathscr{A}$.
5. _______ If $\mathscr{A}$ is a tautology, then $\mathscr{A} \models X$.
6. _______ If $\mathscr{A}$ and $\mathscr{B}$ are satisfiable, then $\mathscr{A} \models \neg\mathscr{B}$.
7. _______ '$((P \to \neg(Q \leftrightarrow \neg R)) \vee \neg T)$' is a sentence of SL.
8. _______ If $\mathscr{A}, \mathscr{B} \models \mathscr{C}$, then the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid.
9. _______ If the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B} \models \mathscr{C}$.
10. _______ If $\mathscr{A}$ is a necessary truth, then $\mathscr{A}$ is a tautology.

B. TRUTH-TABLES AND ENTAILMENT. Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) Construct a truth-table to decide whether the argument's premises entail its conclusion or not. If the premises entail the conclusion, then write 'Entailment'. If they do not, then tell me which valuation shows that the premises don't entail the conclusion.

> If a moral theory is studied empirically, then examples of conduct will be considered. But if examples of conduct are considered, principles for selecting examples are used. But if principles for selecting examples are used, then moral theory is not studied empirically. Therefore, moral theory is not studied empirically.

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If, once translated into SL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises (you may use the derived rules in your proof). If, once translated into SL, the argument's premises do not entail its conclusion, then provide a truth-table to demonstrate that the premises don't entail the conclusion, and tell me which valuation shows that the premises don't entail the conclusion.

> Warren will win the primary only if Biden does not win the primary. If she doesn't win, then Biden will win. So, either Warren will win the primary or Biden will win the primary.

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises (you may use the derived rules in your proof). If the premises don't entail its conclusion, then provide a truth-table and tell me which valuation shows that the premises don't entail the conclusion.

> If Wednesday wins, then either Pugsley or Uncle Fester comes in second place. So, if Wednesday wins and Uncle Fester doesn't come in second place, then Pugsley comes in second place.

E. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences of SL is a theorem. (You can pick either sentence. If you provide two proofs, then only the first will be graded.)

(a) $[(A \rightarrow \neg A) \rightarrow A] \rightarrow A$

(b) $(X \rightarrow Y) \vee (Y \rightarrow X)$

A. TRUE/FALSE.

1. *false* If an argument is valid, then its conclusion is true.
2. *false* The main operator of '$(K \vee L) \rightarrow \neg L$' is '$\neg$'.
3. *true* If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are jointly impossible, then the argument $\mathscr{A}, \mathscr{B} \therefore \neg\mathscr{C}$ is valid.
4. *false* If $\mathscr{A}$ is a contradiction, then $X \models \mathscr{A}$.
5. *false* If $\mathscr{A}$ is a tautology, then $\mathscr{A} \models X$.
6. *false* If $\mathscr{A}$ and $\mathscr{B}$ are satisfiable, then $\mathscr{A} \models \neg\mathscr{B}$.
7. *true* '$((P \rightarrow \neg(Q \leftrightarrow \neg R)) \vee \neg T)$' is a sentence of SL.
8. *true* If $\mathscr{A}, \mathscr{B} \models \mathscr{C}$, then the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid.
9. *false* If the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B} \models \mathscr{C}$.
10. *false* If $\mathscr{A}$ is a necessary truth, then $\mathscr{A}$ is a tautology.

B. TRUTH-TABLES AND ENTAILMENT. Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) Construct a truth-table to decide whether the argument's premises entail its conclusion or not. If the premises entail the conclusion, then write 'Entailment'. If they do not, then tell me which valuation shows that the premises don't entail the conclusion.

> If a moral theory is studied empirically, then examples of conduct will be considered. But if examples of conduct are considered, principles for selecting examples are used. But if principles for selecting examples are used, then moral theory is not studied empirically. Therefore, moral theory is not studied empirically.

Here is the symbolization key:

M = Moral theory is studied empirically
E = Examples of conduct are considered
P = Principles for selecting examples are used

Then, this is the argument:

$$M \rightarrow E\ ,\ E \rightarrow P\ ,\ P \rightarrow \neg M\ \therefore\ \neg M$$

Here is the truth-table:

E	M	P	M	$\rightarrow$	E	E	$\rightarrow$	P	P	$\rightarrow$	$\neg$	M	$\neg$	M
T	T	T	T	**T**	T	T	**T**	T	T	**F**	F	T	**F**	T
T	T	F	T	**T**	T	T	**F**	F	F	**T**	F	T	**F**	T
T	F	T	F	**T**	T	T	**T**	T	T	**T**	T	F	**T**	F
T	F	F	F	**T**	T	T	**F**	F	F	**T**	T	F	**T**	F
F	T	T	T	**F**	F	F	**T**	T	T	**F**	F	T	**F**	T
F	T	F	T	**F**	F	F	**T**	F	F	**T**	F	T	**F**	T
F	F	T	F	**T**	F	F	**T**	T	T	**T**	T	F	**T**	F
F	F	F	F	**T**	F	F	**T**	F	F	**T**	T	F	**T**	F

There is no row on which the premises are all true and the conclusion false, so it is an entailment.

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into SL. If, once translated into SL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises (you may use the derived rules in your proof). If, once translated into SL, the argument's premises do not entail its conclusion, then provide a truth-table to demonstrate that the premises don't entail the conclusion, and tell me which valuation shows that the premises don't entail the conclusion.

> Warren will win the primary only if Biden does not win the primary. If she doesn't win, then Biden will win. So, either Warren will win the primary or Biden will win the primary.

Here is the symbolization key:

W = Warren will win the primary
B = Biden will win the primary

Then, this is the argument:

$$W \to \neg B\,,\ \neg W \to B\ \therefore\ W \vee B$$

The premises of this argument do entail its conclusion, as any of the following natural deduction proofs demonstrate (to be clear: for the midterm, you only have to provide *one* proof).

1	$W \to \neg B$	
2	$\neg W \to B$	
3	$\quad \neg(W \vee B)$	Ass. ($\neg E$)
4	$\quad \neg W \wedge \neg B$	DeM 3
5	$\quad \neg W$	$\wedge E$ 4
6	$\quad B$	$\to E$ 2, 5
7	$\quad \neg B$	$\wedge E$ 4
8	$\quad \bot$	$\bot I$ 6, 7
9	$W \vee B$	$\neg E$ 3–8

1	$W \to \neg B$	
2	$\neg W \to B$	
3	$\quad W$	Ass. (LEM)
4	$\quad W \vee B$	$\vee I$ 3
5	$\quad \neg W$	Ass. (LEM)
6	$\quad B$	$\to E$ 2, 5
7	$\quad W \vee B$	$\vee I$ 6
8	$W \vee B$	LEM 3–4, 5–7

1	$W \to \neg B$	
2	$\neg W \to B$	
3	$\quad \neg(W \vee B)$	Ass. ($\neg E$)
4	$\quad\quad W$	Ass. ($\neg I$)
5	$\quad\quad W \vee B$	$\vee I$ 4
6	$\quad\quad \bot$	$\bot I$ 3, 5
7	$\quad \neg W$	$\neg I$ 4–6
8	$\quad B$	$\to E$ 2, 7
9	$\quad W \vee B$	$\vee I$ 8
10	$\quad \bot$	$\bot I$ 3, 9
11	$W \vee B$	$\neg E$ 3–10

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into SL. If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises (you may use the derived rules in your proof). If the premises don't entail its conclusion, then provide a truth-table and tell me which valuation shows that the premises don't entail the conclusion.

> If Wednesday wins, then either Pugsley or Uncle Fester comes in second place. So, if Wednesday wins and Uncle Fester doesn't come in second place, then Pugsley comes in second place.

Here is the symbolization key:

W	= Wednesday wins
P	= Pugsley comes in second place
F	= Uncle Fester comes in second place

Then, this is the argument:

$$W \rightarrow (P \vee F) \quad \therefore \quad (W \wedge \neg F) \rightarrow P$$

The premises of this argument do entail its conclusion, as either of the following natural deduction proofs demonstrate (to be clear: for the midterm you only have to provide *one* proof).

1	$W \rightarrow (P \vee F)$	
2	$\quad W \wedge \neg F$	Ass. ($\rightarrow I$)
3	$\quad W$	$\wedge E$ 2
4	$\quad P \vee F$	$\rightarrow E$ 1, 3
5	$\quad \neg F$	$\wedge E$ 2
6	$\quad P$	DS 4, 5
7	$(W \wedge \neg F) \rightarrow P$	$\rightarrow I$ 2–6

1	$W \rightarrow (P \vee F)$	
2	$\quad W \wedge \neg F$	Ass. ($\rightarrow I$)
3	$\quad W$	$\wedge E$ 2
4	$\quad P \vee F$	$\rightarrow E$ 1, 3
5	$\quad\quad P$	Ass. ($\vee E$)
6	$\quad\quad P$	R 5
7	$\quad\quad F$	Ass. ($\vee E$)
8	$\quad\quad \neg F$	$\wedge E$ 2
9	$\quad\quad \bot$	$\bot I$ 7, 8
10	$\quad\quad P$	$\bot E$ 9
11	$\quad P$	$\vee E$ 4, 5–6, 7–10
12	$(W \wedge \neg F) \rightarrow P$	$\rightarrow I$ 2–11

E. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences of SL is a theorem. (You can pick either sentence. If you provide two proofs, then only the first will be graded.)

(a) $[(A \to \neg A) \to A] \to A$

1	$(A \to \neg A) \to A$	Ass. ($\to I$)
2	$\quad \neg A$	Ass. ($\neg E$)
3	$\quad\quad A$	Ass. ($\to I$)
4	$\quad\quad \neg A$	R 2
5	$\quad A \to \neg A$	$\to I$ 3–4
6	$\quad A$	$\to E$ 1, 5
7	$\quad \bot$	$\bot I$ 2, 6
8	A	$\neg E$ 2–7
9	$((A \to \neg A) \to A) \to A$	$\to I$ 1–8

(b) $(X \to Y) \lor (Y \to X)$

Either of the following proofs would suffice:

1	$\neg((X \to Y) \lor (Y \to X))$	Ass. ($\neg E$)
2	$\neg(X \to Y) \land \neg(Y \to X)$	DeM 1
3	$\neg(X \to Y)$	$\land E$ 2
4	$\quad Y$	Ass. ($\neg I$)
5	$\quad\quad X$	Ass. ($\to I$)
6	$\quad\quad Y$	R 4
7	$\quad X \to Y$	$\to I$ 5–6
8	$\quad \bot$	$\bot I$ 3, 7
9	$\neg Y$	$\neg I$ 4–8
10	$\quad Y$	Ass. ($\to I$)
11	$\quad \bot$	$\bot I$ 9, 10
12	$\quad X$	$\bot E$ 11
13	$Y \to X$	$\to I$ 10–12
14	$\neg(Y \to X)$	$\land E$ 2
15	$\bot$	$\bot I$ 13, 14
16	$(X \to Y) \lor (Y \to X)$	$\neg E$ 1–15

1	X	Ass. (LEM)
2	$\quad Y$	Ass. ($\to I$)
3	$\quad X$	R 1
4	$Y \to X$	$\to I$ 2–3
5	$(X \to Y) \lor (Y \to X)$	$\lor I$ 4
6	$\neg X$	Ass. (LEM)
7	$\quad X$	Ass. (LEM)
8	$\quad \bot$	$\bot I$ 6, 7
9	$\quad Y$	$\bot E$ 8
10	$X \to Y$	$\to I$ 7–9
11	$(X \to Y) \lor (Y \to X)$	$\lor I$ 10
12	$(X \to Y) \lor (Y \to X)$	LEM 1–5, 6–11

You will have fifty minutes to complete the midterm. There are 5 sections, which means you should budget about 10 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. _______ If an argument is invalid, then it must be that its premises are all true and its conclusion is false.
2. _______ The sentence '$\neg\neg\neg(X \vee A) \rightarrow (Y \leftrightarrow (A \vee B))$' is a negation.
3. _______ If the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is invalid, then $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$, and $\neg\mathscr{C}$ are jointly possible.
4. _______ If $\mathscr{A}$ is a tautology, then $\neg\mathscr{A} \models \mathscr{A}$.
5. _______ If $\mathscr{A}$ is a necessary falsehood, then the argument $\mathscr{A} \therefore \mathscr{A}$ is valid.
6. _______ If $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are satisfiable, then they are jointly possible.
7. _______ '$(\mathscr{A} \rightarrow \neg\mathscr{B})$' is a sentence of SL.
8. _______ If $\mathscr{A}$ and $\mathscr{B}$ are jointly impossible, then $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are jointly impossible.
9. _______ If the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B} \models \mathscr{C}$.
10. _______ If $\mathscr{A}$ is a necessary falsehood, then $\mathscr{A}$ is a contradiction.

For 5 additional bonus points: write out the definition of 'validity' here:

B. TRUTH-TABLES AND ENTAILMENT. Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) Construct a truth-table to decide whether the argument's premises entail its conclusion or not. If the premises entail the conclusion, then write 'Entailment'. If they do not, then tell me which valuation shows that the premises don't entail the conclusion.

> If the park is closed, then I'll come to Karen's barbecue unless it rains. It won't rain. Therefore, if I don't come to Karen's barbecue, then the park isn't closed.

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If, once translated into SL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises (you may use the derived rules in your proof). If, once translated into SL, the argument's premises do not entail its conclusion, then provide a truth-table to demonstrate that the premises don't entail the conclusion, and tell me which valuation shows that the premises don't entail the conclusion.

> If a Democrat won, then, if Clinton didn't win, then Bernie won. Clinton didn't win. Therefore, if Bernie didn't win, then a Democrat didn't win.

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises (you may use the derived rules in your proof). If the premises don't entail its conclusion, then provide a truth-table and tell me which valuation shows that the premises don't entail the conclusion.

> If Heloise loves Abelard, then she dedicates her Philosophy to him. If Heloise dedicates her Philosophy to Abelard, then Abelard loves Heloise. Therefore, Abelard loves Heloise.

E. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences of SL is a theorem. (You can pick either sentence. If you provide two proofs, then only the first will be graded.)

(a) $\neg(A \rightarrow B) \rightarrow A$

(b) $(\neg P \leftrightarrow Q) \rightarrow (P \leftrightarrow \neg Q)$

You will have fifty minutes to complete the midterm. There are 5 sections, which means you should budget about 10 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. __False__ If an argument is invalid, then it must be that its premises are all true and its conclusion is false.
2. __False__ The sentence '$\neg\neg\neg(X \vee A) \rightarrow (Y \leftrightarrow (A \vee B))$' is a negation.
3. __True__ If the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is invalid, then $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$, and $\neg\mathscr{C}$ are jointly possible.
4. __True__ If $\mathscr{A}$ is a tautology, then $\neg\mathscr{A} \models \mathscr{A}$.
5. __True__ If $\mathscr{A}$ is a necessary falsehood, then the argument $\mathscr{A} \therefore \mathscr{A}$ is valid.
6. __False__ If $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are satisfiable, then they are jointly possible.
7. __False__ '$(\mathscr{A} \rightarrow \neg\mathscr{B})$' is a sentence of SL.
8. __True__ If $\mathscr{A}$ and $\mathscr{B}$ are jointly impossible, then $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are jointly impossible.
9. __False__ If the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B} \models \mathscr{C}$.
10. __False__ If $\mathscr{A}$ is a necessary falsehood, then $\mathscr{A}$ is a contradiction.

For 5 additional bonus points: write out the definition of 'validity' here:

An argument is valid iff it is impossible for all of its premises to be true while its conclusion is false.

B. TRUTH-TABLES AND ENTAILMENT. Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) Construct a truth-table to decide whether the argument's premises entail its conclusion or not. If the premises entail the conclusion, then write 'Entailment'. If they do not, then tell me which valuation shows that the premises don't entail the conclusion.

> If the park is closed, then I'll come to Karen's barbecue unless it rains. It won't rain. Therefore, if I don't come to Karen's barbecue, then the park isn't closed.

Here is the symbolization key:

P = The park is closed
K = I will come to Karen's barbecue
R = It will rain

Then, this is the argument:

$$P \rightarrow (K \vee R), \neg R \therefore \neg K \rightarrow \neg P$$

Here's the truth-table.

K	P	R	P	→	(K	∨	R)	¬	R	¬	K	→	¬	R
T	T	T	T	T	T	T	T	F	T	F	T	T	F	T
T	T	F	T	T	T	T	F	T	F	F	T	T	T	F
T	F	T	F	T	T	T	T	F	T	F	T	T	F	T
T	F	F	F	T	T	T	F	T	F	F	T	T	T	F
F	T	T	T	T	F	T	T	F	T	T	F	F	F	T
F	T	F	T	F	F	F	F	T	F	T	F	T	T	F
F	F	T	F	T	F	T	T	F	T	T	F	F	F	T
F	F	F	F	T	F	F	F	T	F	T	F	T	T	F

There's no row where the premises are all true and the conclusion is false, so the argument is an entailment.

c. evaluating arguments (part 1). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If, once translated into SL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises (you may use the derived rules in your proof). If, once translated into SL, the argument's premises do not entail its conclusion, then provide a truth-table to demonstrate that the premises don't entail the conclusion, and tell me which valuation shows that the premises don't entail the conclusion.

> If a Democrat won, then, if Clinton didn't win, then Bernie won. Clinton didn't win. Therefore, if Bernie didn't win, then a Democrat didn't win.

Here is the symbolization key:

D	= A Democrat won
C	= Clinton won
B	= Bernie won

Then, this is the argument:

$$D \to (\neg C \to B), \neg C \therefore \neg B \to \neg D$$

This argument is an entailment, as the following natural deduction proof demonstrates:

1	$D \to (\neg C \to B)$	
2	$\neg C$	
3	$\quad \neg B$	Ass. ($\to I$)
4	$\quad\quad D$	Ass. ($\neg I$)
5	$\quad\quad \neg C \to B$	$\to E$ 1, 4
6	$\quad\quad B$	$\to E$ 2, 5
7	$\quad\quad \bot$	$\bot I$ 3, 6
8	$\quad \neg D$	$\neg I$ 4–7
9	$\neg B \to \neg D$	$\to I$ 3–8

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into SL. (**Be sure to provide a symbolization key, telling me what English statements the statement letters of SL represent**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises (you may use the derived rules in your proof). If the premises don't entail its conclusion, then provide a truth-table and tell me which valuation shows that the premises don't entail the conclusion.

> If Heloise loves Abelard, then she dedicates her Philosophy to him. If Heloise dedicates her Philosophy to Abelard, then Abelard loves Heloise. Therefore, Abelard loves Heloise.

Here is the symbolization key:

H	= Heloise loves Abelard
D	= Heloise dedicates her Philosophy to Abelard
A	= Abelard loves Heloise

Then, this is the argument:

$$H \rightarrow D, D \rightarrow A \therefore A$$

This argument is not an entailment, as the following truth-table demonstrates:

A D H	H → D	D → A	A
T T T	T T T	T T T	T
T T F	F T T	T T T	T
T F T	T F F	F T T	T
T F F	F T F	F T T	T
F T T	T T T	T F F	F
F T F	F T T	T F F	F
F F T	T F F	F T F	F
F F F	F T F	F T F	F

In the final row of the truth-table, the premises are both true but the conclusion is false.

E. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences of SL is a theorem. (You can pick either sentence. If you provide two proofs, then only the first will be graded.)

(a) $\neg(A \to B) \to A$

1	$\mid\ \neg(A \to B)$	Ass. ($\to I$)
2	$\mid\ \mid\ \neg A$	Ass. ($\neg E$)
3	$\mid\ \mid\ \mid\ A$	Ass. ($\to I$)
4	$\mid\ \mid\ \mid\ \bot$	$\bot I$ 2, 3
5	$\mid\ \mid\ \mid\ B$	$\bot E$ 4
6	$\mid\ \mid\ A \to B$	$\to I$ 3–5
7	$\mid\ \mid\ \bot$	$\bot I$ 1, 6
8	$\mid\ A$	$\neg E$ 2–7
9	$\neg(A \to B) \to A$	$\to I$ 1–8

(b) $(\neg P \leftrightarrow Q) \to (P \leftrightarrow \neg Q)$

1	$\mid\ \neg P \leftrightarrow Q$	Ass.($\to I$)
2	$\mid\ \mid\ P$	Ass.($\leftrightarrow I$)
3	$\mid\ \mid\ \mid\ Q$	Ass.($\neg I$)
4	$\mid\ \mid\ \mid\ \neg P$	$\leftrightarrow E$ 1, 3
5	$\mid\ \mid\ \mid\ \bot$	$\bot I$ 2, 4
6	$\mid\ \mid\ \neg Q$	$\neg I$ 3–5
7	$\mid\ \mid\ \neg Q$	Ass.($\leftrightarrow I$)
8	$\mid\ \mid\ \mid\ \neg P$	Ass.($\neg E$)
9	$\mid\ \mid\ \mid\ Q$	$\leftrightarrow E$ 1, 8
10	$\mid\ \mid\ \mid\ \bot$	$\bot I$ 7, 9
11	$\mid\ \mid\ P$	$\neg E$ 8–10
12	$\mid\ P \leftrightarrow \neg Q$	$\leftrightarrow I$ 2–6, 7–11
13	$(\neg P \leftrightarrow Q) \to (P \leftrightarrow \neg Q)$	$\to I$ 1–12

Part III

Predicate Logic

1. Consider the argument:

    Everyone who has a dog is happy. Obama has a dog. So, Obama is happy.

    This argument is valid, but when we translate it into SL, we get: $E, O \therefore H$. This is not an entailment. So SL can't tell us that this argument is valid.

2. For the second half of the semester, we're going to introduce a new, better theory which will tell us that this argument is valid. For this theory, known as *predicate logic*, we're going to introduce a new formal language, called 'PL'. Today, we'll learn a bit about how PL works, and how to translate from English into PL. (Later on, we'll get more precise about the grammar of the language, just like we did with SL.)

Translation into PL

3. When we translated into SL, we did so with the aid of a *symbolization key*. We will also have symbolization keys in PL—except that they will tell us what the relevant *names* and *predicates* of PL mean (and they'll also tell us which *domain* of things we're talking about—more on that later).

    ▹ In PL, we will use the lowercase letters 'a' through 'v' as *names* (we can add subscripts if we need to):

    $$a, b, c, d, \ldots, t, u, v, a_1, b_1, \ldots, v_1, a_2, \ldots$$

    Think of these being like *proper* names in English. Each lowercase letter refers to some particular person, place or thing.

    ▹ In PL, we will use the uppercase letters 'A' through 'Z' as *predicates* (and we can add subscripts if we need to):

    $$A, B, C, \ldots, X, Y, Z, A_1, B_1, \ldots, Z_1, A_2, \ldots$$

    Think of these predicates as a *gappy statement*—they are like statements, but with a name or names missing.

4. Here, then, is a symbolization key (I'll come back to the *domain* below):

domain	: all things in the solar system		
a	: Abelard	L___	: ___ is large
h	: Heloise	B___	: ___ is bald
b	: Barcelona	P___	: ___ loves Philosophy
j	: Jupiter	X___	: ___ is excited

    ▹ A predicate is a gappy statement, so if we fill in its gaps with names, what we get is a statement. For instance, we can fill in the predicate 'B ___' with the name 'a', and we can fill in the predicate 'X' with the name 'h'. Then, we get the statements:

    Ba : Abelard is bald Xh : Heloise is excited

    (a) These kinds of statements are *atomic*. As in SL, we can combine atomic statements with the logical operators '¬', '∧', '∨', '→', '↔' to get more logically complex statements. For instance:

$\neg Bh$	: Heloise isn't bald
$Xh \rightarrow Pa$	: Heloise is excited only if Abelard loves Philosophy
$Pa \wedge Ph$	: Abelard and Heloise love Philosophy
$Lb \vee \neg Lj$	: Barcelona is large unless Jupiter isn't
$\neg Pa \rightarrow \neg Xh$	: Heloise isn't excited if Abelard doesn't love Philosophy
$\neg(Lb \vee Lj)$	: Neither Barcelona nor Jupiter is large

5. We can fill in a predicate's gaps with a *name*, but in PL we will *also* allow ourselves to fill in its gaps with *variables*.

