Natural Deduction Proofs for SL

Day 1

PHIL 500

1	$\neg A \wedge B$	
2	$\neg A$	$\wedge E$ 1
3	$\neg A \vee \neg B$	$\vee I$ 2

The Point of Natural Deduction Proofs

- Consider the argument:

$$
\begin{aligned}
& (P \longleftrightarrow Q) \rightarrow R \\
& R \longleftrightarrow S \\
& S \longleftrightarrow T \\
& T \longleftrightarrow U \\
& U \longleftrightarrow V \\
& \neg V \\
\therefore \quad & (P \wedge \neg Q) \vee (\neg P \wedge Q)
\end{aligned}
$$

The Point of Natural Deduction Proofs

- The premises of this argument entail its conclusion. However, checking this with a truth table would require $2^7 = 128$ rows.

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- Natural deduction will allow us to establish the validity of this argument much more easily.
- They will, at the same time, teach us how to *think through* which sentences of SL entail which other sentences of SL.

Outline

Orientation

Basic Rules

Proof Strategies

Derived Rules

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The Anatomy of a Proof

- a Natural Deduction Proof for SL consists of:

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The Anatomy of a Proof

- a Natural Deduction Proof for SL consists of:
  - a certain number of lines, each one numbered.
  - on each line of the proof, there is a sentence of SL along with a *justification* explaining why we get to write that sentence down on that line—*unless* that sentence is one of the *assumptions*.

The Anatomy of a Proof

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3
5	$B \vee Z$	$\vee I$, 4
6	$(B \vee Z) \vee Y$	$\vee I$, 5

Legality

- If the proof is to be *legal*, then the symbols appearing on each line must be sentences of SL (or a special symbol, $\bot$, which we'll meet later).

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- Each line which isn't an assumption must *follow from* the lines cited in the justification, according to the rule cited in the justification.
- The lines cited must *precede* the line on which the justification is written.
- Only lines preceding a given line are *accessible* from that line; and only *accessible* lines may be legally cited in a justification.

The Point of Proofs

FACT: If there is a *legal* SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its final line*, then $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \models \mathscr{C}$.

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FACT: If $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \models \mathscr{C}$, then there is a *legal* SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its final line*

Notation

- If there is a legal SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its final line*, then I will write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$$

The Point of Proofs

FACT: $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$ iff $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \models \mathscr{C}$.

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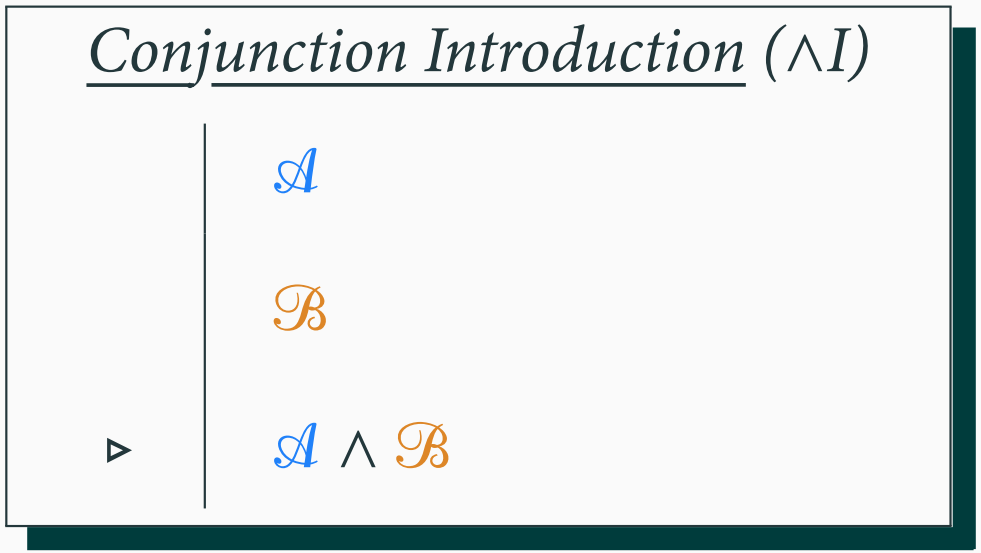
Conjunction Introduction



$$A \to B,\ B \vee Z,\ Z \leftrightarrow Y \vdash [(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$$