   ▹ In PL, we will use the lowercase letters 'w' through 'z' as *variables* (we can add subscripts if we need to):

   $$w, x, y, z, w_1, x_1, y_1, z_1, w_2, \ldots$$

   Think of a variable as a name without a fixed meaning—it can refer to *anything* (in the domain). It is a bit like the English name 'one'. When I say 'One shouldn't chew with one's mouth open', I'm not talking about anyone in particular.

6. The reason that we have variables is that they will allow us to make *general* claims about what *something* or *everything*. To express these kinds of claims, we'll make use of what we'll call *quantifiers*, '∀' and '∃'.

   ▹ To illustrate: let 'G__' be the predicate '__ is green'. Then:

   Everything is green : $\forall x\, Gx$
   Something is green : $\exists z\, Gz$

   (Notice: it doesn't matter which variable we use '$\forall x\, Gx$' means the same thing as '$\forall y\, Gy$', which means the same thing as '$\forall w_9 G w_9$'.)

   ▹ In general, let's write '$\mathscr{A}_x$' for a sentence which has the variable 'x' in it somewhere. Then,

   – '$\forall x\, \mathscr{A}_x$' says that '$\mathscr{A}_x$' is true, *no matter what* we let 'x' refer to.
     That is: '$\forall x\, \mathscr{A}_x$' says that *any* x makes '$\mathscr{A}_x$' true.

   – And '$\exists x\, \mathscr{A}_x$' says that there's *some* thing we could let 'x' refer to which would make '$\mathscr{A}_x$' is true.
     That is: '$\exists x\, \mathscr{A}_x$' says that *some* x makes '$\mathscr{A}_x$' true.

7. We allow variables to refer to *anything* that we might want to talk about. So our symbolization key will also tell us which things we might want to talk about—it will also include a *domain*.

   (a) A *domain* specifies which things we might be talking about. It says which things a name or a variable in our language could refer to.

   (b) So: if something isn't included in our domain, then we can't have a *name* for that thing.

8. Which domain we have in our symbolization key will make a difference with respect to how we translate from English into PL.

   ▹ If our domain is all students at Pitt, and 'D__' is the predicate '__ has a dog', then we should translate 'All students at Pitt have a dog' with '$\forall x\, Dx$'

   ▹ If our domain is all people, 'D__' is the predicate '__ has a dog', and 'P__' is the predicate '__ is a student at Pitt', then we should translate 'All students at Pitt have a dog' with '$\forall x\,(Px \to Dx)$'.

9. In sum, a *symbolization key*:

   ▹ tells us what the domain is (the domain can't be empty)

   ▹ for each relevant name, it gives us something *in the domain* which that name refers to (we allow that one thing could have multiple names, but each name must refer to one and only one thing); and

   ▹ for each relevant predicate, it tells us which gappy statement that predicate represents.

10. We'll talk more about this next class, but here are four important statement forms, and their translation into PL:

All $\mathscr{F}$s are $\mathscr{G}$s	:	$\forall x(\mathscr{F}x \to \mathscr{G}x)$
No $\mathscr{F}$s are $\mathscr{G}$s	:	$\forall x(\mathscr{F}x \to \neg\mathscr{G}x)$
Some $\mathscr{F}$s are $\mathscr{G}$s	:	$\exists x(\mathscr{F}x \wedge \mathscr{G}x)$
Some $\mathscr{F}$s are not $\mathscr{G}$s	:	$\exists x(\mathscr{F}x \wedge \neg\mathscr{G}x)$

Four Important Statement Forms

1. Here are four important statement forms, and their translation into PL:

(**A**)	All $\mathscr{F}$s (in the domain) are $\mathscr{G}$s	:	$\forall x(\mathscr{F}x \rightarrow \mathscr{G}x)$
(**E**)	No $\mathscr{F}$s (in the domain) are $\mathscr{G}$s	:	$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$
(**I**)	Some $\mathscr{F}$s (in the domain)are $\mathscr{G}$s	:	$\exists x(\mathscr{F}x \wedge \mathscr{G}x)$
(**O**)	Some $\mathscr{F}$s (in the domain) are not $\mathscr{G}$s	:	$\exists x(\mathscr{F}x \wedge \neg\mathscr{G}x)$

(a) Remember: any quantified claim in PL (a sentence of the form $\forall x\ \mathscr{A}_x$ or $\exists x\ \mathscr{A}_x$) is made *relative to a domain*.
  - '$\forall x\ \mathscr{A}_x$' says that *everything in the domain* makes $\mathscr{A}_x$ true.
  - '$\exists x\ \mathscr{A}_x$' says that *something in the domain* makes $\mathscr{A}_x$ true.

2. A general strategy for translating English claims into PL: find a statement which means the same thing as the statement you want to translate, but which has one of the four forms above. Then, use the translations provided above, making the appropriate substitutions for '$\mathscr{F}$' and '$\mathscr{G}$'.
  - Note: $\mathscr{F}$ and $\mathscr{G}$ might be complex properties/relations.

3. Consider this symbolization key:

domain	: all people				
F____	:	____ is funny	S____	:	____ is shy
T____	:	____ is tall	Q____	:	____ is quirky

Then, using the procedure above, we can get the following translations:

- Everyone tall is shy : $\forall x(Tx \rightarrow Sx)$

  (The statement has the form (**A**), with $\mathscr{F}x = x$ is tall, and $\mathscr{G}x = x$ is shy)
- No quirky people are funny : $\forall y(Qy \rightarrow \neg Fy)$

  (The statement has the form (**E**), with $\mathscr{F}x = x$ is quirky, and $\mathscr{G}x = x$ is funny)
- Any shy quirky person is funny : $\forall x[(Sx \wedge Qx) \rightarrow Fx]$

  (The statement has the form (**A**), with $\mathscr{F}x = x$ is shy and quirky, and $\mathscr{G}x = x$ is funny)
- Some tall people are shy : $\exists w(Tw \wedge Sw)$

  (The statement has the form (**I**), with $\mathscr{F}x = x$ is tall, and $\mathscr{G}x = x$ is shy)
- No tall people are either funny or quirky : $\forall x[Tx \rightarrow \neg(Fx \vee Qx)]$

  (The statement has the form (**E**), with $\mathscr{F}x = x$ is tall, and $\mathscr{G}x = x$ is either funny or quirky)
- Some tall people are neither funny nor shy : $\exists z[Tz \wedge \neg(Fz \vee Sz)]$

  (The statement has the form (**I**), with $\mathscr{F}x = x$ is tall, and $\mathscr{G}x = x$ is neither funny nor shy)
- Some tall people are funny and some are not : $\exists x(Tx \wedge Fx) \wedge \exists x(Tx \wedge \neg Fx)$

  (The statement is a conjunction of a statement of the form (**I**) and a statement of the form (**O**), where in both cases $\mathscr{F}x = x$ is tall and $\mathscr{G}x = x$ is funny)
- If every quirky person is funny, then no quirky person is shy : $\forall x(Qx \rightarrow Fx) \rightarrow \forall y(Qy \rightarrow \neg Sy)$

  (The statement is a conditional whose antecedent has the form (**A**) and whose consequent has the form (**E**))
- There are unfunny tall people if and only if some tall person is shy : $\exists x(Tx \wedge \neg Fx) \leftrightarrow \exists y(Ty \wedge Sy)$

  (The statement is a biconditional whose left-hand-side has the form (**O**) and whose right-hand-side has the form (**I**))

2-Place Predicates

4. Recall: a *predicate* is a *gappy* statement—it is a statement with some name or names missing.
   (a) If there is *one* name missing—if there is *one* gap—then we will call it a *1-place* predicate
   (b) If there are *two* names missing—if there are *two* gaps—then we will call it a *2-place* predicate
   (c) In general, if there are N names missing—if there are N gaps—then we will call it an N*-place* predicate

5. When we have predicates with two or more places, we need some way of keeping track of which gaps are which. So we'll start writing out our symbolization keys like this:

$$\begin{array}{rlrl} \text{domain} & : \text{all people} & & \\ Lxy & : \underline{\quad}_x \text{ loves } \underline{\quad}_y & Mxy & : \underline{\quad}_y \text{ loves } \underline{\quad}_x \\ a & : \text{Abelard} & h & : \text{Heloise} \end{array}$$

Note the difference between the entry for 'L' and the entry for 'M'. 'Lah' translates 'Abelard loves Heloise', whereas 'Mah' translates 'Heloise loves Abelard', or 'Abelard is loved by Heloise'.

6. With this symbolization key,

$$\begin{array}{rlrl} \text{domain} & : \text{everything in the office} & & \\ Px & : \underline{\quad}_x \text{ is a person} & Lxy & : \underline{\quad}_x \text{ likes } \underline{\quad}_y \\ Ex & : \underline{\quad}_x \text{ is easy going} & Txy & : \underline{\quad}_x \text{ is taller than } \underline{\quad}_y \\ m & : \text{Michael} & s & : \text{Stanley} \end{array}$$

some more translations:

▹ No one likes Michael : $\forall x(Px \to \neg Lxm)$

  (The statement has the form of (**E**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ likes Michael.)

▹ Michael likes everyone : $\forall x(Px \to Lmx)$

  (The statement has the form of (**A**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ is liked by Michael.)

▹ Stanley doesn't like anyone : $\forall x(Px \to \neg Lsx)$

  (The statement has the form of (**E**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x =$ Stanley likes x.)

▹ Michael doesn't like anyone taller than him : $\forall x[(Px \wedge Txm) \to \neg Lmx]$

  (The statement has the form of (**E**), with $\mathcal{F}x = x$ is a person who is taller than Michael, and $\mathcal{G}x =$ Michael likes x.)

▹ Everyone likes everyone : $\forall x\,[Px \to \forall y\,(Py \to Lxy)]$

  (The statement has the form of (**A**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ likes everyone. And 'x likes everyone' is a statement with the form of (**A**), with $\mathcal{F}y = y$ is a person and $\mathcal{G}y = y$ is liked by x.)

▹ Everyone likes someone : $\forall x\,[Px \to \exists y\,(Py \wedge Lxy)]$

  (The statement has the form of (**A**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ likes someone. And 'x likes someone' is a statement with the form of (**I**), with $\mathcal{F}y = y$ is a person and $\mathcal{G}y = y$ is liked by x.)

▹ Someone likes someone : $\exists x\,[Px \wedge \exists y\,(Py \wedge Lxy)]$

  (The statement has the form of (**E**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ likes someone. And 'x likes someone' is a statement with the form of (**I**), with $\mathcal{F}y = y$ is a person and $\mathcal{G}y = y$ is liked by x.)

▹ Someone likes everyone : $\exists x\,[Px \wedge \forall y\,(Py \to Lxy)]$

  (The statement has the form of (**I**), with $\mathcal{F}x = x$ is a person, and $\mathcal{G}x = x$ likes everyone. And 'x likes everyone' is a statement with the form of (**A**), with $\mathcal{F}y = y$ is a person, and $\mathcal{G}y = y$ is liked by x.)

A. Using this symbolization key:

domain	: all American politicians
Dx	: $\_\_\_\_\_\__x$ is a Democrat
Rx	: $\_\_\_\_\_\__x$ is a Republican
Cx	: $\_\_\_\_\_\__x$ is conservative
Lx	: $\_\_\_\_\_\__x$ is liberal
Px	: $\_\_\_\_\_\__x$ is President
e	: Elizabeth Warren
j	: Joe Biden
d	: Donald Trump
m	: Mike Pence

translate the following sentences into PL:

1. Donald Trump is a conservative Republican

   ____

2. Elizabeth Warren is liberal if Biden is

   ____

3. Joe Biden is a liberal Democrat only if Elizabeth Warren is.

   ____

4. Joe Biden is a Democrat, unless he's a Republican

   ____

5. Mike Pence is neither liberal nor President.

   ____

6. If Donald Trump isn't President, then Mike Pence is.

   ____

7. Neither Joe Biden nor Elizabeth Warren are Republicans.

   ____

8. Some Democrat politicians are conservative

   ____

9. Some Republican politicians are liberal

   ____

10. Some politicians are neither liberal nor conservative

   ____

B. Using the following symbolization key:

domain : all jellybeans
Bx : $\underline{\hspace{2em}}_x$ is black
Rx : $\underline{\hspace{2em}}_x$ is red.

translate these sentences of English into PL:

1. All jellybeans are black

   ____

2. Some jellybeans are black

   ____

3. No jellybean is black

   ____

4. Some jellybeans are black and some are red

   ____

5. If all jellybeans are black, then none are red

   ____

C. Using the following symbolization key:

domain : all foods
Jx : $\underline{\hspace{2em}}_x$ is a jellybean
Bx : $\underline{\hspace{2em}}_x$ is black
Rx : $\underline{\hspace{2em}}_x$ is red.

translate these sentences of English into PL:

1. All jellybeans are black

   ____

2. Some jellybeans are black

   ____

3. No jellybean is black

   ____

4. Some jellybeans are black and some are red

   ____

5. If all jellybeans are black, then none are red

   ____

D. Using the following symbolization key:

domain	: all people
Dx	: _______$_x$ is at the door.
Hx	: _______$_x$ is honest.
Lx	: _______$_x$ is likable
Px	: _______$_x$ is a politician
h	: Harrington

translate these sentences of English into PL:

1. All politicians are honest

   ____

2. No politicians are honest

   ____

3. Some politicians are honest

   ____

4. Some politicians are not honest

   ____

5. No honest politician is likable.

   ____

6. Some honest politician is at the door.

   ____

7. There are no honest politicians.

   ____

8. A politician is likeable only if they're not honest.

   ____

9. Harrington's dishonest if anyone is.

   ____

10. If every politician is likable, then no politician is honest.

    ____

11. Some politicians are honest, but none are likable.

    ____

12. A likable politician is not honest.

    ____

E. Using this symbolization key:

domain	: all mammals
Lx	: $\_\_\_\_\_\__x$ is a lion.
Fx	: $\_\_\_\_\_\__x$ is ferocious.
Tx	: $\_\_\_\_\_\__x$ is a tiger
Ax	: $\_\_\_\_\_\__x$ is best avoided
b	: Bruce Willis
d	: Danny DeVito

translate these sentences of English into PL:

1. Lions are ferocious

   ____

2. Lions are ferocious, but tigers are not.

   ____

3. Ferocious lions are best avoided.

   ____

4. Lions and tigers are ferocious.

   ____

5. An honest politician is not likable.

   ____

6. Some, but not all, tigers are ferocious.

   ____

7. Danny DeVito and ferocious lions are best avoided.

   ____

8. Bruce Willis is not ferocious, but he is best avoided.

   ____

9. No tigers are ferocious.

   ____

10. Danny DeVito is a ferocious lion, unless he's a tiger.

    ____

11. Bruce Willis is neither a lion nor a tiger, but he is best avoided if he's ferocious.

    ____

Part A

Symbolize each of the following arguments in PL:[4]

Barbara All G are F. All H are G. So: All H are F

Celarent No G are F. All H are G. So: No H are F

Ferio No G are F. Some H is G. So: Some H is not F

Darii All G are F. Some H is G. So: Some H is F.

Camestres All F are G. No H are G. So: No H are F.

Cesare No F are G. All H are G. So: No H are F.

Baroko All F are G. Some H is not G. So: Some H is not F.

Festino No F are G. Some H are G. So: Some H is not F.

Datisi All G are F. Some G is H. So: Some H is F.

Disamis Some G is F. All G are H. So: Some H is F.

Ferison No G are F. Some G is H. So: Some H is not F.

Bokardo Some G is not F. All G are H. So: Some H is not F.

Camenes All F are G. No G are H So: No H is F.

Dimaris Some F is G. All G are H. So: Some H is F.

Fresison No F are G. Some G is H. So: Some H is not F.

Part B

Using this symbolization key:

domain	: people
Kx	: $\_\_\_\__x$ knows the combination to the safe
Sx	: $\_\_\_\__x$ is a spy
Vx	: $\_\_\_\__x$ is a vegetarian
h	: Hofthor
i	: Ingmar

symbolize the following sentences in PL:

1. Neither Hofthor nor Ingmar is a vegetarian.
2. No spy knows the combination to the safe.
3. No one knows the combination to the safe unless Ingmar does.
4. Hofthor is a spy, but no vegetarian is a spy.

[4] The exercises in parts A through D come from *Forall x: An Introduction to Formal Logic*, by P. D. Magnus and Tim Button.

Part C

Using this symbolization key:

$$
\begin{aligned}
\text{domain} &: \text{all animals} \\
Ax &: \_\_\_\__x \text{ is an alligator} \\
Mx &: \_\_\_\__x \text{ is a monkey} \\
Rx &: \_\_\_\__x \text{ is a reptile} \\
Zx &: \_\_\_\__x \text{ lives at the zoo} \\
a &: \text{Amos} \\
b &: \text{Bouncer} \\
c &: \text{Cleo}
\end{aligned}
$$

symbolize the following sentences in PL:

1. Amos, Bouncer, and Cleo all live at the zoo.
2. Bouncer is a reptile, but not an alligator.
3. Some reptile lives at the zoo.
4. Every alligator is a reptile.
5. Any animal that lives at the zoo is either a monkey or an alligator.
6. There are reptiles which are not alligators.
7. If any animal is a reptile, then Amos is.
8. If any animal is an alligator, then it is a reptile.

Part D

For each argument, write out a symbolization key and symbolize the argument in PL.

1. Willard is a logician. All logicians wear funny hats. So Willard wears a funny hat
2. Nothing on my desk escapes my attention. There is a computer on my desk. As such, there is a computer that does not escape my attention.
3. All my dreams are black and white. Old TV shows are in black and white. Therefore, some of my dreams are old TV shows.
4. Neither Holmes nor Watson has been to Australia. A person could see a kangaroo only if they had been to Australia or to a zoo. Although Watson has not seen a kangaroo, Holmes has. Therefore, Holmes has been to a zoo.
5. No one expects the Spanish Inquisition. No one knows the troubles I've seen. Therefore, anyone who expects the Spanish Inquisition knows the troubles I've seen.
6. All babies are illogical. Nobody who is illogical can manage a crocodile. Berthold is a baby. Therefore, Berthold is unable to manage a crocodile.

Part E

Using this symbolization key,

domain	:	all people
Ex	:	$\_\_\_\_\__x$ is a real estate agent
Lx	:	$\_\_\_\_\__x$ is a lawyer
Px	:	$\_\_\_\_\__x$ is a professor
Nx	:	$\_\_\_\_\__x$ lives next door
Rx	:	$\_\_\_\_\__x$ is rich
Yx	:	$\_\_\_\_\__x$ is a yuppie
f	:	Fred

translate the following sentences of English into PL.[5]

1. If Fred is a yuppie, he's not a professor, and if he's a professor, he's not rich.
2. All real estate agents are yuppies.
3. No professor is rich.
4. No real estate agent is a yuppie.
5. No rich lawyer lives next door.
6. Every rich lawyer is a yuppie.
7. Some rich lawyers are not yuppies.
8. If Fred is rich, then every professor is rich.
9. Some but not all real estate agents are yuppies.
10. If any real estate agent is a yuppie, then all lawyers are.
11. Any real estate agent who isn't a yuppie isn't rich.
12. Any yuppie who is either a real estate agent or a lawyer is rich.

[5] Some but not all of these sentences are lifted from the fifth edition of *The Logic Book*, by Bergmann, Nelson, and Moor.

Part A

Barbara All G are F. All H are G. So: All H are F

$$\forall x(Gx \rightarrow Fx), \forall y(Hy \rightarrow Gy) \therefore \forall z(Hz \rightarrow Fz)$$

Celarent No G are F. All H are G. So: No H are F

$$\forall x(Gx \rightarrow \neg Fx), \forall y(Hy \rightarrow Gy) \therefore \forall z(Hz \rightarrow \neg Fz)$$

Ferio No G are F. Some H is G. So: Some H is not F

$$\forall x(Gx \rightarrow \neg Fx), \exists y(Hy \wedge Gy) \therefore \exists z(Hz \wedge \neg Fz)$$

Darii All G are F. Some H is G. So: Some H is F.

$$\forall x(Gx \rightarrow Fx), \exists y(Hy \wedge Gy) \therefore \exists z(Hz \wedge Fz)$$

Camestres All F are G. No H are G. So: No H are F.

$$\forall x(Fx \rightarrow Gx), \forall y(Hy \rightarrow \neg Gy) \therefore \forall z(Hz \rightarrow \neg Fz)$$

Cesare No F are G. All H are G. So: No H are F.

$$\forall x(Fx \rightarrow \neg Gx), \forall y(Hy \rightarrow Gy) \therefore \forall z(Hz \rightarrow \neg Fz)$$

Baroko All F are G. Some H is not G. So: Some H is not F.

$$\forall x(Fx \rightarrow Gx), \exists y(Hy \wedge \neg Gy) \therefore \exists z(Hz \wedge \neg Fz)$$

Festino No F are G. Some H are G. So: Some H is not F.

$$\forall x(Fx \rightarrow \neg Gx), \exists y(Hy \wedge Gy) \therefore \exists z(Hz \wedge \neg Fz)$$

Datisi All G are F. Some G is H. So: Some H is F.

$$\forall x(Gx \rightarrow Fx), \exists y(Gy \wedge Hy) \therefore \exists z(Hz \wedge Fz)$$

Disamis Some G is F. All G are H. So: Some H is F.

$$\exists x(Gx \rightarrow Fx), \forall y(Gy \rightarrow Hy) \therefore \exists z(Hz \wedge Fz)$$

Ferison No G are F. Some G is H. So: Some H is not F.

$$\forall x(Gx \rightarrow \neg Fx), \exists y(Gy \wedge Hy) \therefore \exists z(Hz \wedge \neg Fz)$$

Bokardo Some G is not F. All G are H. So: Some H is not F.

$$\exists x(Gx \wedge \neg Fx), \forall y(Gy \rightarrow Hy) \therefore \exists z(Hz \wedge \neg Fz)$$

Camenes All F are G. No G are H So: No H is F.

$$\forall x(Fx \rightarrow Gx), \forall y(Gy \rightarrow \neg Hy) \therefore \forall z(Hz \rightarrow \neg Fz)$$

Dimaris Some F is G. All G are H. So: Some H is F.

$$\exists x(Fx \wedge Gx), \forall y(Gy \rightarrow Hy) \therefore \exists z(Hz \wedge Fz)$$

Fresison No F are G. Some G is H. So: Some H is not F.

$$\forall x(Fx \wedge \neg Gx), \exists y(Gy \wedge Hy) \therefore \exists z(Hz \wedge \neg Fz)$$

Part B

1. $\neg(Vh \lor Vi)$
2. $\forall x(Sx \rightarrow \neg Kx)$
3. $\forall x\neg Kx \lor Ki$
4. $Sh \land \forall y(Vy \rightarrow \neg Sy)$

Part C

1. $Za \land (Zb \land Zc)$
2. $Rb \land \neg Ab$
3. $\exists z(Rz \land Zz)$
4. $\forall x(Ax \rightarrow Rx)$
5. $\forall y[Zy \rightarrow (My \lor Ay)]$
6. $\exists z(Rz \land \neg Az)$
7. $\exists xRx \rightarrow Ra$
8. $\forall y(Ay \rightarrow Ry)$

Part D

1. Symbolization key:

   domain : all people
   Lx : $\_\_\_\__x$ is a logician
   Hx : $\_\_\_\__x$ wears a funny hat
   w : Willard

   Argument:

   $$Lw, \forall x(Lx \rightarrow Hx) \therefore Hw$$

2. Symbolization key:

   domain : all things in my room
   Dx : $\_\_\_\__x$ is on my desk
   Cx : $\_\_\_\__x$ is a computer
   Ex : $\_\_\_\__x$ escapes my attention

   Argument:

   $$\forall x(Dx \rightarrow \neg Ex), \exists y(Cy \land Dy) \therefore \exists z(Cz \land \neg Ez)$$

3. Symbolization key:

   domain : all things
   Bx : $\_\_\_\__x$ is in black and white
   Tx : $\_\_\_\__x$ is an old TV show
   Dx : $\_\_\_\__x$ is a dream of mine

   Argument:

   $$\forall x(Dx \rightarrow Bx), \forall y(Ty \rightarrow By) \therefore \exists z(Dz \land Tz)$$

4. Symbolization key:

$$\begin{aligned}\text{domain} &: \text{all people}\\ Ax &: \underline{\quad}_x \text{ has been to Australia}\\ Zx &: \underline{\quad}_x \text{ has been to a zoo}\\ Kx &: \underline{\quad}_x \text{ has seen a kangaroo}\\ h &: \text{Holmes}\\ w &: \text{Watson}\end{aligned}$$

Argument:

$$\neg(Ah \lor Aw), \forall x[Kx \rightarrow (Ax \lor Zx)], \neg Kw \land Kh \therefore Zh$$

5. Symbolization key:

$$\begin{aligned}\text{domain} &: \text{all people}\\ Ex &: \underline{\quad}_x \text{ expects the Spanish Inquisition}\\ Kx &: \underline{\quad}_x \text{ knows the troubles I've seen}\end{aligned}$$

Argument:

$$\forall x \neg Ex, \forall y \neg Ky \therefore \forall z(Ez \rightarrow Kz)$$

6. Symbolization key:

$$\begin{aligned}\text{domain} &: \text{all people}\\ Ix &: \underline{\quad}_x \text{ is illogical}\\ Cx &: \underline{\quad}_x \text{ is capable of managing a crocodile}\\ Bx &: \underline{\quad}_x \text{ is a baby}\\ b &: \text{Berthold}\end{aligned}$$

Argument:

$$\forall x(Bx \rightarrow Ix), \forall y(Iy \rightarrow \neg Cy), Bb \therefore \neg Cb$$

Part E

1. $(Yf \rightarrow \neg Pf) \land (Pf \rightarrow \neg Rf)$
2. $\forall x(Ex \rightarrow Yx)$
3. $\forall y(Py \rightarrow \neg Ry)$
4. $\forall z(Ez \rightarrow \neg Yz)$
5. $\forall x[(Rx \land Lx) \rightarrow \neg Nx]$
6. $\forall x[(Rx \land Lx) \rightarrow Yx]$
7. $\exists z[(Rz \land Lz) \land \neg Yz]$
8. $Rf \rightarrow \forall x(Px \rightarrow Rx)$
9. $\exists x(Ex \land Yx) \land \neg\forall y(Ey \rightarrow Yy)$
10. $\exists x(Ex \land Yx) \rightarrow \forall z(Lz \rightarrow Yz)$
11. $\forall x[(Ex \land \neg Yx) \rightarrow \neg Rx]$
12. $\forall w([Yw \land (Ew \lor Lw)] \rightarrow Rw)$

Vocabulary

1. The vocabulary of PL includes the following symbols:

   (a) For each $N \geqslant 0$, N-place predicates—which are capital letters, perhaps with subscripts.

   $$A, B, C, D, E, \ldots, X, Y, Z, A_1, B_1, \ldots$$

   (b) *names*—which are lowercase letters between a and v, perhaps with subscripts.

   $$a, b, c, d, e, \ldots, t, u, v, a_1, b_1, \ldots$$

   (c) *variables*—which are lowercase letters between w and z, perhaps with subscripts.

   $$w, x, y, z, w_1, x_1, \ldots$$

   (d) logical operators

   $$\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \exists, \forall$$

   (e) parenthases

   $$(\, , \,)$$

   Nothing else is included in the vocabulary of PL.

   ▹ Terminology: we'll call both names and variables *terms*. So, both 'a' and 'x' are *terms*. (A *term* is just a lowercase letter.)

Grammar

2. Any sequence of symbols from the vocabulary of PL is an *expression* of PL. But not all expressions are grammatical sentences. To define which expressions count as grammatical sentences, we begin by defining the notion of an *atomic* sentence.

<u>*Atomic Sentence*</u>

If $\mathscr{R}$ is an N-place predicate and $t_1, t_2, \ldots, t_N$ are N terms, then '$\mathscr{R} t_1 t_2 \ldots t_N$' is an atomic sentence.

Now, we define which expressions count as grammatical sentences with the following rules:

<u>*Rules for Sentences*</u>

$\mathscr{R}$) Every atomic sentence is a sentence

$\neg$) If '$\mathscr{A}$' is a sentence, then '$\neg \mathscr{A}$' is a sentence

$\vee$) If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \vee \mathscr{B})$' is a sentence

$\wedge$) If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \wedge \mathscr{B})$' is a sentence

$\rightarrow$) If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \rightarrow \mathscr{B})$' is a sentence

$\leftrightarrow$) If '$\mathscr{A}$' and '$\mathscr{B}$' are sentences, then '$(\mathscr{A} \leftrightarrow \mathscr{B})$' is a sentence

$\forall$) If '$\mathscr{A}$' is a sentence and 'x' is a variable, then '$\forall x\, \mathscr{A}$' is a sentence

$\exists$) If '$\mathscr{A}$' is a sentence and 'x is a variable, then '$\exists x\, \mathscr{A}$' is a sentence

- ▹ Only things we we can show to be sentences using these rules are sentences. Nothing else is a sentence.

With these rules, we can show that a given expression is a sentence by building it up from atomic sentences.

3. Some conventions: we will allow ourselves to omit any parentheses which were added in the *final* step of building the sentence up according to these rules. Thus, instead of '$(\forall x\, Rxa \to \exists y\, \exists z\, Gyz)$', we will just write '$\forall x\, Rxa \to \exists y\, \exists z\, Gyz$'. And we will allow ourselves to use square brackets to improve readability.

4. With the rules for sentences, we can define some other important syntactic notions.

   (a) A non-atomic sentence's *main operator* is the logical operator whose associated rule would be the last one appealed to, were we building the sentence up according to the rules for sentences.

   (b) '$\mathscr{B}$' is a *subsentence* of '$\mathscr{A}$' iff, in the course of building up '$\mathscr{A}$' by applying the rules for sentences, you'd first have to show that '$\mathscr{B}$' was a sentence.

   - ▹ So, *e.g.*, '$\forall$' is the main operator of '$\forall x\, Fx$', and '$\exists$' is the main operator of '$\exists z\, (Qz \wedge Pza)$'.
   - ▹ A sentence whose main operator is '$\forall$' is a *universal* sentence. And a sentence whose main operator is '$\exists$' is a *existential* sentence.

   (c) The *scope* of an operator (in a sentence) is the sub-sentence for which that operator is the main operator.

$$\underbrace{\forall x\ \underbrace{\exists y\,[\overbrace{\forall w\, Fwx}^{\text{scope of `}\forall w\text{'}} \leftrightarrow \overbrace{\forall z\,(Gxz \to Wzyx)}^{\text{scope of `}\forall z\text{'}}]}_{\text{scope of `}\exists y\text{'}}}_{\text{scope of `}\forall x\text{'}}$$

Free and Bound Variables

> A variable, x, in a sentence of PL, is *bound* iff it occurs within the scope of a quantifier, $\forall x$ or $\exists x$, whose associated variable is x.
>
> A variable, x, in a sentence of PL, is *free* iff it does *not* occur within the scope of a quantifier, $\forall x$ or $\exists x$, whose associated variable is x.

- ▹ For instance, in '$\forall x\, \forall y\, Fy \to \exists z\, Gzx$', the '$y$' in '$Fy$' is bound, the '$z$' in '$Gzx$' is bound, and the '$x$' in '$Gzx$' is free.

5. In a sentence of the form $\forall x \mathscr{A}$ or $\exists x \mathscr{A}$, the quantifier binds every *free* occurrence of x in $\mathscr{A}$. If an occurrence of x in $\mathscr{A}$ is already bound, then the quantifier does not bind it.

   - ▹ For instance, in '$\exists x\, \forall x\, Fx$', the universal quantifier '$\forall x$' binds the 'x' in 'Fx'. The existential quantifier does not bind the 'x' in 'Fx'.

6. If all of the variables in a sentence are bound, then we'll say that that sentence is *closed*.

7. If some of the variables in a sentence are free, then we'll say that that sentence is *open*.

8. When you're translating English sentences into PL, you want to translate them as *closed* sentences.

A. SENTENCES. Which of the following are sentences of PL? (Throughout, G is a 1-place predicate and L and F are 2-place predicates.)

1. ______ $(\forall xy\, Fxy \vee \exists z\, Gz)$
2. ______ $\forall v\, Gv$
3. ______ $Labx$
4. ______ $\exists x\, \forall y\, \mathcal{F}xy$
5. ______ $\forall x\, \exists x\, \forall x\, Fax$
6. ______ $(\forall x\, \exists y\, Lxy \to \exists x\, \forall y\, Lyx)$

B. SYNTAX TREES. Write out the syntax trees for the sentences of PL below.

1. $\forall x\, (Px \to Qx)$

2. $\forall x\, \neg(\exists y\, \forall z\, Rxzy \to \exists z\, Fxz)$

3. $\exists x\, (\forall y\, Fy \to Fx)$

C. QUANTIFIER SCOPE, FREE AND BOUND VARIABLES. For each of the following sentences of PL, say what its main operator is. Then, for each quantifier appearing in the sentence, say what the *scope* of the quantifier is. Then, for each variable, say whether it is free or bound, and if it is bound, say which quantifier binds it.

1. $\forall x\, \exists x\, Fxx \rightarrow \forall y\, Gy$
   (a) MAIN OPERATOR: ________________
   (b) SCOPE OF '$\forall x$': ________________
   (c) SCOPE OF '$\exists x$': ________________
   (d) SCOPE OF '$\forall y$': ________________
   (e) first x in 'Fxx':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________
   (f) second x in 'Fxx':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________
   (g) 'y' in 'Gy':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________

2. $\forall x\, (\forall y\, Fxz \rightarrow \neg \exists z\, \neg Gxz)$
   (a) MAIN OPERATOR: ________________
   (b) SCOPE OF '$\forall x$': ________________
   (c) SCOPE OF '$\forall y$': ________________
   (d) SCOPE OF '$\exists z$': ________________
   (e) x in 'Fxz':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________
   (f) z in 'Fxz':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________
   (g) x in 'Gxz':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________
   (h) z in 'Gxz':
      i. FREE OR BOUND? ________________
      ii. IF BOUND, BOUND BY WHICH QUANTIFIER? ________________

1. Recall, in SL, we gave a semantics in terms of *valuations*.

   ▹ A *valuation* said which atomic statement letters of SL were true and which were false.

   ▹ Given a valuation, our semantics said which sentences of SL were true and which were false.

   ▹ We defined the notion of *entailment*, *satisfiability*, *tautology*, and so on in terms of these valuations.

2. In PL, we will do the same thing, but instead of *valuations*, we will use *interpretations*.

   ▹ An *interpretation* will allow us to figure out which atomic sentences of PL are true and which are false

   ▹ Given an interpretation, our semantics will say which sentences of SL are true and which are false.

   ▹ We will be able to define a notion of *entailment*, *satisfiability*, *tautology*, and so on in terms of these interpretations

Interpretations

An INTERPRETATION, **I**, tells us:

1) what *domain* of things we are talking about;
2) for each relevant term (name or variable), what thing in the domain it refers to; and
3) for each relevant predicate of PL, which things in the domain the predicate is true of

3. Suppose we're interested in the argument $\forall z\,(Dz \rightarrow Hz)$, Do $\therefore$ Ho. Then, any of the following will count as an *interpretation* for that argument:

Domain	: all people	Domain	: Adam, Betty, and Carl	Domain	: 1, 2, 3, and 4
Hx	: $\_\_\_\__x$ is happy	H	: Adam and Betty	H	: 1, 2 and 4
Dx	: $\_\_\_\__x$ has a dog	D	: Adam	D	: 2 and 3
o	: Obama	o	: Adam	o	: 1

   ▹ The first interpretation tells us what H and D are true of by providing a gappy English sentence, with the understanding that the predicate is true of whatever makes that sentence true.

   ▹ The latter two interpretations, on the other hand, just *list off* all of the things that the predicate is true of.

4. If we're specifying which things satisfy a 1-place predicate, we can simply list them. But, in order to specify which things satisfy a 2-place predicate, we have to list them *in order*.

   ▹ We'll do this with angled brackets, '⟨ ⟩'. Thus, the following is an interpretation for the sentence '$\forall x\, Lax \wedge \exists y\, Lyb$':

Domain	: Sammy and Tammy
L	: ⟨ Sammy, Tammy ⟩, ⟨ Sammy, Sammy ⟩
a	: Sammy
b	: Tammy

   The first pair ⟨ Sammy, Tammy ⟩ tells us that Sammy bears the L-relation to Tammy. This is very different from saying that Tammy bears the L-relation to Sammy. This interpretation tells us that the sentence Lab is true, but the sentence Lba is false.

Truth on an Interpretation

5. To say which sentences of PL are true on a given interpretation, we'll first say how to determine which *atomic* sentences are true, and then work our way up to the more complicated sentences.

6. Recall: an *atomic* sentence is an N-place predicate $\mathscr{R}$ followed by N terms, $t_1, t_2, \ldots, t_\mathcal{N}$. We say that an atomic sentence is true on an interpretation iff the things referred to by the terms are true of the predicate they follow (in the order in which they follow the predicate).

> An atomic sentence '$\mathscr{R} t_1 t_2 \ldots t_\mathcal{N}$' is true on **I** iff, according to **I**, $\mathscr{R}$ is true of the objects named by $t_1, t_2, \ldots, t_\mathcal{N}$ (in that order).

7. The semantics for ¬, ∧, ∨, →, and ↔ are the same as they were for SL. In particular:

> ¬) '$\neg\mathscr{A}$' is true on **I** iff '$\mathscr{A}$' is false on **I**
>
> ∧) '$(\mathscr{A} \wedge \mathscr{B})$' is true on **I** iff both '$\mathscr{A}$' and '$\mathscr{B}$' are true on **I**
>
> ∨) '$(\mathscr{A} \vee \mathscr{B})$' is true on **I** iff either '$\mathscr{A}$' or '$\mathscr{B}$' is true on **I**
>
> →) '$(\mathscr{A} \rightarrow \mathscr{B})$' is true on **I** iff either '$\mathscr{A}$' is false on **I** or '$\mathscr{B}$' is true on **I**
>
> ↔) '$(\mathscr{A} \leftrightarrow \mathscr{B})$' is true on **I** iff '$\mathscr{A}$' and '$\mathscr{B}$' have the same truth-value on **I**

8. To introduce the semantics for the quantifiers, we first need the notion of a *modified* interpretation.

   - If **I** is an interpretation, d is something in the domain of **I**, and x is a variable, then the modified interpretation $\mathbf{I}[x : d]$ is the interpretation which is exactly like **I**, except that the variable x refers to d.
   - For instance,

I		$\mathbf{I}[y : \text{Amy}]$		$\mathbf{I}[y : \text{Bruce}]$	
Domain	: Amy and Bruce	Domain	: Amy and Bruce	Domain	: Amy and Bruce
F	: Bruce	F	: Bruce	F	: Bruce
		y	: Amy	y	: Bruce

   (a) Now: to check whether '$\exists y\, Fy$' is true on **I**, we check whether 'Fy' is true on *either* of $\mathbf{I}[y : \text{Amy}]$ or $\mathbf{I}[y : \text{Bruce}]$. Since '$Fy$' is true on $\mathbf{I}[y : \text{Bruce}]$, '$\exists y\, Fy$' is true on **I**.

   (b) To check whether '$\forall y\, Fy$' is true on **I**, we check whether 'Fy' is true on *both* $\mathbf{I}[y : \text{Amy}]$ and $\mathbf{I}[y : \text{Bruce}]$. Since '$Fy$' is false on $\mathbf{I}[y : \text{Amy}]$, '$\forall y\, Fy$' is false on **I**.

9. In general,

> ∃) '$\exists x\, \mathscr{A}$' is true on **I** iff '$\mathscr{A}$' is true on $\mathbf{I}[x : d]$ for ***some d*** in the domain of **I**
>
> ∀) '$\forall x\, \mathscr{A}$' is true on **I** iff '$\mathscr{A}$' is true on $\mathbf{I}[x : d]$ for ***every d*** in the domain of **I**

   - That is: '$\forall x\, \mathscr{A}$' is true on an interpretation iff '$\mathscr{A}$' is true on that interpretation, *no matter what* we let 'x' name.
   - And '$\exists x\, \mathscr{A}$' is true on an interpretation iff there's *something* we can let 'x' name which will make '$\mathscr{A}$' is true on that interpretation.

1. Recall,

> If $\mathbf{I}$ is an interpretation, d is something in the domain of $\mathbf{I}$, and x is a variable, then the modified interpretation $\mathbf{I}[x : d]$ is the interpretation which is exactly like $\mathbf{I}$, except that the variable x refers to d.

> ∃) '$\exists x\, \mathscr{A}$' is true on $\mathbf{I}$ iff '$\mathscr{A}$' is true on $\mathbf{I}[x : d]$ for ***some*** d in the domain of $\mathbf{I}$
>
> ∀) '$\forall x\, \mathscr{A}$' is true on $\mathbf{I}$ iff '$\mathscr{A}$' is true on $\mathbf{I}[x : d]$ for ***every*** d in the domain of $\mathbf{I}$

Four Important Statement Forms

2. I said that '$\forall x(\mathscr{F}x \rightarrow \mathscr{G}x)$' translates 'All $\mathscr{F}$s are $\mathscr{G}$s'. Let's think this through now that we know the semantics for PL.
   - There's one way for 'All $\mathscr{F}$s are $\mathscr{G}$s' to be false. There could be something in the domain which is $\mathscr{F}$ but not $\mathscr{G}$. Otherwise, 'All Fs are $\mathscr{G}$s' is true. We'll show that '$\forall x(\mathscr{F}x \rightarrow \mathscr{G}x)$' is false iff something in the domain is $\mathscr{F}$ but not $\mathscr{G}$.
   - Suppose we have an interpretation, $\mathbf{I}$, in which something in the domain is $\mathscr{F}$ but not $\mathscr{G}$. Just to give it a name, call that thing 'Bob'. Then, in the modified interpretation $\mathbf{I}[x : \text{Bob}]$, '$\mathscr{F}x$' will be true and '$\mathscr{G}x$' will be false. So '$\mathscr{F}x \rightarrow \mathscr{G}x$' will be false on $\mathbf{I}[x : \text{Bob}]$. So '$\forall x(\mathscr{F}x \rightarrow \mathscr{G}x)$' is false on $\mathbf{I}$.
   - Suppose on the other hand we have an interpretation, $\mathbf{I}$, in which everything in the domain is either not $\mathscr{F}$ or it is $\mathscr{G}$. Then, for every modified interpretation $\mathbf{I}[x : d]$, either '$\mathscr{F}x$' is false, or else '$\mathscr{G}x$' is true. So, for every modified interpretation, '$(\mathscr{F}x \rightarrow \mathscr{G}x)$' is true. So '$\forall x(\mathscr{F}x \rightarrow \mathscr{G}x)$' is true on the interpretation $\mathbf{I}$.
3. I said that '$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$' translates 'No $\mathscr{F}$s are $\mathscr{G}$'. Let's think this through with our formal semantics.
   - 'No $\mathscr{F}$s are $\mathscr{G}$'s is false if there is something in the domain which is $\mathscr{F}$ and $\mathscr{G}$. Otherwise it is true. We'll show that '$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$' is false iff there's something in the domain which if $\mathscr{F}$ and $\mathscr{G}$.
   - Suppose we have an interpretation $\mathbf{I}$, and there is something in the domain which is $\mathscr{F}$ and $\mathscr{G}$. Just to give it a name, call that thing 'Fred'. Then, in the modified interpretation $\mathbf{I}[x : \text{Fred}]$, '$\mathscr{F}x$' will be true and '$\mathscr{G}x$' will be true. So '$\neg\mathscr{G}x$' will be false. So '$\mathscr{F}x \rightarrow \neg\mathscr{G}x$' will be false on the modified interpretation $\mathbf{I}[x : \text{Fred}]$. So '$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$' will be false on the interpretation $\mathbf{I}$.
   - Suppose on the other hand that we have an interpretation $\mathbf{I}$ in which nothing in the domain is both $\mathscr{F}$ and $\mathscr{G}$. Then, everything in the domain is either not $\mathscr{F}$ or not $\mathscr{G}$. Then, for every modified interpretation $\mathbf{I}[x : d]$, either '$\mathscr{F}x$' is false or else '$\neg\mathscr{G}x$' is true. So, for every modified interpretation $\mathbf{I}[x : d]$, '$\mathscr{F}x \rightarrow \neg\mathscr{G}x$' is true. So, on the interpretation $\mathbf{I}$, '$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$' is true.
4. I said that '$\exists x(\mathscr{F}x \wedge \mathscr{G}x)$' translates 'Some $\mathscr{F}$s are $\mathscr{G}$s'. Let's think this through with the formal semantics.
   - 'Some $\mathscr{F}$s are $\mathscr{G}$s is true if there's something in the domain which is both $\mathscr{F}$ and $\mathscr{G}$. Otherwise, it's false. We'll show that '$\exists x(\mathscr{F}x \wedge \mathscr{G}x)$' is true iff there's something in the domain which if $\mathscr{F}$ and $\mathscr{G}$.
   - Suppose we have an interpretation $\mathbf{I}$, and there is something in the domain which is $\mathscr{F}$ and $\mathscr{G}$. Just to give it a name, call that thing 'Jim'. Then, in the modified interpretation $\mathbf{I}[x : \text{Jim}]$, '$\mathscr{F}x$' will be true and '$\mathscr{G}x$' will be true. So '$\mathscr{F}x \wedge \mathscr{G}x$' will be true on the modified interpretation $\mathbf{I}[x : \text{Fred}]$. So '$\exists x(\mathscr{F}x \wedge \mathscr{G}x)$' will be true on the interpretation $\mathbf{I}$.
   - Suppose on the other hand that we have an interpretation $\mathbf{I}$ in which nothing in the domain is both $\mathscr{F}$ and $\mathscr{G}$. Then, everything in the domain is either not $\mathscr{F}$ or not $\mathscr{G}$. Then, for every modified interpretation $\mathbf{I}[x : d]$, either '$\mathscr{F}x$' is false or else '$\mathscr{G}x$' is false. So '$\forall x(\mathscr{F}x \rightarrow \neg\mathscr{G}x)$' is true on the interpretation $\mathbf{I}$.

5. Finally, I said that '$\exists x(\mathscr{F}x \wedge \neg \mathscr{G}x)$' translates 'Some $\mathscr{F}$s are not $\mathscr{G}$s'.

- ▹ 'Some $\mathscr{F}$s are not $\mathscr{G}$s is true if there's something in the domain which is both $\mathscr{F}$ and not $\mathscr{G}$. Otherwise, it's false. We'll show that '$\exists x(\mathscr{F}x \wedge \neg \mathscr{G}x)$' is true iff there's something in the domain which if $\mathscr{F}$ and not $\mathscr{G}$.
- ▹ Suppose we have an interpretation, **I**, in which something in the domain is $\mathscr{F}$ but not $\mathscr{G}$. Just to give it a name, call that thing 'Sally'. Then, in the modified interpretation $\mathbf{I}[x : \text{Sally}]$, '$\mathscr{F}x$' will be true and '$\mathscr{G}x$' will be false. So '$\neg \mathscr{G}x$' will be true. So '$\mathscr{F}x \wedge \neg \mathscr{G}x$' will be true on $\mathbf{I}[x : \text{Sally}]$. So '$\exists x(\mathscr{F}x \wedge \neg \mathscr{G}x)$' is true on **I**.
- ▹ Suppose on the other hand we have an interpretation, **I**, in which everything in the domain is either not $\mathscr{F}$ or it is $\mathscr{G}$. Then, for every modified interpretation $\mathbf{I}[x : d]$, either '$\mathscr{F}x$' is false, or else '$\mathscr{G}x$' is true. So, for every modified interpretation $\mathbf{I}[x : d]$, either '$\mathscr{F}x$' is false or else '$\neg \mathscr{G}x$' is false. So, for every modified interpretation, '$(\mathscr{F}x \wedge \neg \mathscr{G}x)$' is false. So '$\exists x(\mathscr{F}x \wedge \neg \mathscr{G}x)$' is false on the interpretation **I**.