1	$A \to B$
2	$B \vee Z$
3	$Z \leftrightarrow Y$

$A \to B,\ B \vee Z,\ Z \leftrightarrow Y \vdash [(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$

1	$A \to B$	
2	$B \vee Z$	
3	$Z \leftrightarrow Y$	
4	$(A \to B) \wedge (Z \leftrightarrow Y)$	$\wedge I$ 1, 3

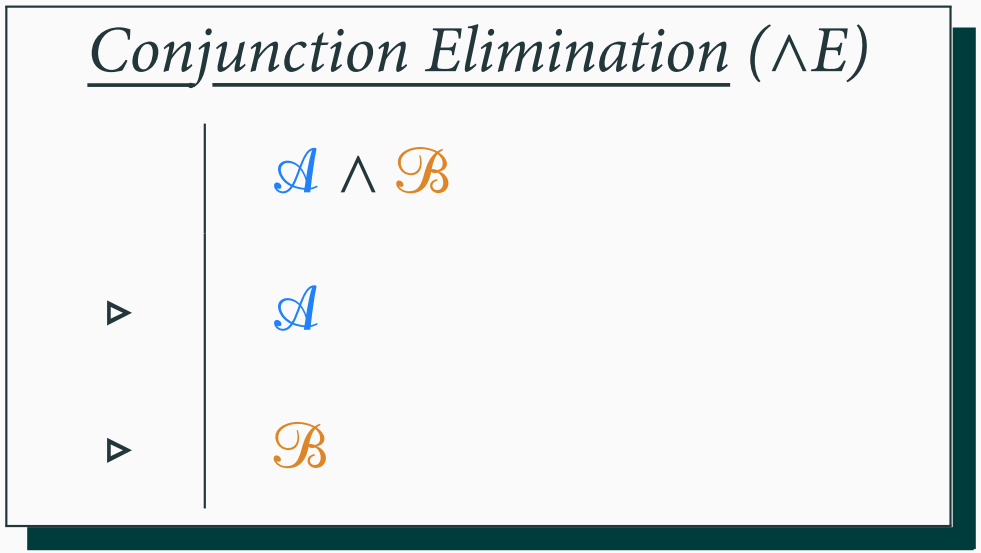
$A \to B,\ B \vee Z,\ Z \leftrightarrow Y \vdash [(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$

1	$A \to B$	
2	$B \vee Z$	
3	$Z \leftrightarrow Y$	
4	$(A \to B) \wedge (Z \leftrightarrow Y)$	$\wedge I$ 1, 3
5	$[(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$	$\wedge I$ 2, 4

$A \to B,\ B \vee Z,\ Z \leftrightarrow Y \vdash [(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$

1	$A \to B$	
2	$B \vee Z$	
3	$Z \leftrightarrow Y$	
4	$(A \to B) \wedge (Z \leftrightarrow Y)$	$\wedge I$ 1, 3
5	$[(A \to B) \wedge (Z \leftrightarrow Y)] \wedge (B \vee Z)$	$\wedge I$ 2, 4

Conjunction Elimination



$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1 $\quad (A \to B) \wedge [C \wedge (D \to F)]$

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1	$(A \to B) \wedge [C \wedge (D \to F)]$	
2	$A \to B$	$\wedge E$ 1

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1	$(A \to B) \wedge [C \wedge (D \to F)]$	
2	$A \to B$	$\wedge E$ 1
3	$C \wedge (D \to F)$	$\wedge E$ 1

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1	$(A \to B) \wedge [C \wedge (D \to F)]$	
2	$A \to B$	$\wedge E$ 1
3	$C \wedge (D \to F)$	$\wedge E$ 1
4	$D \to F$	$\wedge E$ 3