Overlapping Quantifiers

6. Submitted for your approval: if our domain contains only people, and if 'Lxy' translates '$\_\_\_\__x$ loves $\_\_\_\__y$', then the following four sentences of PL should be translated into English as follows:

$\forall x\, \exists y\, Lxy$	Everyone loves someone
$\exists y\, \forall x\, Lxy$	Someone is loved by everyone
$\exists y\, \forall x\, Lyx$	Someone loves everyone
$\forall x\, \exists y\, Lyx$	Everyone is loved by someone

7. Let's think each of these through, starting with '$\forall x\, \exists y\, Lxy$'.

- ▹ '$\forall x\, \exists y\, Lxy$' says that '$\exists y\, Lxy$' is true *no matter who* we let 'x' refer to.
- ▹ Let 'x' refer to Abelard. Then, '$\exists y\, Lxy$' says that Abelard loves *someone*.
- ▹ '$\forall x\, \exists y\, Lxy$' says that this isn't just true of Abelard—it is true of *everyone*.
- ▹ So: '$\forall x\, \exists y\, Lxy$' says that *everyone* loves someone.

8. Next, consider '$\exists y\, \forall x\, Lxy$'.

- ▹ '$\exists y\, \forall x\, Lxy$' says that '$\forall x\, Lxy$' is true when we let '$y$' refer to *some* particular person.
- ▹ Let 'y' refer to Abelard. Then, '$\forall x\, Lxy$' says that Abelard is loved by everyone.
- ▹ '$\exists y\, \forall x\, Lxy$' says that this isn't necessarily true of Abelard—but it's true of *someone*.
- ▹ So, '$\exists y\, \forall x\, Lxy$' says that *someone* is loved by everyone.

9. Next, consider, '$\exists y\, \forall x\, Lyx$'.

- ▹ '$\exists y\, \forall x\, Lyx$' says that '$\forall x\, Lyx$' is true when we let '$y$' refer to *some* particular person.
- ▹ Let 'y' refer to Abelard. Then, '$\forall x\, Lyx$' says that Abelard loves everyone.
- ▹ '$\exists y\, \forall x\, Lyx$' says that this isn't necessarily true of Abelard—but it's true of *someone*.
- ▹ So '$\exists y\, \forall x\, Lyx$' says that *someone* loves everyone.

10. Finally: '$\forall x\, \exists y\, Lyx$'.

- ▹ '$\forall x\, \exists y\, Lyx$' says that '$\exists y\, Lyx$' is true *no matter who* we let 'x' refer to.
- ▹ Let 'x refer to Abelard. Then, '$\exists y\, Lyx$' says that Abelard is loved by someone.
- ▹ '$\forall x\, \exists y\, Lyx$' says that this isn't just true of Abelard—it's true of *everyone*.
- ▹ So '$\forall x\, \exists y\, Lyx$' says that *everyone* is loved by someone.

Part A

A. Using the following symbolization key:

domain	: all people
Rxy	: $\_\_\_\_\_\__x$ respects $\_\_\_\_\_\__y$
Fx	: $\_\_\_\_\_\__x$ is a farmer
Px	: $\_\_\_\_\_\__x$ is a professor
Ax	: $\_\_\_\_\_\__x$ is an actor
a	: Albert
l	: Laurence Olivier

translate these sentences of English into PL:

1. If Albert respects any actor, then he respects Laurence Olivier
2. Every actor respects all actors
3. Everyone respects someone
4. Laurence Olivier doesn't respect any actor
5. Some farmer respects every professor
6. Someone is respected by everyone
7. Anyone who respects all professors is a professor themselves
8. No professor respects every actor
9. Someone respects everyone
10. All farmers respect Albert
11. Albert respects all farmers
12. Everyone is respected by someone

and translate these sentences of PL into *idiomatic* English:

13. $\forall x \, \exists y \, Rxy$
14. $\forall x \, \exists y \, Ryx$
15. $\exists y \, \forall x \, Rxy$
16. $\exists y \, \forall x \, Ryx$
17. $\forall x \, [Fx \rightarrow \forall y \, (Py \rightarrow \neg Rxy)]$
18. $\exists x \, [Ax \wedge \exists y \, (Fx \wedge Rxy)]$
19. $\forall x \, [\exists y \, (Fy \wedge Rxy) \rightarrow Fx]$

20. $\exists z\, [\exists y\, (Ay \wedge Rzy) \wedge \exists x\, (\neg Ax \wedge Rzx)]$

21. $\forall x\, (Ax \rightarrow \neg Fx)$

22. $\exists x\, [Ax \wedge \forall y\, (Py \rightarrow Ryx)]$

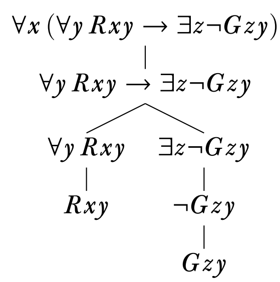
23. $\forall x\, \forall y\, Rxy$

24. $\forall x\, \forall y\, [(Px \wedge Py) \rightarrow \neg Rxy]$

Part B

Which of the following expressions are sentences of PL? For each expression which is a sentence, write out its syntax tree and identify its main operator. For each quantifier in the sentence, say what that quantifier's *scope* is, and say which variables (if any) it *binds*.

For instance, if you are given the expression '$\forall xy\, Ryx$' you should say that it is *not* a sentence. And if we are given the expression '$\forall x\, (\forall y\, Rxy \rightarrow \exists z \neg Gzy)$', you should say that it is a *sentence*, and provide the following syntax tree:



The main operator of this sentence is the universal quantifier '$\forall x$'. The scope of '$\forall x$' is the entire sentence, '$\forall x\, (\forall y\, Rxy \rightarrow \forall z \neg Gzy)$'. It binds the '$x$' in '$Rxy$'. The scope of '$\forall y$' is '$\forall y\, Rxy$'. It binds the '$y$' in '$Rxy$'. It *doesn't* bind the 'y' in 'Gzy'. The scope of '$\exists z$' is '$\exists z\, \neg Gzy$'. It binds the 'z' in 'Gzy'.

1. $\forall x\, \exists x\, Rxx$
2. $\forall y\, \exists x\, Rxy$
3. $\forall x\, (\exists y\, Rxy \rightarrow \forall y\, \neg Ryx)$
4. $\forall w\, (\forall x\, (\forall y\, (Fy)))$
5. $\forall w\, \forall x\, \forall y\, Fy$
6. $\forall x\, \exists y\, (Rxz \rightarrow \exists z\, (Gyz \wedge Pyyz))$
7. $(Rabcyw \rightarrow \exists w\, Rabcyw)$
8. $\forall x\, \exists x\, (Qax \rightarrow \forall z\, Qzz)$

Part C

Consider the following interpretation:[6]

[6] Parts C and D come from *Forall x*, by Magnus and Button.

- ▹ The domain comprises only Corwin and Benedict
- ▹ 'Ax' is to be true of both Corwin and Benedict
- ▹ 'Bx' is to be true of Benedict only
- ▹ 'Nx' is to be true of no one
- ▹ 'c' is to refer to Corwin

Determine whether each of the following sentences is true or false in that interpretation:

1. Bc
2. $Ac \leftrightarrow \neg Nc$
3. $Nc \rightarrow (Ac \vee Bc)$
4. $\forall x\, Ax$
5. $\forall x \neg Bx$
6. $\exists x(Ax \wedge Bx)$
7. $\exists x(Ax \rightarrow Nx)$
8. $\forall x(Nx \vee \neg Nx)$

Part D

Consider the following interpretation:

- ▹ The domain comprises only Lemmy, Courtney and Eddy
- ▹ 'Gx' is to be true of Lemmy, Courtney and Eddy.
- ▹ 'Hx' is to be true of and only of Courtney
- ▹ 'Mx' is to be true of and only of Lemmy and Eddy
- ▹ 'c' is to refer to Courtney
- ▹ 'e' is to refer to Eddy

Determine whether each of the following sentences is true or false in that interpretation:

1. Hc
2. He
3. $Mc \vee Me$
4. $Gc \vee \neg Gc$
5. $Mc \rightarrow Gc$
6. $\exists x\, Hx$

7. $\forall x\, Hx$
8. $\exists x \neg Mx$
9. $\exists x(Hx \wedge Gx)$
10. $\exists x(Mx \wedge Gx)$
11. $\forall x(Hx \vee Mx)$
12. $\exists x\, Hx \wedge \exists x\, Mx$
13. $\forall x(Hx \leftrightarrow \neg Mx)$
14. $\exists x\, Gx \wedge \exists x \neg Gx$
15. $\forall x \exists y(Gx \wedge Hy)$

Part 1

A. Using the following symbolization key:

$$\begin{aligned} \text{domain} &: \text{all people} \\ Rxy &: \underline{\quad\quad}_x \text{ respects } \underline{\quad\quad}_y \\ Fx &: \underline{\quad\quad}_x \text{ is a farmer} \\ Px &: \underline{\quad\quad}_x \text{ is a professor} \\ Ax &: \underline{\quad\quad}_x \text{ is an actor} \\ a &: \text{Albert} \\ l &: \text{Laurence Olivier} \end{aligned}$$

translate these sentences of English into PL:

1. If Albert respects any actor, then he respects Laurence Olivier

$$\exists x(Ax \wedge Rax) \rightarrow Ral$$

2. Every actor respects all actors

$$\forall x[Ax \rightarrow \forall y(Ay \rightarrow Rxy)]$$
$$\text{or: } \forall x \forall y[(Ax \wedge Ay) \rightarrow Rxy]$$

3. Everyone respects someone

$$\forall x \exists y Rxy$$

4. Laurence Olivier doesn't respect any actor

$$\forall x(Ax \rightarrow \neg Rlx)$$
$$\text{or}: \quad \neg \exists x(Ax \wedge Rlx)$$

5. Some farmer respects every professor

$$\exists x[Fx \wedge \forall y(Py \rightarrow Rxy)]$$

6. Someone is respected by everyone

$$\exists x \forall y Ryx$$

7. Anyone who respects all professors is a professor themselves

$$\forall x(\forall y(Py \rightarrow Rxy) \rightarrow Px)$$

8. No professor respects every actor

$$\forall x[Px \rightarrow \neg \forall y(Ay \rightarrow Rxy)]$$
$$\text{or}: \quad \neg \exists x(Px \wedge \forall y(Ay \rightarrow Rxy))]$$

9. Someone respects everyone

$$\exists x \forall y Rxy$$

10. All farmers respect Albert

$$\forall x(Fx \rightarrow Rxa)$$

11. Albert respects all farmers

$$\forall x(Fx \rightarrow Rax)$$

12. Everyone is respected by someone

$$\forall x \exists y Ryx$$

and translate these sentences of PL into *idiomatic* English:

13. $\forall x\, \exists y\, Rxy$

    Everyone respects someone

14. $\forall x\, \exists y\, Ryx$

    Everyone is respected by someone

15. $\exists y\, \forall x\, Rxy$

    Someone is respected by everyone

16. $\exists y\, \forall x\, Ryx$

    Someone respects everyone

17. $\forall x\, [Fx \rightarrow \forall y\, (Py \rightarrow \neg Rxy)]$

    Farmers don't respect any professors.

18. $\exists x\, [Ax \wedge \exists y\, (Fy \wedge Rxy)]$

    Some actor respects some farmer

    (Note: in the problem set, there was a typo, and I wrote this as '$\exists x\, [Ax \wedge \exists y(Fx \wedge Rxy)]$'. The right way to translate the sentence with the typo is: 'Some actor is a farmer who respects someone'.)

19. $\forall x\, [\exists y\, (Fy \wedge Rxy) \rightarrow Fx]$

    Anyone who respects a farmer is a farmer themselves. (Or: Only farmers respect farmers.)

20. $\exists z\, [\exists y\, (Ay \wedge Rzy) \wedge \exists x\, (\neg Ax \wedge Rzx)]$

    Some people respect both actors and non-actors

21. $\forall x\, (Ax \rightarrow \neg Fx)$

    No actor is a farmer

22. $\exists x\, [Ax \wedge \forall y\, (Py \rightarrow Ryx)]$

    Some actor is respected by every professor.

23. $\forall x\, \forall y\, Rxy$

    Everyone respects everyone.

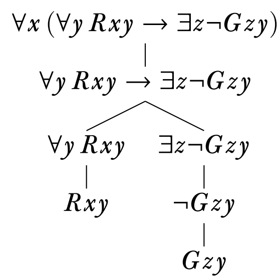
24. $\forall x\, \forall y\, [(Px \wedge Py) \rightarrow \neg Rxy]$

    No professor respects a professor.

Part 2

Which of the following expressions are sentences of PL? For each expression which is a sentence, write out its syntax tree and identify its main operator. For each quantifier in the sentence, say what that quantifier's *scope* is, and say which variables (if any) it *binds*.

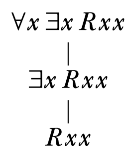
For instance, if you are given the expression '$\forall xy\, Ryx$' you should say that it is *not* a sentence. And if we are given the expression '$\forall x\, (\forall y\, Rxy \rightarrow \exists z \neg Gzy)$', you should say that it is a *sentence*, and provide the following syntax tree:



The main operator of this sentence is the universal quantifier '$\forall x$'. The scope of '$\forall x$' is the entire sentence, '$\forall x\, (\forall y\, Rxy \rightarrow \forall z \neg Gzy)$'. It binds the '$x$' in '$Rxy$'. The scope of '$\forall y$' is '$\forall y\, Rxy$'. It binds the '$y$' in '$Rxy$'. It *doesn't* bind the 'y' in 'Gzy'. The scope of '$\exists z$' is '$\exists z\, \neg Gzy$'. It binds the 'z' in 'Gzy'.

1. $\forall x\, \exists x\, Rxx$

   This is a sentence. Here is its syntax tree:

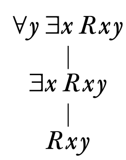


   Its main operator is '$\forall x$'.

   The scope of '$\forall x$' is '$\forall x \exists x Rxx$'. '$\forall x$' doesn't bind any variables. The scope of '$\exists x$' is '$\exists x Rxx$'. '$\exists x$' binds both 'x's in 'Rxx'.

2. $\forall y\, \exists x\, Rxy$

   (a) This is a sentence. Here is its syntax tree:

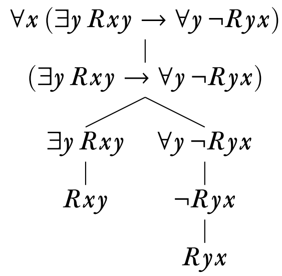


   Its main operator is '$\forall y$'.

   The scope of '$\forall y$' is '$\forall y\, \exists x\, Rxy$'. '$\forall y$' binds the '$y$' in '$Rxy$'. The scope of '$\exists x$' is '$\exists x Rxy$'. '$\exists x$' binds the '$x$' in '$Rxy$'.

3. $\forall x\, (\exists y\, Rxy \rightarrow \forall y\, \neg Ryx)$

   This is a sentence. Here is its syntax tree:



Its main operator is '$\forall x$'.

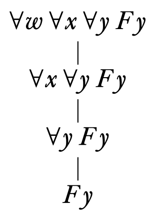
The scope of '$\forall x$' is the entire sentence, '$\forall x\,(\exists y\, Rxy \to \forall y\, \neg Ryx)$'. It binds the '$x$' in '$Rxy$' and the '$x$' in '$Ryx$'. The scope of '$\exists y$' is '$\exists y Rxy$'. It binds the '$y$' in '$Rxy$'. The scope of '$\forall y$' is '$\forall y \neg Ryx$'. It binds the '$y$' in '$Ryx$'.

4. $\forall w\,(\forall x\,(\forall y\,(Fy)))$

   This is not a sentence.

5. $\forall w\,\forall x\,\forall y\, Fy$

   This is a sentence. Here is its syntax tree:

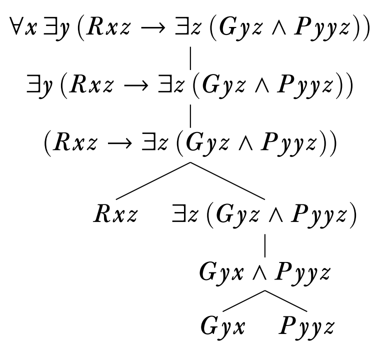


Its main operator is '$\forall w$'.

The scope of '$\forall w$' is the entire sentence $\forall w\,\forall x\,\forall y\, Fy$. It doesn't bind any variables. The scope of '$\forall x$' is '$\forall x\,\forall y\, Fy$'. It doesn't bind any variables. The scope of '$\forall y$' is '$\forall y\, Fy$'. It binds the 'y' in 'Fy'.

6. $\forall x\,\exists y\,(Rxz \to \exists z\,(Gyz \land Pyyz))$

   This is a sentence. Here is its syntax tree:



Its main operator is '$\forall x$'.

The scope of '$\forall x$' is the entire sentence, '$\forall x\,\exists y\,(Rxz \to \exists z\,(Gyz \land Pyyz))$'. It binds the '$x$' in '$Rxz$'. The scope of '$\exists y$' is '$\exists y\,(Rxz \to \exists z\,(Gyz \land Pyyz))$'. It binds the '$y$' in '$Gyz$' and both '$y$'s in '$Pyyz$'. The scope of '$\exists z$' is '$\exists z\,(Gyz \land Pyyz)$'. It binds the '$z$' in '$Gyz$' and the '$z$' in '$Pyyz$'.

7. $(Rabcyw \to \exists w\, Rabcyw)$

   This is a sentence. Here is its syntax tree:

$$(Rabcyw \to \exists w\, Rabcyw)$$

$$Rabcyw \qquad \exists w\, Rabcyw$$

$$Rabcyw$$

It's main operator is '$\to$'.
The scope of '$\exists w$' is '$\exists w\, Rabcyw$'. It binds the the final 'w' in the sentence.

8. $\forall x\, \exists x\, (Qax \to \forall z\, Qzz)$

   This is a sentence. Here is its syntax tree:

   $$\forall x\, \exists x\, (Qax \to \forall z\, Qzz)$$

   $$\exists x\, (Qax \to \forall z\, Qxx)$$

   $$(Qax \to \forall z\, Qzz)$$

   $$Qax \qquad \forall z\, Qzz$$

   $$Qzz$$

   Its main operator is '$\forall x$'.
   The scope of '$\forall x$' is the entire sentence, '$\forall x\, \exists x\, (Qax \to \forall z\, Qzz)$'. It doesn't bind any variables. The scope of '$\exists x$' is '$\exists x\, (Qax \to \forall z\, Qzz)$'. It binds the '$x$' in '$Qax$'. The scope of '$\forall z$' is '$\forall z\, Qzz$'. It binds both '$z$'s in '$Qzz$'.

Part 3

Complete exercises A and B in chapter 26 of *Forall x*.

Part A

Domain:	Corwin and Benedict
A :	Corwin and Benedict
B :	Benedict
N :	
c :	Corwin

1. 'Bc' is **false** on the interpretation
2. '$Ac \leftrightarrow \neg Nc$' is **true** on the interpretation
3. '$Nc \to (Ac \vee Bc)$' is **true** on the interpretation
4. '$\forall x Ax$' is **true** on the interpretation
5. '$\forall x \neg Bx$' is **false** on the interpretation
6. '$\exists x(Ax \wedge Bx)$' is **true** on the interpretation
7. '$\exists x(Ax \to Nx)$' is **false** on the interpretation
8. '$\forall x(Nx \vee \neg Nx)$' is **true** on the interpretation
9. '$\exists x Bx \to \forall Ax$' is **true** on the interpretation

Part B

Domain:	Lemmy, Courtney, and Eddy
G :	Lemmy, Courtney, and Eddy
H :	Courtney
M :	Lemmy and Eddy
c :	Courtney
e :	Eddy

1. 'Hc' is **true** on the interpretation
2. 'He' is **false** on the interpretation
3. '$Mc \vee Me$' is **true** on the interpretation
4. '$Gc \vee \neg Gc$' is **true** on the interpretation
5. '$Mc \rightarrow Gc$' is **true** on the interpretation
6. '$\exists x Hx$' is **true** on the interpretation
7. '$\forall x Hx$' is **false** on the interpretation
8. '$\exists x \neg Mx$' is **true** on the interpretation
9. '$\exists x(Hx \wedge Gx)$' is **true** on the interpretation
10. '$\exists x(Mx \wedge Gx)$' is **true** on the interpretation
11. '$\forall x(Hx \vee Mx)$' is **true** on the interpretation
12. '$\exists x Hx \wedge \exists x Mx$' is **true** on the interpretation
13. '$\forall x(Hx \leftrightarrow \neg Mx)$' is **true** on the interpretation
14. '$\exists x Gx \wedge \exists x \neg Gx$' is **false** on the interpretation
15. '$\forall x \exists y(Gx \wedge Hy)$' is **true** on the interpretation

1. Recall, an argument is *valid* iff there is no possibility in which its premises are all true and its conclusion is false.

2. And, in SL, we came up with a formal surrogate for validity—which we called *entailment* (in SL)—by exchanging the notion of a *possibility* for the notion of a *valuation*.

   ▹ A *valuation* was an assignment of truth-values to the atomic statement letters of SL.

   ▹ An argument is an *entailment* (in SL) iff there is no valuation which makes all of its premises true while its conclusion is false.

   ▹ That is: $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *entails* $\mathscr{C}$ iff there is no valuation which makes all of $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ true and $\mathscr{C}$ false.

3. For PL, we will similarly get a formal surrogate of validity—which we will also call *entailment* (in PL)—by exchanging the notion of a possibility for the notion of an *interpretation*.

> Entailment (in PL)
>
> $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ *entail* $\mathscr{C}$ in PL,
>
> $$\mathscr{A}_1, \mathscr{A}_2, \ldots \mathscr{A}_N \models \mathscr{C}$$
>
> iff there is no **interpretation** on which $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are all true and $\mathscr{C}$ is false.

   ▹ For the remainder of the course, whenever we say 'entail', we will mean 'entail *in PL*'.

   ▹ For the remainder of the course, whenever we write '$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \models \mathscr{C}$', we will mean that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ entail $\mathscr{C}$ *in PL*.

4. If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ entail $\mathscr{C}$, then the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is valid. However, just because the argument $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \therefore \mathscr{C}$ is valid, this doesn't mean that $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ entail $\mathscr{C}$.

   ▹ Consider the argument 'Everything in my house is red. Therefore, everything in my house is colored.' This argument is valid, but when we translate it into PL, we will get the argument '$\forall x\,(Hx \to Rx) \therefore \forall y\,(Hy \to Cy)$'. And this argument is not an entailment. For the following interpretation makes its premises true and its conclusion false:

$$\begin{aligned} \text{domain} &: 1 \\ H &: 1 \\ R &: 1 \\ C &: \end{aligned}$$

   ▹ Given what R and C *mean in English*, this interpretation doesn't represent a genuine possibility. It is a *bogus* possibility. So, even though there's an interpretation which makes the premise true and the conclusion false, it doesn't follow that there is any *possibility* which makes the premise true and the conclusion false.

   ▹ In general, every possibility corresponds to *some* interpretation;[7] however, not every interpretation corresponds to some possibility.

   ▹ In general, then: if we can show that something holds for *all* interpretations, then we know that it holds for all possibilities. But, if we've only shown that something holds for *some* interpretation, that doesn't tell us that it holds for any possibility. (Though an interpretation which shows that an argument isn't an entailment can help us look for a possibility which shows that an argument isn't valid.)

5. In SL, we were able to check every relevant valuation (every row of the truth-table) to *prove* that an argument was an entailment (in SL). In PL, there's no way to check every possible interpretation (since there are infinitely many interpretations for any given argument).

   ▹ To show that an argument is an entailment in PL, we will use natural deduction proofs.

[7] This shouldn't seem obvious to you. If it does, think more explicitly about what it says. Try to argue against it. Then think about how you would argue *for* it.

Proving an Argument isn't an Entailment

6. Consider this argument:

> Someone is fast, and someone is tall. So someone is fast and tall.

We can translate this into PL using this symbolization key:

domain	: all people
Fx	: $\_\_\_\_\__x$ is fast
Tx	: $\_\_\_\_\__x$ is tall

The translation is:

$$\exists x\, Fx \wedge \exists y\, Ty \;\therefore\; \exists z\,(Fz \wedge Tz)$$

To show that this argument is not an entailment, we can provide a single interpretation which makes its premise true and its conclusion false. Here's one that does the trick:

domain	: Amy, Bill
F	: Amy
T	: Bill

- ▹ On this interpretation, '$\exists x\, Fx$' is true, since 'Fx' is true if we let 'x' be a name for Amy. And '$\exists y\, Ty$' is true, since 'Ty' is true if we let y be a name for Bill. So '$\exists x\, Fx \wedge \exists y\, Ty$' is true. But no matter what we let z be a name for, '$Fz \wedge Tz$' will be false. So '$\exists z\,(Fz \wedge Tz)$' is false. So this interpretation makes the premise true and the conclusion false.

7. Another example:

> Abelard loves everyone who loves him. Heloise doesn't love Abelard. So Abelard doesn't love Heloise.

We can translate this into PL using this symbolization key:

domain	: all people
Lxy	: $\_\_\_\_\__x$ loves $\_\_\_\_\__y$
a	: Abelard
h	: Heloise

The translation is:

$$\forall x\,(Lxa \to Lax),\ \neg Lha \;\therefore\; \neg Lah$$

To show that this argument is not an entailment, we provide an interpretation which makes its premise true and its conclusion false. Here's one:

domain	: Abelard, Heloise
L	: $\langle$ Abelard, Heloise $\rangle$
a	: Abelard
h	: Heloise

- ▹ On this interpretation, '$\forall x\,(Lxa \to Lax)$' is true. For we could either let 'x' name Abelard or Heloise. If 'x' names Abelard, then 'Lxa' is false. So '$Lxa \to Lax$' is true. If 'x' names Heloise, then 'Lxa' is false. So '$Lxa \to Lax$' is true. So, no matter what 'x' names, '$Lxa \to Lax$' is true.
- ▹ And, on this interpretation, Heloise doesn't bear the L-relation to Abelard. So 'Lha' is false. So '$\neg Lha$' is true.
- ▹ However, Abelard *does* bear the L-relation to Heloise. So 'Lah' is true. So '$\neg Lah$' is false.
- ▹ So this interpretation makes the premises true and the conclusion false.

Satisfiability

1. Recall, a collection of sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are *jointly possible* iff there is some *possibility* in which $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ are all true. And we say that a collection of sentences are *jointly impossible* iff there is no possibility in which they are all true.

2. And, in SL, we came up with a formal surrogate for joint (im)possibility—which we called *(un)satisfiability* (in SL)—by exchanging the notion of a *possibility* for the notion of a *valuation*.
   - ▹ A *valuation* was an assignment of truth-values to the atomic statement letters of SL.
   - ▹ A collection of sentences were *satisfiable* iff there is some valuation which makes them all true.
   - ▹ And a collection of sentences were *unsatisfiable* iff there is no valuation which makes them all true.

3. For PL, we will similarly get a formal surrogate of joint (im)possibility—which we will also call *(un)satisfiability* (in PL)—by exchanging the notion of a possibility for the notion of an *interpretation*.

> Satisfiability (in PL)
>
> A collection of sentences are *satisfiable* in PL iff there is some **interpretation** which makes each of them true.
>
> A collection of sentences is *unsatisfiable* in PL iff there is no **interpretation** which makes all of them true.

   - ▹ For the remainder of the course, whenever we say 'satisfiable', we will mean 'satisfiable *in PL*'.
   - ▹ And, for the remainder of the course, whenever we say 'unsatisfiable', we will be 'unsatisfiable *in PL*'.

4. If a collection of sentences are unsatisfiable, then they are jointly impossible. However, just because a collection of sentences are *satisfiable*, it doesn't necessarily follow that they are jointly possible.
   - ▹ Consider the pair of sentences 'Sabeen is taller than Luella' and 'Luella is taller than Sabeen'.
   - ▹ If we translate these sentences into PL, we get 'Tsl' and 'Tls'. But here is an interpretation on which both of these sentences are true:

$$\begin{aligned} \text{domain} &: 1 \\ T &: \langle 1, 1 \rangle \\ s &: 1 \\ l &: 1 \end{aligned}$$

   - ▹ Given what T means *in English*, this interpretation doesn't represent a genuine possibility—for nothing is taller than itself. It is a *bogus* possibility. So, even though there's an interpretation which makes both 'Tsl' and 'Tls' true, it doesn't follow that there is any *possibility* which makes both of these claims true.
   - ▹ In general, every possibility corresponds to *some* interpretation; however, not every interpretation corresponds to some possibility.
   - ▹ In general, then: if we can show that something holds for *all* interpretations, then we can conclude that it holds for all possibilities. But if we've only shown that something holds for *one* interpretation, it doesn't follow that it holds for any possibility (it could have been a bogus interpretation).

5. In SL, we were able to check every relevant valuation (every row of the truth-table) to *prove* that a collection of sentences were unsatisfiable (in SL). In PL, there's no way to check every possible interpretation, since there are infinitely many of them.

- ▹ To show that sentences are *unsatisfiable* in PL, we'll have to use natural deduction proofs. (If we can derive $\bot$ from the sentences, then they are unsatisfiable).

6. However, in PL, we can show that a collection of sentences are *satisfiable*, it will suffice to provide a single interpretation which makes all of the sentences true.

   - ▹ For instance, consider the pair of sentences '$\neg\forall x\, Lxx$' and '$\exists y \forall x Lxy$'. And consider this interpretation:

$$\begin{aligned} \text{domain} &: 1, 2 \\ L &: \langle 1, 2 \rangle, \langle 2, 2 \rangle \end{aligned}$$

   - ▹ This interpretation makes both '$\neg\forall x\, Lxx$' and '$\exists y \forall x Lxy$' true. So these sentences are *satisfiable*.

Tautologies and Contradictions

7. Recall, a sentence is a *necessary truth* iff it is true in every possibility. A sentence is a *necessary falsehood* iff it is false in every possibility. And a sentence is a *contingency* iff it is true in some possibilities and false in some other possibilities.

8. And, in SL, we came up with a formal surrogate for necessary truths and necessary falsehoods—which we called *tautologies* (in SL) and *contradictions* (in SL)—by exchanging the notion of a *possibility* for the notion of a *valuation*.

9. For PL, we will similarly get a formal surrogate of necessary truths and falsehoods—which we will also call *tautologies* and *contradictions* (in PL)—by exchanging the notion of a possibility for the notion of an *interpretation*.

> Tautology (in PL)
>
> A sentence is a *tautology* in PL iff it is true in every **interpretation**.

> Contradiction (in PL)
>
> A sentence is a *contradiction* in PL iff it is false in every **interpretation**.

   - ▹ And a sentence is *neither a contradiction nor a tautology* iff it is true in some interpretations and false in some interpretations.
   - ▹ For the remainder of the course, whenever we say 'tautology' or 'contradiction', we will mean *in PL*.

10. In SL, we were able to check every relevant valuation to see whether or not a sentence was a tautology or a contradiction or neither. In PL, this is impossible, since there are infinitely many interpretations—we can't check them all. So, in PL, we will show that a sentence is a tautology or a contradiction by using *natural deduction* proofs.

   - ▹ To show that '$\mathscr{A}$' is a tautology, we can show that $\vdash \mathscr{A}$, or that $\neg\mathscr{A} \vdash \bot$. And to show that '$\mathscr{A}$' is a contradiction, we can show that $\mathscr{A} \vdash \bot$.

11. However, to show that a sentence is *neither a tautology nor a contradiction*, we can simply produce two interpretations: one which makes the sentence true, and one which makes the sentence false.

   - ▹ For instance, consider the sentence 'Someone hates everyone who loves them'. If we translate this into PL (with a domain of people), we will get the sentence '$\exists x \forall y (Lyx \to Hxy)$'. Then, consider these two interpretations:

$$\begin{aligned} \text{domain} &: 1, 2 \\ L &: \langle 2, 1 \rangle, \langle 1, 1 \rangle \\ H &: \langle 1, 2 \rangle, \langle 1, 1 \rangle \end{aligned} \qquad\qquad \begin{aligned} \text{domain} &: 1 \\ L &: \langle 1, 1 \rangle \\ H &: \end{aligned}$$

   The interpretation on the left makes '$\exists x \forall y (Lyx \to Hxy)$' true, and the interpretation on the right makes it false. So it is true on some interpretation and false on some interpretation. So it is neither a tautology nor a contradiction.

Part A

Provide an interpretation to show each of the following:

1. $\forall x(Px \rightarrow Qx) \not\models \exists y(Py \land Qy)$
2. $\forall x(Ax \rightarrow Bx), \forall x(Ax \rightarrow Cx) \not\models \exists y(By \land Cy)$
3. $Wa, Wb, Wc \not\models \forall w Ww$
4. $Zab, \exists y Zya \not\models Zba$
5. $\exists z(Fz \land Gz), \exists x Gx \rightarrow \exists y Hy, \not\models \exists w(Fw \land Hw)$
6. $\forall x Uxa, \forall y Oay \not\models \forall x Oxx$
7. $\exists x(Jx \land Kx), \exists x \neg Kx, \exists x \neg Jx \not\models \exists z(\neg Jz \land \neg Kz)$
8. $Lab \rightarrow \forall x Lxb, \exists x Lxb \not\models Lbb$

Part B

Using the following symbolization key:

domain	: all people
Dx	: $\_\_\_\_\__x$ is a Democrat
Rx	: $\_\_\_\_\__x$ is a Republican
Cx	: $\_\_\_\_\__x$ is conservative
Ix	: $\_\_\_\_\__x$ has done something illegal
Px	: $\_\_\_\_\__x$ is the President
e	: Elizabeth Warren
j	: Joe Biden
d	: Donald Trump

translate these statements into PL, and then show that they are neither tautologies nor contradictions.

1. All Democrats have done something illegal.
2. If some Democrat has done something illegal, then Trump hasn't done something illegal.
3. Elizabeth Warren is a conservative Republican.
4. Some conservative Democrat is President.
5. Some Democrat is the President and some Republican is the President.
6. No one is the President.
7. All conservative Republicans have done something illegal.
8. No conservative Republican has done something illegal.
9. If Joe Biden has done something illegal, then no Republican has done something illegal.

Part C

Using the following symbolization key,

domain	: all people
Fxy	: $\_\_\_\__x$ is friends with $\_\_\_\__y$
Ax	: $\_\_\_\__x$ is an athlete
Px	: $\_\_\_\__x$ is a philosopher
Qx	: $\_\_\_\__x$ is quirky
k	: Kanye
l	: Lewis

translate the following collections of sentences into PL and then show that they are satisfiable.

1. No philosopher is quirky. Every philosopher is quirky.
2. Kanye is a philosopher. No philosopher is friends with Kanye. Some philosopher is friends with themselves.
3. Kanye is a philosopher. Lewis is a philosopher. Not everyone is a philosopher.
4. Everyone who is friends with Lewis is friends with a philosopher. No one is friends with Kanye.
5. Some athlete is a philosopher. No philosophers are quirky. Some athlete is quirky.
6. All quirky people are friends with some quirky person. Kanye is quirky. Kanye is friends with Lewis.
7. If anyone is quirky, Lewis is quirky. Lewis isn't quirky. Lewis is friends with himself.
8. No quirky philosopher is friends with a quirky philosopher. All athletes are friends with some athlete. Lewis is neither an athlete nor a philosopher.
9. Lewis is friends with everybody unless he's not friends with Kanye. Someone is friends with everyone.

Part A

1. $\forall x(Px \to Qx) \not\models \exists y(Py \land Qy)$

$$\begin{array}{rl} \text{domain} & : 1 \\ P & : \\ Q & : \end{array}$$

2. $\forall x(Ax \to Bx), \forall x(Ax \to Cx) \not\models \exists y(By \land Cy)$

$$\begin{array}{rl} \text{domain} & : 1 \\ A & : \\ B & : \\ C & : \end{array}$$

3. $Wa, Wb, Wc \not\models \forall w Ww$

$$\begin{array}{rl} \text{domain} & : 1, 2 \\ W & : 1 \\ a & : 1 \\ b & : 1 \\ c & : 1 \end{array}$$

4. $Zab, \exists y Zya \not\models Zba$

$$\begin{array}{rl} \text{domain} & : 1, 2 \\ Z & : \langle 1, 2 \rangle, \langle 1, 1 \rangle \\ a & : 1 \\ b & : 2 \end{array}$$

5. $\exists z(Fz \land Gz), \exists x Gx \to \exists y Hy, \not\models \exists w(Fw \land Hw)$

$$\begin{array}{rl} \text{domain} & : 1, 2 \\ F & : 1 \\ G & : 1 \\ H & : 2 \end{array}$$

6. $\forall x Uxa, \forall y Oay \not\models \forall x Oxx$

$$\begin{array}{rl} \text{domain} & : 1, 2 \\ U & : \langle 1, 1 \rangle, \langle 2, 1 \rangle \\ O & : \langle 1, 1 \rangle, \langle 1, 2 \rangle \\ a & : 1 \end{array}$$

7. $\exists x(Jx \land Kx), \exists x \neg Kx, \exists x \neg Jx \not\models \exists z(\neg Jz \land \neg Kz)$

$$\begin{array}{rl} \text{domain} & : 1, 2, 3 \\ J & : 1, 2 \\ K & : 1, 3 \end{array}$$

8. $Lab \rightarrow \forall x Lxb, \exists x Lxb \not\models Lbb$

$$\begin{aligned} \text{domain} &: 1, 2, 3 \\ L &: \langle 3, 2 \rangle \\ a &: 1 \\ b &: 2 \end{aligned}$$

Part B

Using the following symbolization key:

domain	: all people
Dx	: _____$_x$ is a Democrat
Rx	: _______$_x$ is a Republican
Cx	: _______$_x$ is conservative
Ix	: _______$_x$ has done something illegal
Px	: _____$_x$ is the President
e	: Elizabeth Warren
j	: Joe Biden
d	: Donald Trump

translate these statements into PL, and then show that they are neither tautologies nor contradictions.

1. All Democrats have done something illegal.

$$\forall x(Dx \rightarrow Ix)$$

true		false	
domain	: 1	domain	: 1
D	: 1	D	: 1
I	: 1	I	:

2. If some Democrat has done something illegal, then Trump hasn't done something illegal.

$$\exists x(Dx \wedge Ix) \rightarrow \neg Id$$

true		false	
domain	: 1	domain	: 1
D	:	D	: 1
I	:	I	: 1
d	: 1	d	: 1

3. Elizabeth Warren is a conservative Republican.

$$Ce \wedge Re$$

true			
		domain	: 1
		C	: 1
		R	: 1
		e	: 1

false

domain	: 1
C	:
R	:
e	: 1

4. Some conservative Democrat is President.

$$\exists x[(Cx \land Dx) \land Px]$$

true

domain	: 1
C	: 1
D	: 1
P	: 1

false

domain	: 1
C	:
D	:
P	:

5. Some Democrat is the President and some Republican is the President.

$$\exists x(Dx \land Px) \land \exists y(Ry \land Py)$$

true

domain	: 1
D	: 1
R	: 1
P	: 1

false

domain	: 1
D	:
R	:
P	:

6. No one is the President.

$$\forall x \neg Px$$

true

domain	: 1
P	:

false

domain	: 1
P	: 1

7. All conservative Republicans have done something illegal.

$$\forall x[(Cx \land Rx) \to Ix]$$

true

domain	: 1
C	:
R	:
I	:

false

domain	: 1
C	: 1
R	: 1
I	:

8. No conservative Republican has done something illegal.

$$\forall x[(Cx \land Rx) \to \neg Ix]$$

true

domain	: 1
C	: 1
R	: 1
I	:

false

domain	: 1
C	: 1
R	: 1
I	: 1

9. If Joe Biden has done something illegal, then no Republican has done something illegal.

$$Ij \rightarrow \forall x(Rx \rightarrow \neg Ix)$$

<u>true</u>

domain	: 1
I	:
R	:
j	: 1

<u>false</u>

domain	: 1
I	: 1
R	: 1
j	: 1

Part C

Using the following symbolization key,

domain	: all people
Fxy	: $\_\_\_\__x$ is friends with $\_\_\_\__y$
Ax	: $\_\_\_\__x$ is an athlete
Px	: $\_\_\_\__x$ is a philosopher
Qx	: $\_\_\_\__x$ is quirky
k	: Kanye
l	: Lewis

translate the following collections of sentences into PL and then show that they are satisfiable.

1. No philosopher is quirky. Every philosopher is quirky.

$$\forall x(Px \rightarrow \neg Qx),\ \forall y(Py \rightarrow Qy)$$

domain	: 1
P	:
Q	:

2. Kanye is a philosopher. No philosopher is friends with Kanye. Some philosopher is friends with themselves.

$$Pk,\ \forall x(Px \rightarrow \neg Fxk),\ \exists y(Py \wedge Fyy)$$

domain	: 1, 2
P	: 1, 2
F	: $\langle 2, 2 \rangle$
k	: 1

3. Kanye is a philosopher. Lewis is a philosopher. Not everyone is a philosopher.

$$Pk,\ Pl,\ \neg\forall x Px$$

domain	: 1, 2
P	: 1
k	: 1
l	: 1

4. Everyone who is friends with Lewis is friends with a philosopher. No one is friends with Kanye.

$$\forall x[Fxl \to \exists y(Py \land Fxy)],\ \forall y \neg Fyk$$

$$\begin{aligned} \text{domain} &: 1 \\ P &: \\ F &: \\ k &: 1 \\ l &: 1 \end{aligned}$$

5. Some athlete is a philosopher. No philosophers are quirky. Some athlete is quirky.

$$\exists x(Ax \land Px),\ \forall y(Py \to \neg Qy),\ \exists z(Az \land Qz)$$

$$\begin{aligned} \text{domain} &: 1, 2 \\ A &: 1, 2 \\ P &: 1 \\ Q &: 2 \end{aligned}$$

6. All quirky people are friends with some quirky person. Kanye is quirky. Kanye is friends with Lewis.

$$\forall x\,[Qx \to \exists y(Qy \land Fxy)],\ Qk,\ Fkl$$

$$\begin{aligned} \text{domain} &: 1, 2 \\ Q &: 1 \\ F &: \langle 1, 1 \rangle \\ k &: 1 \\ l &: 1 \end{aligned}$$

7. If anyone is quirky, Lewis is quirky. Lewis isn't quirky. Lewis is friends with himself.

$$\exists x\, Qx \to Ql,\ \neg Ql,\ Fll$$

$$\begin{aligned} \text{domain} &: 1 \\ Q &: \\ F &: \langle 1, 1 \rangle \\ l &: 1 \end{aligned}$$

8. No quirky philosopher is friends with a quirky philosopher. All athletes are friends with some athlete. Lewis is neither an athlete nor a philosopher.

$$\forall x\{(Qx \land Px) \to \neg\exists y[(Qy \land Py) \land Fxy]\},\ \forall z[Az \to \exists y(Ay \land Fzy)],\ \neg(Al \lor Pl)$$

$$\begin{aligned} \text{domain} &: 1 \\ Q &: \\ A &: \\ P &: \\ F &: \\ l &: 1 \end{aligned}$$

9. Lewis is friends with everybody unless he's not friends with Kanye. Someone is friends with everyone.

$$\forall x Flx \lor \neg Flk,\ \exists x \forall y Fxy$$

$$\begin{aligned} \text{domain} &: 1 \\ F &: \langle 1, 1 \rangle \\ k &: 1 \\ l &: 1 \end{aligned}$$

1. To get a natural deduction system for PL, we will introduce 4 new rules—an introduction and an elimination rule for both the universal and the existential quantifier.

2. Just like in SL, you will be able to prove something in this natural deduction system if and only if it is an entailment.

   ▹ Let's write

   $$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$$

   to mean that there is a legal natural deduction proof which has $\mathscr{C}$ written on a scope line with $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions. Then, this new natural deduction system will have the following property:

   $$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C} \text{ if and only if } \mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vDash \mathscr{C}$$

Universal Elimination

3. The elimination rule for the universal quantifier is:

Universal Elimination ($\forall E$)

$$\forall x\, \mathscr{A}(\ldots x \ldots x \ldots)$$

$$\triangleright \quad \mathscr{A}(\ldots n \ldots n \ldots)$$

This says: if you have accessible a sentence of the form '$\forall x\, \mathscr{A}(\ldots x \ldots x \ldots)$', and '$n$' is a name, then you are allowed to write down '$\mathscr{A}(\ldots n \ldots n \ldots)$'—the result of going through '$\mathscr{A}(\ldots x \ldots x \ldots)$' and replacing every free occurrence of 'x' with the same name 'n'.

▹ '$\mathscr{A}(\ldots x \ldots x \ldots)$' stands for a sentence in which the variable 'x' occurs free. We write '$(\ldots x \ldots x \ldots)$' because '$x$' might occur free more than once.

4. The following is a legal proof:

1	$\forall y\,(Sby \leftrightarrow \neg Syy)$	
2	$Sbb \leftrightarrow \neg Sbb$	$\forall E$ 1
3	$\quad Sbb$	Ass ($\neg I$)
4	$\quad \neg Sbb$	$\leftrightarrow E$ 2, 3
5	$\quad \bot$	$\bot I$ 3, 4
6	$\neg Sbb$	$\neg I$ 3–4
7	Sbb	$\leftrightarrow E$ 2, 6
8	$\bot$	$\bot I$ 6, 7

On line 2, we used $\forall E$ to replace *every free* occurrence of 'y' with the same name: 'b'. Notice that 'b' already occurs in '$\forall y\,(Sby \leftrightarrow \neg Syy)$'. No matter—we're still allowed to use $\forall E$.

▹ This shows that $\forall y(Sby \leftrightarrow \neg Syy) \vdash \bot$. So: $\forall y(Sby \leftrightarrow \neg Syy) \vDash \bot$. So '$\forall y(Sby \leftrightarrow \neg Syy)$' is a contradiction.

5. The following are *not* legal applications of $\forall E$:

m.	$\forall z(Fz \rightarrow Gz)$	
k.	$Fa \rightarrow Gz$	$\forall E$ m $\longleftarrow$ **MISTAKE!**

m.	$\forall z(Fz \rightarrow \exists z\, Gz)$	
k.	$Fa \rightarrow \exists z\, Ga$	$\forall E$ m $\longleftarrow$ **MISTAKE!**

m.	$\forall z(Fz \rightarrow Gz)$	
k.	$Fa \rightarrow Gb$	$\forall E$ m $\longleftarrow$ **MISTAKE!**

m.	$\forall x Fx \rightarrow Ga$	
k.	$Fa \rightarrow Ga$	$\forall E$ m $\longleftarrow$ **MISTAKE!**

- ‘$Fa \rightarrow Gz$’ does not replace ***every*** free ‘z’ with the same name.
- ‘$Fa \rightarrow Gb$’ does not replace every free ‘z’ with the ***same*** name.
- ‘$Fa \rightarrow \exists z Ga$’ does not replace every ***free*** ‘z’ with the same name—the ‘z’ in ‘$\exists z Gz$’ is not free.
- In general, rules may not be applied to sub-sentences. Since ‘$\forall x$’ is not the main operator of ‘$\forall x Fx \rightarrow Ga$’, you may not use $\forall E$.

6. Another legal proof:

1	$\quad\forall x \forall y\, Rxy$	Ass ($\rightarrow I$)
2	$\quad\forall y\, Ray$	$\forall E$ 1
3	$\quad Raa$	$\forall E$ 2
4	$\forall x \forall y\, Rxy \rightarrow Raa$	$\rightarrow I$ 1–3

Since ‘$\forall x \forall y\, Rxy \rightarrow Raa$’ appears outside of the scope of any assumptions, $\vdash \forall x \forall y\, Rxy \rightarrow Raa$. So $\models \forall x \forall y\, Rxy \rightarrow Raa$. So ‘$\forall x \forall y\, Rxy \rightarrow Raa$’ is a tautology.