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1	$(A \to B) \wedge [C \wedge (D \to F)]$	
2	$A \to B$	$\wedge E$ 1
3	$C \wedge (D \to F)$	$\wedge E$ 1
4	$D \to F$	$\wedge E$ 3
5	$(A \to B) \wedge (D \to F)$	$\wedge I$ 2, 4

$(A \to B) \wedge [C \wedge (D \to F)] \quad \vdash \quad (A \to B) \wedge (D \to F)$

1	$(A \to B) \wedge [C \wedge (D \to F)]$	
2	$A \to B$	$\wedge E$ 1
3	$C \wedge (D \to F)$	$\wedge E$ 1
4	$D \to F$	$\wedge E$ 3
5	$(A \to B) \wedge (D \to F)$	$\wedge I$ 2, 4

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3
5	C	$\wedge E$, 3

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3
5	C	$\wedge E$, 3
6	$A \wedge B$	$\wedge I$ 2, 4

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3
5	C	$\wedge E$, 3
6	$A \wedge B$	$\wedge I$ 2, 4
7	$(A \wedge B) \wedge C$	$\wedge I$ 5, 6

$A \wedge (B \wedge C) \quad \vdash \quad (A \wedge B) \wedge C$

1	$A \wedge (B \wedge C)$	
2	A	$\wedge E$, 1
3	$B \wedge C$	$\wedge E$, 1
4	B	$\wedge E$, 3
5	C	$\wedge E$, 3
6	$A \wedge B$	$\wedge I$ 2, 4
7	$(A \wedge B) \wedge C$	$\wedge I$ 5, 6

Legal Proofs

- **Rules *may not* be applied to sub-sentences.**

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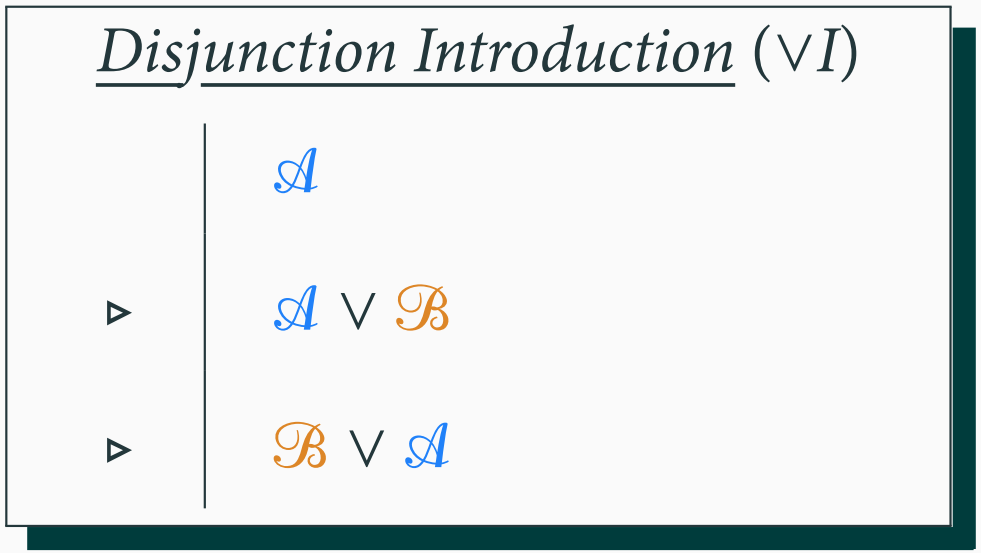
- **Rules *may not* be applied to sub-sentences.**

1	$P \rightarrow (P \wedge Q)$	
2	Q	$\wedge E$, 1 ⟵ **MISTAKE!!!**

Legal Proofs

P	Q	$P \rightarrow (P \wedge Q)$	Q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

Disjuction Introduction



$P \wedge (Q \rightarrow R) \quad \vdash \quad (P \vee X) \wedge [(Q \rightarrow R) \vee X]$