Existential Introduction

7. The introduction rule for the existential quantifier:

Existential Introduction ($\exists I$)

	$\mathscr{A}(\ldots n \ldots n \ldots)$
▹	$\exists x\, \mathscr{A}(\ldots x \ldots n \ldots)$

This says: if you have accessible a sentence of the form ‘$\mathscr{A}(\ldots n \ldots n \ldots)$’, where ‘$n$’ is a name, then you are allowed to write down ‘$\exists x\, \mathscr{A}(\ldots x \ldots n \ldots)$’—where ‘$\mathscr{A}(\ldots x \ldots n \ldots)$’ is the result of going through $\mathscr{A}(\ldots n \ldots n \ldots)$’ and replacing *some* ‘$n$’s with the variable ‘$x$’ (you needn't replace *all* of the ‘n’s).

8. Some legal proofs utilizing this rule:

1	$\forall x\, Rxx$	
2	Ree	$\forall E$ 1
3	$\exists y\, Rey$	$\exists I$ 2
4	$\exists x \exists y\, Rxy$	$\exists I$ 3

1	$\forall x \forall y\, Zxy$	
2	$\forall y\, Zny$	$\forall E$ 1
3	Znn	$\forall E$ 2
4	$\exists w\, Zww$	$\exists I$ 3

9. This is *not* a legal proof:

$$\begin{array}{l|ll} 1 & Rab & \\ \hline 2 & \exists z\, Rzz & \exists I\ 1 \longleftarrow \text{MISTAKE!} \end{array}$$

- ▹ Each occurrence of 'z' in '$\exists z\, Rzz$' must replace the ***same*** name in 'Rab'. But here, the first 'z' replaces 'a', and the second replaces 'b'.

Universal Introduction

10. The introduction rule for the universal quantifier:

> <u>Universal Introduction</u> ($\forall I$)
>
> $$\begin{array}{r|l} & \mathscr{A}(\ldots n \ldots n \ldots) \\ \triangleright & \forall x\, \mathscr{A}(\ldots x \ldots x \ldots) \end{array}$$
>
> *provided that: n does not appear in any open assumption*

- ▹ An assumption is *open* at a line k iff its vertical scope line extends to line k. So the proviso says: if, at your current line, you haven't assumed anything about the name 'n', then you may take it to name *anything* in the domain.
- ▹ However, when you introduce the universal quantifier, you have to get rid of the name 'n'. It cannot appear anywhere in $\mathscr{A}(\ldots x \ldots x \ldots)$.

11. The following proofs utilizing $\forall I$ are *not* legal:

$$\begin{array}{l|ll} 1 & \exists x\, Rxa & \\ \hline 2 & \forall y\, \exists x\, Rxy & \forall I\ 1 \longleftarrow \text{MISTAKE!} \end{array}$$

$$\begin{array}{l|ll} 1 & \exists x\, Rxx & \\ \hline 2 & Rcc & \exists E\ 1 \\ 3 & \forall x\, Rxc & \forall I\ 2 \longleftarrow \text{MISTAKE!} \end{array}$$

- ▹ On the left, the name 'a' appears in an open assumption on line 1
- ▹ On the right, the name 'c' appears in '$\forall x\, Rxc$'. To use $\forall I$, you need to replace *every* occurrence of the name with the same variable.

12. Here is a legal natural deduction proof:

$$\begin{array}{l|ll} 1 & \forall y\, Lay & \\ \hline 2 & Lab & \forall E\ 1 \\ 3 & \exists y\, Lyb & \exists I\ 2 \\ 4 & \forall x\, \exists y\, Lyx & \forall I\ 3 \end{array}$$

- ▹ So: $\forall y\, Lay \vdash \forall x\, \exists y\, Lyx$.
- ▹ So: $\forall y\, Lay \vDash \forall x\, \exists y\, Lyx$.
- ▹ So there's no interpretation which makes '$\forall y\, Lay$' true and which makes '$\forall x\, \exists y\, Lyx$' false.

13. Consider the following arguments (and their translations into PL, given a natural symbolization key):

> All cats are fluffy. Nothing fluffy is scary. So no cats are scary.
> $\forall x(Cx \to Fx), \forall y(Fy \to \neg Sy) \therefore \forall z(Cz \to \neg Sz)$
>
> Nothing finite is perfect. Everything imperfect has a creator. Therefore anything without a creator is infinite.
> $\forall x(Fx \to \neg Px), \forall y(\neg Py \to Cy) \therefore \forall z(\neg Cz \to \neg Fz)$

These arguments are valid. We can prove this by providing the following natural deduction proofs:

1	$\forall x(Cx \to Fx)$	
2	$\forall y(Fy \to \neg Sy)$	
3	$\quad Cb$	Ass ($\to I$)
4	$\quad Cb \to Fb$	$\forall E$ 1
5	$\quad Fb$	$\to E$ 3, 4
6	$\quad Fb \to \neg Sb$	$\forall E$ 2
7	$\quad \neg Sb$	$\to E$ 5, 6
8	$Cb \to \neg Sb$	$\to I$ 3–7
9	$\forall z(Cz \to \neg Sz)$	$\forall I$ 8

1	$\forall x(Fx \to \neg Px)$	
2	$\forall y(\neg Py \to Cy)$	
3	$\quad \neg Cd$	Ass ($\to I$)
4	$\quad \neg Pd \to Cd$	$\forall E$ 2
5	$\quad \neg\neg Pd$	MT 3, 4
6	$\quad Fd \to \neg Pd$	$\forall E$ 1
7	$\quad \neg Fd$	MT 5, 6
8	$\neg Cd \to \neg Fd$	$\to I$ 3–7
9	$\forall z(\neg Cz \to \neg Fz)$	$\forall I$ 8

Existential Elimination

14. Finally, the elimination rule for the existential quantifier:

> <u>Existential Elimination</u> (<u>$\exists E$</u>)
>
> $\exists x\, \mathscr{A}(\ldots x \ldots x \ldots)$
> $\quad \mathscr{A}(\ldots n \ldots n \ldots)$
> $\quad \vdots$
> $\quad \mathscr{C}$
> ▹ $\mathscr{C}$
>
> *provided that: n does not appear outside of the sub-proof (in particular: n does not appear in $\mathscr{C}$)*

The rule says: if you know that *something* is $\mathscr{A}$, then you may introduce a name for that thing—so long as it is a completely *new* name, and so long as you get rid of that name before you draw any conclusions.

- ▹ Intuitively, what's going on here is this: we know that *something* in the domain makes '$\mathscr{A}$' true—but we don't know what exactly it is. Nonetheless, we can introduce a name for that thing, and then reason about it using that name—so long as we don't forget that the name we chose was completely arbitrary, and so long as we get rid of it entirely before drawing any conclusions.
- ▹ The reasoning used in $\exists E$ is like the reasoning used in the following:

Someone's committed a murder. I don't know who it is—but whoever it is, let's call them 'Jack the Ripper'. The murder was committed in Whitechapel, but murders are committed away from the murderer's home. So Jack the Ripper doesn't live in Whitechapel. So: someone who doesn't live in Whitechapel committed a murder.

The speaker *introduces a name* for the person who committed the murder, and then reasons about this individual using that name. However, when they reach their conclusion—that some who doesn't live in Whitechapel committed a murder—they get rid of the name, and don't state their conclusions using it.

(a) if 'Mx' says '$\underline{\quad\quad}_x$ committed the murder', and 'Lx' says '$\underline{\quad\quad}_x$ lives in Whitechapel', then this is a formalization of the reasoning above which utilizes $\exists E$:

1	$\forall y(My \rightarrow \neg Ly)$	
2	$\exists x Mx$	
3	Mj	Ass ($\exists E$)
4	$Mj \rightarrow \neg Lj$	$\forall E$ 1
5	$\neg Lj$	$\rightarrow E$ 3, 4
6	$Mj \wedge \neg Lj$	$\wedge I$ 3, 5
7	$\exists x(Mx \wedge \neg Lx)$	$\exists I$ 6
8	$\exists x(Mx \wedge \neg Lx)$	$\exists E$ 2, 3–7

15. The following uses of $\exists E$ are *not* legal:

1	$\exists x Rxe$	
2	Ree	Ass ($\exists E$)
3	$\exists y Rey$	$\exists I$ 2
4	$\exists y Rey$	$\exists E$ 1, 2–3 ⟵ **MISTAKE!**

1	$\exists x Rxx$	
2	Rpp	Ass ($\exists E$)
3	$\exists y Rpy$	$\exists I$ 2
4	$\exists y Rpy$	$\exists E$ 1, 2–3 ⟵ **MISTAKE!**

- On the left: the name 'e' appears outside of the sub-proof. (It appears on line 1.)
- On the right: the name 'p' appears outside of the sub-proof. (It appears on line 4.)

16. Consider the following argument (and its translation into PL, given a natural symbolization key):

Someone loves everyone. So everyone has someone who loves them.
$\exists x \forall y\, Lxy \;\therefore\; \forall y \exists x\, Lxy$

This argument is valid. We can prove this by providing the following natural deduction proof:

1	$\exists x \forall y Lxy$	
2	$\forall y Lay$	Ass ($\exists E$)
3	Lab	$\forall E$ 2
4	$\exists x Lxb$	$\exists I$ 3
5	$\forall y \exists x Lxy$	$\forall I$ 4
6	$\forall y \exists x Lxy$	$\exists E$ 1, 2–5

17. Or, consider the following argument (and its translation into PL):

> Some pundits argue in bad faith. No one who argues in bad faith is worth arguing with. So some pundits are not worth arguing with.
> $\exists x(Px \wedge Ax),\ \forall y(Ay \rightarrow \neg Wy)\ \therefore\ \exists z(Pz \wedge \neg Wz)$

This argument is also valid. We can show that it's valid with the following natural deduction proof:

1	$\exists x(Px \wedge Ax)$	
2	$\forall y(Ay \rightarrow \neg Wy)$	
3	$\quad Pm \wedge Am$	Ass ($\exists E$)
4	$\quad Am$	$\wedge E$ 3
5	$\quad Am \rightarrow \neg Wm$	$\forall E$ 2
6	$\quad \neg Wm$	$\rightarrow E$ 4, 5
7	$\quad Pm$	$\wedge E$ 3
8	$\quad Pm \wedge \neg Wm$	$\wedge I$ 6, 7
9	$\quad \exists z(Pz \wedge \neg Wz)$	$\exists I$ 8
10	$\exists z(Pz \wedge \neg Wz)$	$\exists E$ 1, 3–9

Summary of Rules

Universal Introduction ($\forall I$)

$\mathscr{A}(\ldots n \ldots n \ldots)$
▹ $\forall x\, \mathscr{A}(\ldots x \ldots x \ldots)$

provided that: n does not appear in any open assumption

Existential Introduction ($\exists I$)

$\mathscr{A}(\ldots n \ldots n \ldots)$
▹ $\exists x\, \mathscr{A}(\ldots x \ldots n \ldots)$

Existential Elimination ($\exists E$)

$\exists x\, \mathscr{A}(\ldots x \ldots x \ldots)$
$\quad \mathscr{A}(\ldots n \ldots n \ldots)$
$\quad \vdots$
$\quad \mathscr{C}$
▹ $\mathscr{C}$

provided that: n does not appear outside of the sub-proof (in particular: n does not appear in $\mathscr{C}$)

Universal Elimination ($\forall E$)

$\forall x\, \mathscr{A}(\ldots x \ldots x \ldots)$
▹ $\mathscr{A}(\ldots n \ldots n \ldots)$

Part A

Explain why these two 'proofs' are *incorrect*. Also, provide interpretations which would invalidity the fallacious argument forms the 'proofs' enshrine.[8]

1	$\forall x\, Rxx$	
2	Raa	$\forall E$ 1
3	$\forall y Ray$	$\forall I$ 2
4	$\forall x \forall y Rxy$	$\forall I$ 3

1	$\forall x \exists y\, Rxy$	
2	$\exists y\, Ray$	$\forall E$ 1
3	$\quad Raa$	Ass ($\exists E$)
4	$\quad \exists x Rxx$	$\exists I$ 3
5	$\exists x Rxx$	$\exists E$ 2, 3–4

Part B

The following three proofs are missing their justifications (rule and line numbers). Add them to turn them into *bona fide* proofs.

1	$\forall x \exists y (Rxy \vee Ryx)$
2	$\forall x \neg Rmx$
3	$\exists y (Rmy \vee Rym)$
4	$\quad Rma \vee Ram$
5	$\quad \neg Rma$
6	$\quad Ram$
7	$\quad \exists x Rxm$
8	$\exists x Rxm$

1	$\forall x (\exists y Lxy \rightarrow \forall z Lzx)$
2	Lab
3	$\exists y Lay \rightarrow \forall z Lza$
4	$\exists y Lay$
5	$\forall z Lza$
6	Lca
7	$\exists y Lcy \rightarrow \forall z Lzc$
8	$\exists y Lcy$
9	$\forall z Lzc$
10	Lcc
11	$\forall x Lxx$

[8] These exercises come from *Forall x: An Introduction to Formal Logic*, by P. D. Magnus and Tim Button.

1	$\forall x(Jx \to Kx)$
2	$\exists x \forall y Lxy$
3	$\forall x Jx$
4	$\quad \forall y Lay$
5	$\quad Laa$
6	$\quad Ja$
7	$\quad Ja \to Ka$
8	$\quad Ka$
9	$\quad Ka \wedge Laa$
10	$\quad \exists x(Kx \wedge Lxx)$
11	$\exists x(Kx \wedge Lxx)$

Part C

Translate these arguments into PL and provide a natural deduction proof to show that they are valid.

Barbara All G are F. All H are G. So: All H are F

Celarent No G are F. All H are G. So: No H are F

Ferio No G are F. Some H is G. So: Some H is not F

Darii All G are F. Some H is G. So: Some H is F.

Camestres All F are G. No H are G. So: No H are F.

Cesare No F are G. All H are G. So: No H are F.

Baroko All F are G. Some H is not G. So: Some H is not F.

Festino No F are G. Some H are G. So: Some H is not F.

Datisi All G are F. Some G is H. So: Some H is F.

Disamis Some G is F. All G are H. So: Some H is F.

Ferison No G are F. Some G is H. So: Some H is not F.

Bokardo Some G is not F. All G are H. So: Some H is not F.

Camenes All F are G. No G are H So: No H is F.

Dimaris Some F is G. All G are H. So: Some H is F.

Fresison No F are G. Some G is H. So: Some H is not F.

Part A

1	$\forall xRxx$	
2	Raa	$\forall E$ 1
3	$\forall yRay$	$\forall I$ 2 ⟵ **MISTAKE!!!**
4	$\forall x\forall yRxy$	$\forall I$ 3

All occurrences of 'a' must be replaced with the variable 'y' on line 3. The following interpretation shows that the argument $\forall xRxx \therefore \forall x\forall yRxy$' is not an entailment.

domain : 1, 2
R : $\langle 1, 1 \rangle, \langle 2, 2 \rangle$

1	$\forall x\exists yRxy$	
2	$\exists yRay$	$\forall E$ 1
3	$\quad Raa$	Ass ($\exists E$)
4	$\quad \exists xRxx$	$\exists I$ 3
5	$\exists xRxx$	$\exists E$ 2, 3–4 ⟵ **MISTAKE!**

The name 'a' appears outside of the subproof 3–4 (on line 2). The following interpretation shows that the argument $\forall x\exists yRxy \therefore \exists xRxx$' is not an entailment.

domain : 1, 2
R : $\langle 1, 2 \rangle, \langle 2, 1 \rangle$

Part B

1	$\forall x\exists y(Rxy \vee Ryx)$	
2	$\forall x\neg Rmx$	
3	$\exists y(Rmy \vee Rym)$	$\forall E$ 1
4	$\quad Rma \vee Ram$	Ass ($\exists E$)
5	$\quad \neg Rma$	$\forall E$ 2
6	$\quad Ram$	DS 4, 5
7	$\quad \exists xRxm$	$\exists I$ 6
8	$\exists xRxm$	$\exists E$ 3, 4–7

1	$\forall x(\exists yLxy \rightarrow \forall zLzx)$	
2	Lab	
3	$\exists yLay \rightarrow \forall zLza$	$\forall E$ 1
4	$\exists yLay$	$\exists I$ 2
5	$\forall zLza$	$\rightarrow E$ 3, 4
6	Lca	$\forall E$ 5
7	$\exists yLcy \rightarrow \forall zLzc$	$\forall E$ 1
8	$\exists yLcy$	$\exists I$ 6
9	$\forall zLzc$	$\rightarrow E$ 7, 8
10	Lcc	$\forall E$ 9
11	$\forall xLxx$	$\forall I$ 10

1	$\forall x(Jx \to Kx)$	
2	$\exists x \forall y Lxy$	
3	$\forall x Jx$	
4	$\quad \forall y Lay$	Ass ($\exists E$)
5	$\quad Laa$	$\forall E$ 4
6	$\quad Ja$	$\forall E$ 3
7	$\quad Ja \to Ka$	$\forall E$ 1
8	$\quad Ka$	$\to E$ 6, 7
9	$\quad Ka \land Laa$	$\land I$ 5, 8
10	$\quad \exists x(Kx \land Lxx)$	$\exists I$ 9
11	$\exists x(Kx \land Lxx)$	$\exists E$ 2, 4–10

Part C

Barbara All G are F. All H are G. So: All H are F

1	$\forall x(Gx \to Fx)$	
2	$\forall y(Hy \to Gy)$	
3	$\quad Ha$	Ass ($\to I$)
4	$\quad Ha \to Ga$	$\forall E$ 2
5	$\quad Ga$	$\to E$ 3, 4
6	$\quad Ga \to Fa$	$\forall E$ 1
7	$\quad Fa$	$\to E$ 5, 6
8	$Ha \to Fa$	$\to I$ 3–7
9	$\forall z(Hz \to Fz)$	$\forall I$ 8

Celarent No G are F. All H are G. So: No H are F

1	$\forall x(Gx \to \neg Fx)$	
2	$\forall y(Hy \to Gy)$	
3	$\quad Ha$	Ass ($\to I$)
4	$\quad Ha \to Ga$	$\forall E$ 2
5	$\quad Ga$	$\to E$ 3, 4
6	$\quad Ga \to \neg Fa$	$\forall E$ 1
7	$\quad \neg Fa$	$\to E$ 5, 6
8	$Ha \to \neg Fa$	$\to I$ 3–7
9	$\forall z(Hz \to \neg Fz)$	$\forall I$ 8

Ferio No G are F. Some H is G. So: Some H is not F

1	$\forall x(Gx \to \neg Fx)$	
2	$\exists y(Hy \wedge Gy)$	
3	$\quad Ha \wedge Ga$	Ass ($\exists E$)
4	$\quad Ga$	$\wedge E$ 3
5	$\quad Ga \to \neg Fa$	$\forall E$ 1
6	$\quad \neg Fa$	$\to oE$ 4, 5
7	$\quad Ha$	$\wedge E$ 3
8	$\quad Ha \wedge \neg Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge \neg Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge \neg Fz)$	$\exists E$ 2, 3–9

Darii All G are F. Some H is G. So: Some H is F.

1	$\forall x(Gx \to Fx)$	
2	$\exists y(Hy \wedge Gy)$	
3	$\quad Ha \wedge Ga$	Ass ($\exists E$)
4	$\quad Ga$	$\wedge E$ 3
5	$\quad Ga \to Fa$	$\forall E$ 1
6	$\quad Fa$	$\to E$ 4, 5
7	$\quad Ha$	$\wedge E$ 3
8	$\quad Ha \wedge Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge Fz)$	$\exists E$ 2, 3–9

Camestres All F are G. No H are G. So: No H are F.

1	$\forall x(Fx \to Gx)$	
2	$\forall y(Hy \to \neg Gy)$	
3	$\quad Ha$	Ass ($\to I$)
4	$\quad Ha \to \neg Ga$	$\forall E$ 2
5	$\quad \neg Ga$	$\to E$ 3, 4
6	$\quad Fa \to Ga$	$\forall E$ 1
7	$\quad \neg Fa$	MT 5, 6
8	$Ha \to \neg Fa$	$\to I$ 3–7
9	$\forall z(Hz \to \neg Fz)$	$\forall I$ 8

Cesare No F are G. All H are G. So: No H are F.

1	$\forall x(Fx \to \neg Gx)$	
2	$\forall y(Hy \to Gy)$	
3	$\quad Ha$	Ass ($\to I$)
4	$\quad Ha \to Ga$	$\forall E$ 2
5	$\quad Ga$	$\to E$ 3, 4
6	$\quad Fa \to \neg Ga$	$\forall E$ 1
7	$\quad\quad Fa$	Ass ($\neg I$)
8	$\quad\quad \neg Ga$	$\to E$ 6, 7
9	$\quad\quad \bot$	$\bot I$ 5, 8
10	$\quad \neg Fa$	$\neg I$ 7–9
11	$Ha \to \neg Fa$	$\to I$ 3–10
12	$\forall z(Hz \to \neg Fz)$	$\forall I$ 11

Baroko All F are G. Some H is not G. So: Some H is not F.

1	$\forall x(Fx \to Gx)$	
2	$\exists y(Hy \wedge \neg Gy)$	
3	$\quad Ha \wedge \neg Ga$	Ass ($\exists E$)
4	$\quad \neg Ga$	$\wedge E$ 3
5	$\quad Fa \to Ga$	$\forall E$ 1
6	$\quad \neg Fa$	MT 4, 5
7	$\quad Ha$	$\wedge E$ 3
8	$\quad Ha \wedge \neg Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge \neg Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge \neg Fz)$	$\exists E$ 2, 3–9

Festino No F are G. Some H are G. So: Some H is not F.

1	$\forall x(Fx \rightarrow \neg Gx)$	
2	$\exists y(Hy \wedge Gy)$	
3	$\quad Hn \wedge Gn$	Ass ($\exists E$)
4	$\quad Fn \rightarrow \neg Gn$	$\forall E$ 1
5	$\quad\quad Fn$	Ass ($\neg I$)
6	$\quad\quad \neg Gn$	$\rightarrow E$ 4, 5
7	$\quad\quad Gn$	$\wedge E$ 3
8	$\quad\quad \bot$	$\bot I$ 6, 7
9	$\quad \neg Fn$	$\neg I$ 5–8
10	$\quad Hn$	$\wedge E$ 3
11	$\quad Hn \wedge \neg Fn$	$\wedge I$ 9, 10
12	$\quad \exists z(Hz \wedge \neg Fz)$	$\exists I$ 11
13	$\exists z(Hz \wedge \neg Fz)$	$\exists E$ 2, 3–12

Datisi All G are F. Some G is H. So: Some H is F.

1	$\forall x(Gx \rightarrow Fx)$	
2	$\exists y(Gy \wedge Hy)$	
3	$\quad Ga \wedge Ha$	Ass ($\exists E$)
4	$\quad Ga \rightarrow Fa$	$\forall E$ 1
5	$\quad Ga$	$\wedge E$ 3
6	$\quad Fa$	$\rightarrow E$ 4, 5
7	$\quad Ha$	$\wedge E$ 3
8	$\quad Ha \wedge Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge Fz)$	$\exists E$ 2, 3–9

Disamis Some G is F. All G are H. So: Some H is F.

1	$\exists x(Gx \rightarrow Fx)$	
2	$\forall y(Gy \rightarrow Hy)$	
3	$\quad Ga \wedge Fa$	Ass ($\exists E$)
4	$\quad Ga \rightarrow Ha$	$\forall E$ 1
5	$\quad Ga$	$\wedge E$ 3
6	$\quad Ha$	$\rightarrow E$ 4, 5
7	$\quad Fa$	$\wedge E$ 3
8	$\quad Ha \wedge Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge Fz)$	$\exists E$ 2, 3–9

Ferison No G are F. Some G is H. So: Some H is not F.