1 $\quad$ $P \wedge (Q \rightarrow R)$

$P \land (Q \to R) \quad \vdash \quad (P \lor X) \land [(Q \to R) \lor X]$

1	$P \land (Q \to R)$	
2	P	$\land E$, 1

$P \wedge (Q \rightarrow R) \quad \vdash \quad (P \vee X) \wedge [(Q \rightarrow R) \vee X]$

1	$P \wedge (Q \rightarrow R)$	
2	P	$\wedge E$, 1
3	$Q \rightarrow R$	$\wedge E$, 1

$P \land (Q \to R) \quad \vdash \quad (P \lor X) \land [(Q \to R) \lor X]$

1	$P \land (Q \to R)$	
2	P	$\land E$, 1
3	$Q \to R$	$\land E$, 1
4	$P \lor X$	$\lor I$, 2

$P \wedge (Q \rightarrow R) \quad \vdash \quad (P \vee X) \wedge [(Q \rightarrow R) \vee X]$

1	$P \wedge (Q \rightarrow R)$	
2	P	$\wedge E$, 1
3	$Q \rightarrow R$	$\wedge E$, 1
4	$P \vee X$	$\vee I$, 2
5	$(Q \rightarrow R) \vee X$	$\vee I$, 3

$P \wedge (Q \rightarrow R) \quad \vdash \quad (P \vee X) \wedge [(Q \rightarrow R) \vee X]$

1	$P \wedge (Q \rightarrow R)$	
2	P	$\wedge E$, 1
3	$Q \rightarrow R$	$\wedge E$, 1
4	$P \vee X$	$\vee I$, 2
5	$(Q \rightarrow R) \vee X$	$\vee I$, 3
6	$(P \vee X) \wedge [(Q \rightarrow R) \vee X]$	$\wedge I$ 4, 5

$P \wedge (Q \rightarrow R) \quad \vdash \quad (P \vee X) \wedge [(Q \rightarrow R) \vee X]$

1	$P \wedge (Q \rightarrow R)$	
2	P	$\wedge E$, 1
3	$Q \rightarrow R$	$\wedge E$, 1
4	$P \vee X$	$\vee I$, 2
5	$(Q \rightarrow R) \vee X$	$\vee I$, 3
6	$(P \vee X) \wedge [(Q \rightarrow R) \vee X]$	$\wedge I$ 4, 5

Legal Proofs

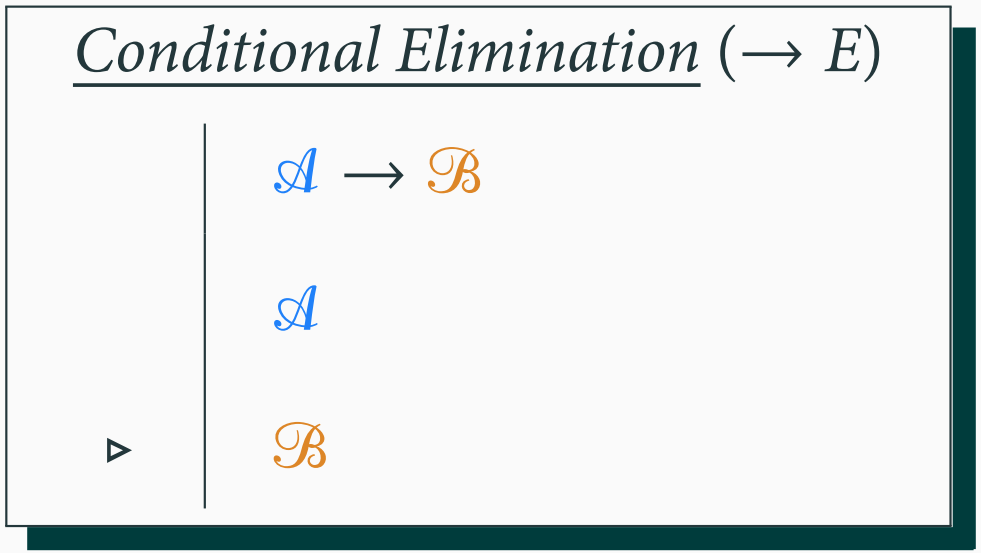
Rules *may not* be applied to sub-sentences.

1	$A \rightarrow C$		
2	$(A \vee B) \rightarrow C$	$\vee I$, 1	⟵ **MISTAKE!!!**

Legal Proofs

A	B	C	$A \to C$	$(A \vee B) \to C$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	T

Conditional Elimination



$$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \quad \vdash \quad S$$

1

$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \quad \vdash \quad S$

1	P
2	$P \to Q$
3	$Q \to R$
4	$R \to S$

$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \ \vdash\ S$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	$R \to S$	
5	Q	$\to E$ 1, 2

$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \ \vdash\ S$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	$R \to S$	
5	Q	$\to E$ 1, 2
6	R	$\to E$ 3, 5

$$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \ \vdash\ S$$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	$R \to S$	
5	Q	$\to E\ 1, 2$
6	R	$\to E\ 3, 5$
7	S	$\to E\ 4, 6$

$$P\,,\ P \to Q\,,\ Q \to R\,,\ R \to S \ \vdash\ S$$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	$R \to S$	
5	Q	$\to E\ 1, 2$
6	R	$\to E\ 3, 5$
7	S	$\to E\ 4, 6$

$$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$$

1	A
2	$(A \to B) \wedge (A \to C)$

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2
4	B	$\to E$ 1, 3

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2
4	B	$\to E$ 1, 3
5	$A \to C$	$\wedge E$, 2

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2
4	B	$\to E$ 1, 3
5	$A \to C$	$\wedge E$, 2
6	C	$\to E$ 1, 5

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2
4	B	$\to E$ 1, 3
5	$A \to C$	$\wedge E$, 2
6	C	$\to E$ 1, 5
7	$B \wedge C$	$\wedge I$ 4, 6

$A\ ,\ (A \to B) \wedge (A \to C) \quad \vdash \quad B \wedge C$

1	A	
2	$(A \to B) \wedge (A \to C)$	
3	$A \to B$	$\wedge E$, 2
4	B	$\to E$ 1, 3
5	$A \to C$	$\wedge E$, 2
6	C	$\to E$ 1, 5
7	$B \wedge C$	$\wedge I$ 4, 6

Natural Deduction Proofs for SL

Day 2

PHIL 500

1	P	Ass. $(\rightarrow I)$
2	$P \vee Q$	$\vee I$ 1
3	$P \rightarrow (P \vee Q)$	$\rightarrow I$ 1–2

New Due Date

Problem Set #4 is due on 10/9

$A \wedge B\ ,\ (A \vee C) \rightarrow ((D \rightarrow E) \wedge F \quad \vdash \quad D \rightarrow E$

1		$A \wedge B$
2		$(A \vee C) \rightarrow ((D \rightarrow E) \wedge F)$

$A \wedge B\ ,\ (A \vee C) \rightarrow ((D \rightarrow E) \wedge F \quad \vdash \quad D \rightarrow E$

1		$A \wedge B$	
2		$(A \vee C) \rightarrow ((D \rightarrow E) \wedge F)$	
	⊢		
3		A	$\wedge E$ 1

$A \wedge B\ ,\ (A \vee C) \rightarrow ((D \rightarrow E) \wedge F \quad \vdash \quad D \rightarrow E$

1	$A \wedge B$	
2	$(A \vee C) \rightarrow ((D \rightarrow E) \wedge F)$	
3	A	$\wedge E$ 1
4	$A \vee C$	$\vee I$ 3

$A \land B\ ,\ (A \lor C) \to ((D \to E) \land F \quad \vdash \quad D \to E$

1	$A \land B$	
2	$(A \lor C) \to ((D \to E) \land F)$	
3	A	$\land E$ 1
4	$A \lor C$	$\lor I$ 3
5	$(D \to E) \land F$	$\to E$ 2, 4

$A \land B\ ,\ (A \lor C) \to ((D \to E) \land F \quad \vdash \quad D \to E$

1	$A \land B$	
2	$(A \lor C) \to ((D \to E) \land F)$	
3	A	$\land E$ 1
4	$A \lor C$	$\lor I$ 3
5	$(D \to E) \land F$	$\to E$ 2, 4
6	$D \to E$	$\land E$ 5