1	$\forall x(Gx \rightarrow \neg Fx)$	
2	$\exists y(Gy \wedge Hy)$	
3	$\quad Ga \wedge Ha$	Ass ($\exists E$)
4	$\quad Ga \rightarrow \neg Fa$	$\forall E$ 1
5	$\quad Ga$	$\wedge E$ 3
6	$\quad \neg Fa$	$\rightarrow E$ 4, 5
7	$\quad Ha$	$\wedge E$ 3
8	$\quad Ha \wedge \neg Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge \neg Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge \neg Fz)$	$\exists E$ 2, 3–9

Bokardo Some G is not F. All G are H. So: Some H is not F.

1	$\exists x(Gx \wedge \neg Fx)$	
2	$\forall y(Gy \rightarrow Hy)$	
3	$\quad Ga \wedge \neg Fa$	Ass ($\exists E$)
4	$\quad Ga \rightarrow Ha$	$\forall E$ 1
5	$\quad Ga$	$\wedge E$ 3
6	$\quad Ha$	$\rightarrow E$ 4, 5
7	$\quad \neg Fa$	$\wedge E$ 3
8	$\quad Ha \wedge \neg Fa$	$\wedge I$ 6, 7
9	$\quad \exists z(Hz \wedge \neg Fz)$	$\exists I$ 8
10	$\exists z(Hz \wedge \neg Fz)$	$\exists E$ 2, 3–9

Camenes All F are G. No G are H So: No H is F.

1	$\forall x(Fx \to Gx)$	
2	$\forall y(Gy \to \neg Hy)$	
3	$\quad Ha$	Ass ($\to I$)
4	$\quad\quad Fa$	Ass ($\neg I$)
5	$\quad\quad Fa \to Ga$	$\forall E$ 1
6	$\quad\quad Ga$	$\to E$ 4, 5
7	$\quad\quad Ga \to \neg Ha$	$\forall E$ 2
8	$\quad\quad \neg Ha$	6, 7
9	$\quad\quad \bot$	$\bot I$ 3, 8
10	$\quad \neg Fa$	$\neg I$ 4–9
11	$Ha \to \neg Fa$	$\to I$ 3–10
12	$\forall z(Hz \to \neg Fz)$	$\forall I$ 11

Dimaris Some F is G. All G are H. So: Some H is F.

1	$\exists x(Fx \land Gx)$	
2	$\forall y(Gy \to Hy)$	
3	$\quad Fj \land Gj$	Ass ($\exists E$)
4	$\quad Gj$	$\land E$ 3
5	$\quad Gj \to Hj$	$\forall E$ 2
6	$\quad Hj$	$\to E$ 4, 5
7	$\quad Fj$	$\land E$ 3
8	$\quad Hj \land Fj$	$\land I$ 6, 7
9	$\quad \exists z(Hz \land Fz)$	$\exists I$ 8
10	$\exists z(Hz \land Fz)$	$\exists E$ 1, 3–9

Fresison No F are G. Some G is H. So: Some H is not F.

1	$\forall x(Fx \to \neg Gx)$	
2	$\exists y(Gy \land Hy)$	
3	$\quad Gk \land Hk$	Ass ($\exists E$)
4	$\quad\quad Fk$	Ass ($\neg I$)
5	$\quad\quad Fk \to \neg Gk$	$\forall E$ 1
6	$\quad\quad \neg Gk$	$\to E$ 4, 5
7	$\quad\quad Gk$	$\land E$ 3
8	$\quad\quad \bot$	$\bot I$ 6, 7
9	$\quad \neg Fk$	$\neg I$ 4–8
10	$\quad Hk$	$\land E$ 3
11	$\quad Hk \land \neg Fk$	$\land I$ 9, 10
12	$\quad \exists z(Hz \land \neg Fz)$	$\exists I$ 11
13	$\exists z(Hz \land \neg Fz)$	$\exists E$ 2, 3–12

Derived Rules

1. For natural deduction in PL, we will have 4 derived rules which will allow us to move negations inside of and outside of the quantifiers. Since doing so *changes* the quantifer from an existential to a universal, or from a universal to an existential, we will call this rule 'Change of Quantifier', or 'CQ':

Change of Quantifiers (CQ)		
$\neg\forall x\mathcal{A}$	$\lhd\ \rhd$	$\exists x\neg\mathcal{A}$
$\neg\exists x\mathcal{A}$	$\lhd\ \rhd$	$\forall x\neg\mathcal{A}$

These rules are not needed—in fact, they *follow from* the other rules, in the following sense: anything we could prove *with* the rule CQ, we could prove without it, too. For instance, if '$\mathcal{A}$' is 'Fx', then the following four derivations show how we could achieve any of the four inference rules allowed by CQ with just the other rules:

1	$\neg\forall xFx$	
2	$\quad\neg\exists x\neg Fx$	Ass ($\neg E$)
3	$\quad\quad\neg Fa$	Ass ($\neg E$)
4	$\quad\quad\exists x\neg Fx$	$\exists I$ 3
5	$\quad\quad\bot$	$\bot I$ 2, 4
6	$\quad Fa$	$\neg E$ 3–5
7	$\quad\forall xFx$	$\forall I$ 6
8	$\quad\bot$	$\bot I$ 1, 7
9	$\exists x\neg Fx$	$\neg E$ 2–8

1	$\exists x\neg Fx$	
2	$\quad\neg Fa$	Ass ($\exists E$)
3	$\quad\quad\forall xFx$	Ass ($\neg I$)
4	$\quad\quad Fa$	$\forall E$ 3
5	$\quad\quad\bot$	$\bot I$ 2, 4
6	$\quad\neg\forall xFx$	$\neg I$ 3–5
7	$\neg\forall xFx$	$\exists E$ 1, 2–6

1	$\neg\exists xFx$	
2	$\quad Fa$	Ass ($\neg I$)
3	$\quad\exists xFx$	$\exists I$ 2
4	$\quad\bot$	$\bot I$ 1, 3
5	$\neg Fa$	$\neg I$ 2–4
6	$\forall x\neg Fx$	$\forall I$ 5

1	$\forall x\neg Fx$	
2	$\quad\exists xFx$	Ass ($\neg I$)
3	$\quad\quad Fa$	Ass ($\exists E$)
4	$\quad\quad\neg Fa$	$\forall E$ 1
5	$\quad\quad\bot$	$\bot I$ 3, 4
6	$\quad\bot$	$\exists E$ 2, 3–5
7	$\neg\exists xFx$	$\neg I$ 2–6

Strategies for PL Natural Deduction

2. Recall the strategies for natural deduction proofs from SL—they will still apply when we're doing natural deduction in PL:

- ▹ Come up with a 'big picture' strategy (form a *main* goal, and then about how to achieve it)

  When forming your strategy, *think about meaning*—if your strategy requires you to prove something that *doesn't follow*, then you should abandon that strategy.

- ▹ Form 'sub-goals' which will help you achieve your main goal, given your 'big-picture' strategy
- ▹ Try to use the introduction rule for the main operator of the sentence you want to write down
- ▹ Try to use the elimination rule for the main operator of a sentence you have accessible
- ▹ If all else fails, try negative elimination
- ▹ If you don't have any ideas, *just do something*

3. To these tips, let me add four new tips, unique to the rules for the quantifiers:

4. Tip #1: When using $\forall E$, use your (sub-)oal as a guide for which name to instantiate.

   - ▹ For instance, if you have accessible '$\forall y(Py \rightarrow Qy)$' and '$Ps$', and your goal is to obtain 'Qs', then it makes sense to use $\forall E$ to write down '$Ps \rightarrow Qs$'. It wouldn't make much sense to write down '$Pa \rightarrow Qa$' though the rule $\forall E$ allows that.

5. Tip #2: If your goal is an existentially quantified sentence, '$\exists x \mathcal{A}(...x...x...)$', set yourself the sub-goal of deriving '$\mathcal{A}(...n...n...)$', for some name 'n'.

   - ▹ For instance, if you're trying to show that $\forall xRxa \vdash \exists yRyy$, then it's a good idea to first try to get the sentence 'Raa' so that you can use $\exists I$ to write down '$\exists yRyy$'. Here's a proof that takes this strategy:

1	$\forall xRxa$	
2	Raa	$\forall E$ 1
3	$\exists yRyy$	$\exists I$ 2

6. Tip #3: If your goal is a universally quantified sentence, '$\forall x \mathcal{A}(...x...x..)$', then set yourself the sub-goal of deriving '$\mathcal{A}(...n...n...)$', for some name n which doesn't appear in any open assumptions.

   - ▹ For instance, if you're trying to show that $\forall x(Px \rightarrow Qx), \forall y(Qy \rightarrow \neg Ry) \vdash \forall z(Pz \rightarrow \neg Rz)$, then you should set yourself the sub-goal of deriving '$Pk \rightarrow \neg Rk$' on the main scope line. Here's a derivation which takes this strategy:

1	$\forall x(Px \rightarrow Qx)$	
2	$\forall y(Qy \rightarrow \neg Ry)$	Goal: $\forall z(Pz \rightarrow \neg Rz)$
3	Pk	Ass ($\rightarrow I$) [subgoal: $\neg Rk$]
4	$Pk \rightarrow Qk$	$\forall E$ 1
5	Qk	$\rightarrow E$ 3, 4
6	$Qk \rightarrow \neg Rk$	$\forall E$ 2
7	$\neg Rk$	$\rightarrow E$ 5, 6
8	$Pk \rightarrow \neg Rk$	$\rightarrow I$ 3–7
9	$\forall z(Pz \rightarrow \neg Rz)$	$\forall I$ 8

7. Tip #4: When you have an existentially quantified sentence '$\exists x \mathcal{A}(...x...x...)$', start a sub-proof with '$\mathcal{A}(...n...n...)$' (for some *new* name 'n') before using the rule $\forall E$.

   - ▹ That is: eliminate existential quantifiers *before* you eliminate universal quantifiers.

Your challenge, should you choose to accept it, is to provide natural deduction proofs to establish the following claims. You should feel free to use the derived rules. For each natural deduction, if you complete it correctly, you will earn the indicated number of points—these points will be added to your final grade (your grade on the final will be out of 100 points).

1. Show $\forall x Fx \;\vdash\; Fa$ ***without using the rule*** $\forall E$. (1/2 pt.)
2. Show $Fa \;\vdash\; \exists x Fx$ ***without using the rule*** $\exists I$. (1/2 pt.)
3. Show $\exists x Fx \;\vdash\; \exists y Fy$ ***without using the rules*** $\exists E$ ***or*** $\exists I$. (1 pt.)
4. Show $\forall x Fx \;\vdash\; \forall y Fy$ ***without using either of the rules*** $\forall E$ ***or*** $\forall I$. (1 pt.)
5. $\exists x \forall y (Rxy \leftrightarrow \neg Ryy) \;\vdash\; \bot$ (1 pt.)
6. $\forall x \forall y (Fxy \leftrightarrow \neg Gyx), \exists z \exists w Gzw \;\vdash\; \exists x \exists y \neg Fxy$ (1 pt.)
7. $\forall x\,(Px \vee Qx) \;\vdash\; \forall x\, Px \vee \exists y\, Qy$ (1 pt.)
8. $\forall x (Wx \wedge \exists y \neg Txy) \;\dashv\vdash\; \neg \exists x (\neg Wx \vee \forall y Txy)$ (2 pts.)
9. $\vdash\; \exists z\,(Qzz \rightarrow \forall y\, Qyy)$ (3 pts.)
10. $\exists x (Fx \rightarrow Ga) \;\dashv\vdash\; \forall x Fx \rightarrow Ga$ (3 pts.)
11. $\exists x (Ga \rightarrow Fx) \;\dashv\vdash\; Ga \rightarrow \exists x Fx$ (3 pts.)

1. Show $\forall x F x \vdash Fa$ ***without using the rule*** $\forall E$. (1/2 pt.)

1	$\forall x F x$	
2	$\quad \mid \neg F a$	Ass ($\neg E$)
3	$\quad \mid \exists x \neg F x$	$\exists I$ 2
4	$\quad \mid \neg \forall x F x$	CQ 3
5	$\quad \mid \bot$	$\bot I$ 1, 4
6	$F a$	

2. Show $Fa \vdash \exists x F x$ ***without using the rule*** $\exists I$ (1/2 pt.)

1	$F a$	
2	$\quad \mid \neg \exists x F x$	Ass ($\neg E$)
3	$\quad \mid \forall x \neg F x$	CQ 2
4	$\quad \mid \neg F a$	$\forall E$ 3
5	$\quad \mid \bot$	$\bot I$ 1, 4
6	$\exists x F x$	$\neg E$ 2–5

3. Show $\exists x F x \vdash \exists y F y$ ***without using the rules*** $\exists E$ ***or*** $\exists I$ (1/2 pt.)

1	$\exists x F x$	
2	$\quad \mid \neg \exists y F y$	Ass ($\neg E$)
3	$\quad \mid \forall y \neg F y$	CQ 2
4	$\quad \mid \neg F a$	$\forall E$ 3
5	$\quad \mid \forall x \neg F x$	$\forall I$ 4
6	$\quad \mid \neg \exists x F x$	CQ 5
7	$\quad \mid \bot$	$\bot I$ 1, 6
8	$\exists y F y$	$\neg E$ 2–7

4. Show $\forall xFx \vdash \forall yFy$ ***without using either of the rules*** $\forall E$ ***or*** $\forall I$ (1 pt.)

1	$\forall xFx$	
2	$\quad \neg\forall yFy$	Ass ($\neg E$)
3	$\quad \exists y\neg Fy$	CQ 2
4	$\quad\quad \neg Fk$	Ass ($\exists E$)
5	$\quad\quad \exists x\neg Fx$	$\exists I$ 4
6	$\quad\quad \neg\forall xFx$	CQ 5
7	$\quad\quad \bot$	$\bot I$ 1, 6
8	$\quad \bot$	$\exists E$ 3, 4–7
9	$\forall yFy$	$\neg E$ 2–8

5. $\exists x\forall y(Rxy \leftrightarrow \neg Ryy) \vdash \bot$ (1 pt.)

1	$\exists x\forall y(Rxy \leftrightarrow \neg Ryy)$	
2	$\quad \forall y(Ray \leftrightarrow \neg Ryy)$	Ass ($\exists E$)
3	$\quad Raa \leftrightarrow \neg Raa$	$\forall E$ 2
4	$\quad\quad Raa$	Ass ($\neg I$)
5	$\quad\quad \neg Raa$	$\leftrightarrow E$ 3, 4
6	$\quad\quad \bot$	$\bot I$ 4, 5
7	$\quad \neg Raa$	$\neg I$ 4–6
8	$\quad Raa$	$\leftrightarrow E$ 3, 7
9	$\quad \bot$	$\bot I$ 7, 8
10	$\bot$	$\exists E$ 1, 2–9

6. $\forall x \forall y(Fxy \leftrightarrow \neg Gyx)$, $\exists z \exists w Gzw \vdash \exists x \exists y \neg Fxy$ (1 pt.)

1	$\forall x \forall y(Fxy \leftrightarrow \neg Gyx)$	
2	$\exists z \exists w Gzw$	
3	$\exists w Gaw$	Ass ($\exists E$)
4	Gab	Ass ($\exists E$)
5	$\forall y(Fby \leftrightarrow \neg Gyb)$	$\forall E$ 1
6	$Fab \leftrightarrow \neg Gab$	$\forall E$ 5
7	Fab	Ass ($\neg I$)
8	$\neg Gab$	$\leftrightarrow E$ 6, 7
9	$\bot$	$\bot I$ 4, 8
10	$\neg Fab$	$\neg I$ 7–9
11	$\exists y \neg Fay$	$\exists I$ 10
12	$\exists x \exists y \neg Fxy$	$\exists I$ 11
13	$\exists x \exists y \neg Fxy$	$\exists E$ 3, 4–12
14	$\exists x \exists y \neg Fxy$	$\exists E$ 2, 3–13

7. $\forall x\,(Px \lor Qx) \vdash \forall x\, Px \lor \exists y\, Qy$ (1 pt.)

1	$\forall x(Px \lor Qx)$	
2	$\neg(\forall x Px \lor \exists y Qy)$	Ass ($\neg E$)
3	$\neg \forall x Px \land \neg \exists y Qy$	DeM 2
4	$\neg \forall x Px$	$\land E$ 3
5	$\neg \exists y Qy$	$\land E$ 3
6	$\exists x \neg Px$	CQ 4
7	$\forall y \neg Qy$	CQ 5
8	$\neg Pc$	Ass ($\exists E$)
9	$Pc \lor Qc$	$\forall E$ 1
10	Qc	DS 8, 9
11	$\neg Qc$	$\forall E$ 7
12	$\bot$	$\bot I$ 10, 11
13	$\bot$	$\exists E$ 6, 8–12
14	$\forall x Px \lor \exists y Qy$	$\neg E$ 2–13

8. $\forall x(Wx \land \exists y \neg Txy) \dashv\vdash \neg\exists x(\neg Wx \lor \forall y Txy)$ (2 pts.)

1	$\forall x(Wx \land \exists y \neg Txy)$	
2	$\exists x(\neg Wx \lor \forall y Txy)$	Ass ($\neg I$)
3	$\neg Wc \lor \forall y Tcy$	Ass ($\exists E$)
4	$Wc \land \exists y \neg Tcy$	$\forall E$ 1
5	Wc	$\land E$ 4
6	$\neg Wc$	Ass ($\neg I$)
7	$\bot$	$\bot I$ 5, 6
8	$\neg\neg Wc$	$\neg I$ 6–7
9	$\forall y Tcy$	DS 3, 8
10	$\exists y \neg Tcy$	$\land E$ 4
11	$\neg\forall y Tcy$	CQ 10
12	$\bot$	$\bot I$ 9, 11
13	$\bot$	$\exists E$ 2, 3–12
14	$\neg\exists x(\neg Wx \lor \forall y Txy)$	$\neg I$ 2–13

1	$\neg\exists x(\neg Wx \lor \forall y Txy)$	
2	$\forall x \neg(\neg Wx \lor \forall y Txy)$	CQ 1
3	$\neg(\neg Wa \lor \forall y Tay)$	$\forall E$ 2
4	$\neg\neg Wa \land \neg\forall y Tay$	DeM 3
5	$\neg\neg Wa$	$\land E$ 4
6	Wa	DNE 5
7	$\neg\forall y Tay$	$\land E$ 4
8	$\exists y \neg Tay$	CQ 7
9	$Wa \land \exists y \neg Tay$	$\land I$ 6, 8
10	$\forall x(Wx \land \exists y \neg Txy)$	$\forall I$ 9

9. $\vdash \exists z\,(Qzz \to \forall y\, Qyy)$ (3 pts.)

1	$\neg\exists z(Qzz \to \forall y Qyy)$	Ass ($\neg E$)
2	$\forall z \neg(Qzz \to \forall y Qyy)$	CQ 1
3	$\neg(Qaa \to \forall y Qyy)$	$\forall E$ 2
4	$\neg Qaa$	Ass ($\neg E$)
5	Qaa	Ass ($\to I$)
6	$\bot$	$\bot I$ 4, 5
7	$\forall y Qyy$	$\bot E$ 6
8	$Qaa \to \forall y Qyy$	$\to I$ 5–7
9	$\bot$	$\bot I$ 3, 8
10	Qaa	$\neg E$ 4–9
11	$\forall y Qyy$	$\forall I$ 10
12	Qaa	Ass ($\to I$)
13	$\forall y Qyy$	R 11
14	$Qaa \to \forall y Qyy$	$\to I$ 12–13
15	$\bot$	$\bot I$ 3, 14
16	$\exists z(Qzz \to \forall y Qyy)$	$\neg E$ 1–15

10. $\exists x(Fx \to Ga) \dashv\vdash \forall xFx \to Ga$ (3 pts.)

1	$\exists x(Fx \to Ga)$	
2	$\quad \forall xFx$	Ass $(\to I)$
3	$\quad\quad Fb \to Ga$	Ass $(\exists E)$
4	$\quad\quad Fb$	$\forall E$ 2
5	$\quad\quad Ga$	$\to E$ 3, 4
6	$\quad Ga$	$\exists E$ 1, 3–5
7	$\forall xFx \to Ga$	$\to I$ 2–6

1	$\forall xFx \to Ga$	
2	$\quad \neg\exists x(Fx \to Ga)$	Ass $(\neg E)$
3	$\quad \forall x\neg(Fx \to Ga)$	CQ 2
4	$\quad \neg(Fk \to Ga)$	$\forall E$ 3
5	$\quad\quad \neg Fk$	Ass $(\neg E)$
6	$\quad\quad\quad Fk$	Ass $(\to I)$
7	$\quad\quad\quad \bot$	$\bot I$ 5, 6
8	$\quad\quad\quad Ga$	$\bot E$ 7
9	$\quad\quad Fk \to Ga$	$\to I$ 6–8
10	$\quad\quad \bot$	$\bot I$ 4, 9
11	$\quad Fk$	$\neg E$ 5–10
12	$\quad \forall xFx$	$\forall I$ 11
13	$\quad\quad Ga$	Ass $(\neg I)$
14	$\quad\quad\quad Fk$	Ass $(\to I)$
15	$\quad\quad\quad Ga$	R 13
16	$\quad\quad Fk \to Ga$	$\to I$ 14–15
17	$\quad\quad \bot$	$\bot I$ 4, 16
18	$\quad \neg Ga$	$\neg I$ 13–17
19	$\quad Ga$	$\to E$ 1, 12
20	$\quad \bot$	$\bot I$ 18, 19
21	$\exists x(Fx \to Ga)$	$\neg E$ 2–20

11. $\exists x(Ga \to Fx) \dashv\vdash Ga \to \exists xFx$ (3 pts.)

1	$\exists x(Ga \to Fx)$	
2	$Ga \to Fh$	Ass ($\exists E$)
3	Ga	Ass ($\to I$)
4	Fh	$\to E$ 2, 3
5	$\exists xFx$	$\exists I$ 4
6	$Ga \to \exists xFx$	$\to I$ 3–5
7	$Ga \to \exists xFx$	$\exists E$ 1, 2–6

1	$Ga \to \exists xFx$	
2	$\neg\exists x(Ga \to Fx)$	Ass ($\neg E$)
3	$\forall x\neg(Ga \to Fx)$	CQ 2
4	Ga	Ass ($\to I$)
5	$\exists xFx$	$\to E$ 1, 4
6	Fk	Ass ($\exists E$)
7	Ga	Ass ($\to I$)
8	Fk	R 6
9	$Ga \to Fk$	$\to I$ 7–8
10	$\neg(Ga \to Fk)$	$\forall E$ 3
11	$\bot$	$\bot I$ 9, 10
12	$\bot$	$\exists E$ 5, 6–11
13	Fj	$\bot E$ 12
14	$Ga \to Fj$	$\to I$ 4–13
15	$\neg(Ga \to Fj)$	$\forall E$ 3
16	$\bot$	$\bot I$ 14, 15
17	$\exists x(Ga \to Fx)$	$\neg E$ 2–16

Final

You will have 110 minutes to complete the final. There are 6 sections, which means you should budget about 18 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. _______ An invalid argument can have true premises and a true conclusion.
2. _______ The main operator of '$\forall x\, \exists y\, (Rxy \wedge \neg Ryx) \vee \exists y\, Qy$' is '$\forall x$'.
3. _______ If a collection of sentences are satisfiable, then they are jointly possible.
4. _______ If $\mathscr{A}$ is a contradiction, then $\mathscr{A} \vdash \neg(Fa \vee \neg Fa)$.
5. _______ If the argument $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B}$, and $\neg\mathscr{C}$ are jointly impossible.
6. _______ '$\forall x \neg(\forall y\, Qy)$' is a sentence of PL.
7. _______ If '$\mathscr{A}$' is a tautology, then '$\neg\mathscr{A} \rightarrow \mathscr{A}$' is a tautology.
8. _______ If $\mathscr{A}, \mathscr{B} \models \mathscr{C}$, then $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid.
9. _______ If $\mathscr{A}, \mathscr{B} \therefore \mathscr{C}$ is valid, then $\mathscr{A}, \mathscr{B} \models \mathscr{C}$.
10. _______ If $\mathscr{A} \vdash \neg\mathscr{A}$, then $\mathscr{A}$ is a contradiction.

B. INTERPRETATIONS AND ENTAILMENT. Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) Then, provide an interpretation which shows that the argument is not an entailment.

> Everyone who works hard deserves success. Some people who achieve success deserve it. So some people who achieve success work hard.

c. evaluating arguments (part 1). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) If, once translated into PL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises. If, once translated into PL, the argument's premises do not entail its conclusion, then provide an interpretation which shows this.

Everyone loves Mary. Anyone who loves Mary loves Barbara. So everyone loves Barbara.

d. evaluating arguments (part 2). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises. If the premises don't entail its conclusion, then provide an interpretation which shows this.

Some cats are fluffy. No dogs are fluffy. So no cats are dogs.

E. SYLLOGISMS. Using this symbolization key:

domain	: all questions
Tx	: ____$_x$ is a matter of taste
Sx	: ____$_x$ is subjective
Mx	: ____$_x$ is a moral question

translate *one* of the following syllogisms into PL and provide a natural deduction proof to show that the premises entail the conclusion.

(a) All matters of taste are subjective. No moral questions are subjective. So no moral question is a matter of taste.

(b) Some matters of taste are subjective. No moral questions are subjective. So not all matters of taste are moral questions.

F. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences is a theorem.

(a) $\exists z\, \neg Qz \rightarrow \neg\forall x\, Qx$

(b) $\forall x\, Rxx \rightarrow \forall x\, \exists y\, Ryx$

You will have 110 minutes to complete the final. There are 6 sections, which means you should budget about 18 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. False If an argument has a false conclusion, then it is invalid.
2. False The main operator of '$\neg\forall x Fx \rightarrow \exists y Ry$' is '$\neg$'.
3. True If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are unsatisfiable, then $\mathcal{A}, \mathcal{B} \models \neg\mathcal{C}$.
4. True If $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N \models \mathcal{C}$, then $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_N \vdash \mathcal{C}$.
5. True If $\mathcal{A}$ is a tautology, then $\mathcal{B} \models \mathcal{A}$.
6. False '$\forall x(\mathcal{F}x \rightarrow \neg\mathcal{G}x)$' is a sentence of PL.
7. True If '$\mathcal{A}$' is a contradiction, then '$\mathcal{A} \rightarrow \neg\mathcal{A}$' is a tautology.
8. True If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are unsatisfiable, then $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are jointly impossible.
9. False If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are jointly impossible, then $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are unsatisfiable.
10. True If $\mathcal{A} \vdash \bot$, then $\mathcal{A}$ is a contradiction.

B. INTERPRETATIONS AND ENTAILMENT. Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) Then, provide an interpretation which shows that the argument is not an entailment.

> Everyone who works hard succeeds. Some people succeed. Therefore, some people work hard.

Here is the symbolization key:

$$\begin{aligned} \text{domain} &: \text{all people} \\ Wx &: \underline{\quad\quad}_x \text{ works hard} \\ Sx &: \underline{\quad\quad}_x \text{ succeeds} \end{aligned}$$

Then, here is the argument in PL:

$$\forall x(Wx \rightarrow Sx), \exists xSx \therefore \exists xWx$$

The following interpretation makes the premises true and the conclusion false, and so shows that the argument is not an entailment.

$$\begin{aligned} \text{domain} &: \text{Bill} \\ W &: \\ S &: \text{Bill} \end{aligned}$$

To see that '$\forall x(Wx \rightarrow Sx)$' is true, note that '$Wx \rightarrow Sx$' is true if we let '$x$' be name for Bill. For, if 'x' is a name for Bill, then 'Wx' is false, since Bill is not W. And if 'Wx' is false, then '$Wx \rightarrow Sx$' is true. Since Bill is the only thing in our domain, '$Wx \rightarrow Sx$' is true *no matter what* we let 'x' name.

To see that '$\exists xSx$' is true, note that 'Sx' is true if we let 'x' be a name for Bill, since Bill is S. So there's *something* we could let 'x' be a name for which would make 'Sx' true. So '$\exists x\, Sx$' is true.

To see that '$\exists x\, Wx$' is false, note that 'Wx' is false if we let 'x' be a name for Bill—since Bill is not W. And Bill is the only thing in our domain. So there's nothing we could let 'x' be a name for which would make 'Wx' true. So '$\exists x\, Wx$' is false.

So this interpretation makes the premises true and the conclusion false. So the argument isn't an entailment.

[*Note: on the final, you don't have to explain why the interpretation makes the premises true and the conclusion false—it's enough to provide an interpretation which* <u>*does*</u> *make the premises true and the conclusion false.*]

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key.**) If, once translated into PL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises. If, once translated into PL, the argument's premises do not entail its conclusion, then provide an interpretation which shows this.

Every quirky philosopher is funny. No one who is funny is quirky. So no philosopher is quirky.

Here is the symbolization key:

domain	:	all people
Px	:	$\_\_\_\_\__x$ is a philosopher
Qx	:	$\_\_\_\_\__x$ is quirky
Fx	:	$\_\_\_\_\__x$ is funny

Then, here is the argument in PL:

$$\forall x\,[(Qx \land Px) \to Fx],\ \forall y(Fy \to \neg Qy)\ \therefore\ \forall z(Pz \to \neg Qz)$$

The following natural deduction proof shows that this argument is an entailment.

1	$\forall x\,[(Qx \land Px) \to Fx]$	
2	$\forall y(Fy \to \neg Qy)$	
3	$\quad Pa$	Ass. ($\to I$)
4	$\quad\quad Qa$	Ass. ($\neg I$)
5	$\quad\quad Qa \land Pa$	$\land I$ 3, 4
6	$\quad\quad (Qa \land Pa) \to Fa$	$\forall E$ 1 ($x : a$)
7	$\quad\quad Fa$	$\to E$ 5, 6
8	$\quad\quad Fa \to \neg Qa$	$\forall E$ 2 ($y : a$)
9	$\quad\quad \neg Qa$	$\to E$ 7, 8
10	$\quad\quad \bot$	$\bot I$ 4, 9
11	$\quad \neg Qa$	$\neg I$ 4–10
12	$Pa \to \neg Qa$	$\to I$ 3–11
13	$\forall z(Pz \to \neg Qz)$	$\forall I$ 12

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key.**) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises. If the premises don't entail its conclusion, then provide an interpretation which shows this.

Not all wrong choices are bad. All bad choices are regrettable. So not all wrong choices are regrettable.

Here is the symbolization key:

domain : all choices
Bx : $\_\_\_\__x$ is bad
Wx : $\_\_\_\__x$ is wrong
Rx : $\_\_\_\__x$ is regrettable

Then, here is the argument in PL:

$$\neg\forall x(Wx \to Bx),\ \forall y(By \to Ry)\ \therefore\ \neg\forall z(Wz \to Rz)$$

The following interpretation makes the premises true and the conclusion false, and so shows that the argument is not an entailment.

domain : 1
B :
W : 1
R : 1

To see that '$\neg\forall x(Wx \to Bx)$' is true, note that '$Wx \to Bx$' is false if we let 'x' be name for 1. For, if 'x' is a name for 1, then 'Wx' is true, since 1 is W. And 'Bx' is false, since 1 is not B. So '$Wx \to Bx$' is false. So there's something we can let 'x' name which makes '$Wx \to Bx$' false. So '$\forall x(Wx \to Bx)$' is false. So '$\neg\forall x(Wx \to Bx)$' is true.

To see that '$\forall y(By \to Ry)$' is true, note that 'By' is false if we let 'y' be a name for 1, since 1 is not B. And if 'By' is false, then '$By \to Ry$' is true. Since 1 is the only thing in our domain, '$By \to Ry$' is true *no matter what* we let 'y' name. So '$\forall y(By \to Ry)$' is true.

To see that '$\neg\forall z(Wz \to Rz)$' is true, note that '$\forall z(Wz \to Rz)$' is true. For if we let 'z' be a name for 1, then 'Wz' is false—since 1 is not W. And if 'Wz' is false, then '$Wz \to Rz$' is true. Since 1 is the only thing in our domain, '$Wz \to Rz$' is true *no matter what* we let 'z' name. So '$\forall z(Wz \to Rz)$' is true. So '$\neg\forall z(Wz \to Rz)$' is false.

So this interpretation makes the premises true and the conclusion false. So the argument isn't an entailment.

[*Note: on the final, you don't have to explain why the interpretation makes the premises true and the conclusion false—it's enough to provide an interpretation which* <u>*does*</u> *make the premises true and the conclusion false.*]

E. SYLLOGISMS. Using this symbolization key:

$$\begin{aligned} \text{domain} &: \text{all events} \\ Mx &: \underline{\quad}_x \text{ is mental} \\ Px &: \underline{\quad}_x \text{ is physical} \\ Cx &: \underline{\quad}_x \text{ is conscious} \end{aligned}$$

translate *one* of the following syllogisms into PL and provide a natural deduction proof to show that the premises entail the conclusion. (For the purposes of translating these, assume that to be unconscious just is to *not* be conscious.)

(a) No physical events are conscious. No unconscious event is mental. So no physical event is mental.

Here is the translation:

$$\forall x(Px \rightarrow \neg Cx),\ \forall y(\neg Cy \rightarrow \neg My)\ \therefore\ \forall z(Pz \rightarrow \neg Mz)$$

The following natural deduction proof shows that the argument is an entailment:

1	$\forall x(Px \rightarrow \neg Cx)$	
2	$\forall y(\neg Cy \rightarrow \neg My)$	
3	| Pb	Ass. ($\rightarrow I$)
4	| $Pb \rightarrow \neg Cb$	$\forall E$ 1 ($x : b$)
5	| $\neg Cb$	$\rightarrow E$ 3, 4
6	| $\neg Cb \rightarrow \neg Mb$	$\forall E$ 2 ($y : b$)
7	| $\neg Mb$	$\rightarrow E$ 5, 6
8	$Pb \rightarrow \neg Mb$	$\rightarrow I$ 3–7
9	$\forall z(Pz \rightarrow \neg Mz)$	$\forall I$ 8

(b) All mental events are physical. Some mental events are conscious. So some physical events are conscious.

Here is the translation:

$$\forall x(Mx \to Px),\ \exists y(My \wedge Cy)\ \therefore\ \exists z(Pz \wedge Cz)$$

The following natural deduction proof shows that the argument is an entailment:

1	$\forall x(Mx \to Px)$	
2	$\exists y(My \wedge Cy)$	
3	$\quad Mj \wedge Cj$	Ass. ($\exists E$)
4	$\quad Mj \to Pj$	$\forall E$ 1 $(x : j)$
5	$\quad Mj$	$\wedge E$ 3
6	$\quad Pj$	$\to E$ 4, 5
7	$\quad Cj$	$\wedge E$ 3
8	$\quad Pj \wedge Cj$	$\wedge I$ 6, 7
9	$\quad \exists z(Pz \wedge Cz)$	$\exists I$ 8
10	$\exists z(Pz \wedge Cz)$	$\exists E$ 2, 3–9

F. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences is a theorem.

(a) $\exists x\,\forall y\,Rxy \rightarrow \forall y\,\exists x\,Rxy$

1	$\exists x \forall y Rxy$	Ass ($\rightarrow I$)
2	$\quad \forall y Ray$	Ass ($\exists E$)
3	$\quad Rab$	$\forall E$ 2 ($y : b$)
4	$\quad \exists x Rxb$	$\exists I$ 3
5	$\quad \forall y \exists x Rxy$	$\forall I$ 4
6	$\forall y \exists x Rxy$	$\exists E$ 1, 2–5
7	$\exists x \forall y Rxy \rightarrow \forall y \exists x Rxy$	$\rightarrow I$ 1–6

(b) $\forall x\,Fx \vee \exists y\,\neg Fy$

1	$\neg(\forall x Fx \vee \exists x \neg Fx)$	Ass ($\neg E$)
2	$\neg \forall x Fx \wedge \neg \exists x \neg Fx$	DeM 1
3	$\neg \forall x Fx$	$\wedge E$ 2
4	$\exists x \neg Fx$	CQ 3
5	$\neg \exists x \neg Fx$	$\wedge E$ 2
6	$\bot$	$\bot I$ 4, 5
7	$\forall x Fx \vee \exists x \neg Fx$	$\neg E$ 1–6

1	$\forall x Fx$	Ass (LEM)
2	$\forall x \vee \exists x \neg Fx$	$\vee I$ 1
3	$\neg \forall x Fx$	Ass (LEM)
4	$\exists x \neg Fx$	CQ 3
5	$\forall x Fx \vee \exists x \neg Fx$	$\vee I$ 4
6	$\forall x Fx \vee \exists x \neg Fx$	LEM 1–2, 3–5

You will have 110 minutes to complete the final. There are 6 sections, which means you should budget about 18 minutes per section.

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. _______ If an argument is valid, then it has a true conclusion.
2. _______ The main operator of '$\forall z(\forall y\, Rzy \rightarrow Rzz) \vee Rab$' is '$\forall z$'.
3. _______ If $\mathscr{A}, \mathscr{B} \models \mathscr{C}$, then $\mathscr{A}, \mathscr{B}$, and $\neg\mathscr{C}$ are unsatisfiable.
4. _______ If $\mathscr{A} \vdash \mathscr{C}$, then $\mathscr{A}, \mathscr{B} \vdash \mathscr{C}$.
5. _______ If $\mathscr{A}$ is a tautology, then $\mathscr{A} \models \neg\mathscr{A}$.
6. _______ '$\forall x(Fx \wedge Hx \rightarrow Gx)$' is a sentence of PL.
7. _______ If '$\mathscr{A}$' is neither a tautology nor a contradiction, then '$\mathscr{A} \rightarrow \neg\mathscr{A}$' is neither a tautology nor a contradiction.
8. _______ If $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are unsatisfiable, then $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are jointly impossible.
9. _______ If $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are jointly impossible, then $\mathscr{A}, \mathscr{B}$, and $\mathscr{C}$ are unsatisfiable.
10. _______ If $\mathscr{A}$ is a contradiction, then $\mathscr{A} \vdash \neg(Qb \rightarrow \forall z\, Fnz)$.

B. INTERPRETATIONS AND ENTAILMENT. Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) Then, provide an interpretation which shows that the argument is not an entailment.

All dogs are pets. All friendly animals are pets. So some dogs are friendly.

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) If, once translated into PL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises. If, once translated into PL, the argument's premises do not entail its conclusion, then provide an interpretation which shows this.

Wednesday hates Pugsly. Everyone Wednesday hates hates her back. So Pusgly hates someone.

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises. If the premises don't entail its conclusion, then provide an interpretation which shows this.

> Some countries are not fun to visit. Any country too dangerous to visit is not fun to visit. So some countries are too dangerous to visit.

E. SYLLOGISMS. Using this symbolization key:

domain	: all animals
Ox	: $\_\_\_\_\__x$ is an orangutan
Fx	: $\_\_\_\_\__x$ is friendly
Dx	: $\_\_\_\_\__x$ is a dolphin

translate *one* of the following syllogisms into PL and provide a natural deduction proof to show that the premises entail the conclusion. (For the purposes of translating these, assume that to be unconscious just is to *not* be conscious.)

(a) All orangutans are friendly. No dolphins are friendly. So no orangutans are dolphins.

(b) Some dolphins are friendly. All friendly animals are orangutans. So some dolphins are orangutans.

F. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences is a theorem.

(a) $\forall y Lay \rightarrow \forall x \exists y Lyx$

(b) $\exists y(Fy \wedge \neg Gy) \rightarrow \neg \forall x(Fx \rightarrow Gx)$

A. TRUE/FALSE. If a statement below is true, write 'T' in the provided space. If it is false, write 'F'. (Write legibly. If I cannot tell whether you have written 'T' or 'F', then you will get the question wrong. You may write '1' for 'true' and '0' for 'false', if you wish.)

1. **false** If an argument is valid, then it has a true conclusion.
2. **false** The main operator of '$\forall z(\forall y\, Rzy \rightarrow Rzz) \vee Rab$' is '$\forall z$'.
3. **true** If $\mathscr{A}, \mathscr{B} \models \mathscr{C}$, then $\mathscr{A}$, $\mathscr{B}$, and $\neg\mathscr{C}$ are unsatisfiable.
4. **true** If $\mathscr{A} \vdash \mathscr{C}$, then $\mathscr{A}, \mathscr{B} \vdash \mathscr{C}$.
5. **false** If $\mathscr{A}$ is a tautology, then $\mathscr{A} \models \neg\mathscr{A}$.
6. **false** '$\forall x(Fx \wedge Hx \rightarrow Gx)$' is a sentence of PL.
7. **true** If '$\mathscr{A}$' is neither a tautology nor a contradiction, then '$\mathscr{A} \rightarrow \neg\mathscr{A}$' is neither a tautology nor a contradiction.
8. **true** If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are unsatisfiable, then $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are jointly impossible.
9. **false** If $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are jointly impossible, then $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ are unsatisfiable.
10. **true** If $\mathscr{A}$ is a contradiction, then $\mathscr{A} \vdash \neg(Qb \rightarrow \forall z\, Fnz)$.

B. INTERPRETATIONS AND ENTAILMENT. Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) Then, provide an interpretation which shows that the argument is not an entailment.

All dogs are pets. All friendly animals are pets. So some dogs are friendly.

Given the following symbolization key,

domain	:	animals
Dx	:	$\_\_\_\__x$ is a dog
Fx	:	$\_\_\_\__x$ is friendly
Px	:	$\_\_\_\__x$ is a pet

we may translate the argument into PL as follows:

$$\forall x(Dx \rightarrow Px),\ \forall y(Fy \rightarrow Py)\ \therefore\ \exists z(Dz \wedge Fz)$$

The following interpretation shows that this argument is not an entailment:

domain	:	Amy
D	:	
F	:	
P	:	

C. EVALUATING ARGUMENTS (PART 1). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key.**) If, once translated into PL, the premises entail the conclusion, then provide a natural deduction proof showing that you can prove the conclusion from the premises. If, once translated into PL, the argument's premises do not entail its conclusion, then provide an interpretation which shows this.

Wednesday hates Pugsly. Everyone Wednesday hates hates her back. So Pusgly hates someone.

Given the following symbolization key,

domain	:	people
Hxy	:	$\_\_\_\__x$ hates $\_\_\_\__y$
n	:	Wednesday
p	:	Pugsly

we may translate the argument into PL as follows:

$$Hnp, \forall w(Hnw \to Hwn) \therefore \exists x Hpx$$

This argument is an entailment, as the following natural deduction proof demonstrates:

1	Hnp	
2	$\forall w(Hnw \to Hwn)$	
3	$Hnp \to Hpn$	$\forall E$ 2
4	Hpn	$\to E$ 1, 3
5	$\exists x Hpx$	$\exists I$ 4

D. EVALUATING ARGUMENTS (PART 2). Translate the premises and the conclusion of the argument below into PL. (**Be sure to provide a symbolization key**.) If the premises entail the conclusion, then provide a natural deduction proof showing that you can prove its conclusion from its premises. If the premises don't entail its conclusion, then provide an interpretation which shows this.

> Some countries are not fun to visit. Any country too dangerous to visit is not fun to visit. So some countries are too dangerous to visit.

Given the following symbolization key,

domain	:	countries
Fx	:	$\_\_\_\_\__x$ is fun to visit
Dx	:	$\_\_\_\_\__x$ is too dangerous to visit

we may translate the argument into PL as follows:

$$\exists x \neg Fx,\ \forall y(Dy \rightarrow \neg Fy)\ \therefore\ \exists y Dy$$

The following interpretation shows that this argument is not an entailment:

domain	:	1
F	:	
D	:	

Alternatively, we could have used a symbolization key like this:

domain	:	places
Cx	:	$\_\_\_\_\__x$ is a country
Fx	:	$\_\_\_\_\__x$ is fun to visit
Dx	:	$\_\_\_\_\__x$ is too dangerous to visit

Then, we'd have the following argument:

$$\exists x(Cx \wedge \neg Fx),\ \forall y((Cy \wedge Dy) \rightarrow \neg Fy)\ \therefore\ \exists y(Cy \wedge Dy)$$

And *this* interpretation shows that it is not an entailment:

domain	:	1
C	:	1
F	:	
D	:	

E. SYLLOGISMS. Using this symbolization key:

domain	:	all animals
Ox	:	$\_\_\_\_\__x$ is an orangutan
Fx	:	$\_\_\_\_\__x$ is friendly
Dx	:	$\_\_\_\_\__x$ is a dolphin

translate *one* of the following syllogisms into PL and provide a natural deduction proof to show that the premises entail the conclusion. (For the purposes of translating these, assume that to be unconscious just is to *not* be conscious.)

(a) All orangutans are friendly. No dolphins are friendly. So no orangutans are dolphins.

$$\forall y(Oy \rightarrow Fy),\ \forall x(Dx \rightarrow \neg Fx)\ \therefore\ \forall w(Ow \rightarrow \neg Dw)$$

1	$\forall y(Oy \rightarrow Fy)$	
2	$\forall x(Dx \rightarrow \neg Fx)$	
3	$\quad Oa$	Ass ($\rightarrow I$)
4	$\quad Oa \rightarrow Fa$	$\forall E$ 1
5	$\quad Fa$	$\rightarrow E$ 3, 4
6	$\quad\quad Da$	Ass ($\neg I$)
7	$\quad\quad Da \rightarrow \neg Fa$	$\forall E$ 2
8	$\quad\quad \neg Fa$	$\rightarrow E$ 6, 7
9	$\quad\quad \bot$	$\bot I$ 5, 8
10	$\quad \neg Da$	$\neg I$ 6–9
11	$Oa \rightarrow \neg Da$	$\rightarrow I$ 3–10
12	$\forall w(Ow \rightarrow \neg Dw)$	$\forall I$ 11

(b) Some dolphins are friendly. All friendly animals are orangutans. So some dolphins are orangutans.

$$\exists z(Dz \wedge Fz),\ \forall y(Fy \rightarrow Oy)\ \therefore\ \exists x(Dx \wedge Ox)$$

1	$\exists z(Dz \wedge Fz)$	
2	$\forall y(Fy \rightarrow Oy)$	
3	$\quad Dj \wedge Fj$	Ass ($\exists E$)
4	$\quad Fj$	$\wedge E$ 3
5	$\quad Fj \rightarrow Oj$	$\forall E$ 2
6	$\quad Oj$	$\rightarrow E$ 4, 5
7	$\quad Dj$	$\wedge E$ 3
8	$\quad Dj \wedge Oj$	$\wedge I$ 6, 7
9	$\quad \exists x(Dx \wedge Ox)$	$\exists I$ 8
10	$\exists x(Dx \wedge Ox)$	$\exists E$ 1, 3–9

F. THEOREMS. Provide a natural deduction proof to show that *one* of the following sentences is a theorem.

(a) $\forall y Lay \to \forall x \exists y Lyx$

1	$\forall y Lay$	Ass ($\to I$)
2	Lab	$\forall E$ 1
3	$\exists y Lyb$	$\exists I$ 2
4	$\forall x \exists y Lyx$	$\forall I$ 3
5	$\forall y Lay \to \forall x \exists y Lyx$	$\to I$ 1–4

(b) $\exists y(Fy \land \neg Gy) \to \neg \forall x(Fx \to Gx)$

1	$\exists y(Fy \land \neg Gy)$	Ass ($\to I$)
2	$\forall x(Fx \to Gx)$	Ass ($\neg I$)
3	$Fn \land \neg Gn$	Ass ($\exists E$)
4	Fn	$\land E$ 3
5	$Fn \to Gn$	$\forall E$ 2
6	Gn	$\to E$ 4, 5
7	$\neg Gn$	$\land E$ 3
8	$\bot$	$\bot I$ 6, 7
9	$\bot$	$\exists E$ 1, 3–8
10	$\neg \forall x(Fx \to Gx)$	$\neg I$ 2–9
11	$\exists y(Fy \land \neg Gy) \to \neg \forall x(Fx \to Gx)$	$\to I$ 1–10