$A \land B\ ,\ (A \lor C) \to ((D \to E) \land F \quad \vdash \quad D \to E$

1	$A \land B$	
2	$(A \lor C) \to ((D \to E) \land F)$	
3	A	$\land E$ 1
4	$A \lor C$	$\lor I$ 3
5	$(D \to E) \land F$	$\to E$ 2, 4
6	$D \to E$	$\land E$ 5

Legal Proofs

Rules *may not* be applied to sub-sentences.

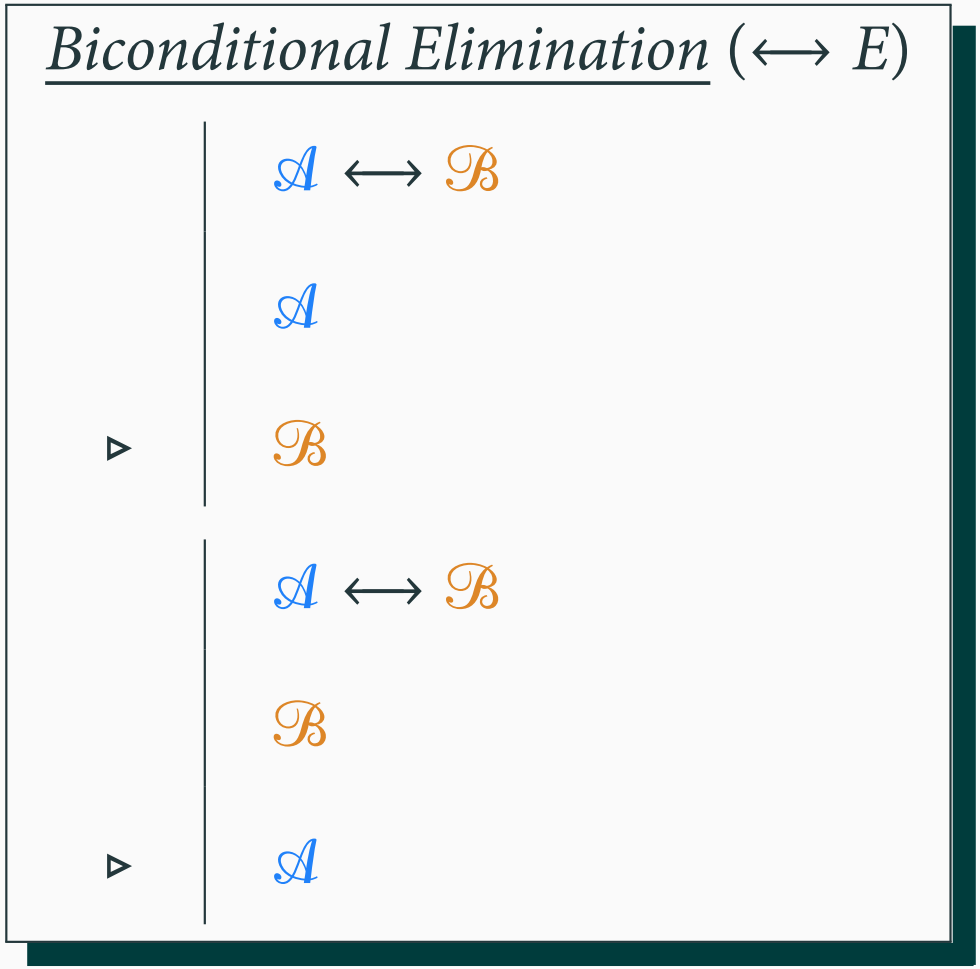
1	$(A \to C) \vee D$	
2	A	
3	C	$\to E$ 1, 2 ⟵ **MISTAKE!!!**

Legal Proofs

Rules *may not* be applied to sub-sentences.

1	$(A \to C) \vee D$		
2	A		
3	$C \vee D$	$\to E$ 1, 2	⟵ **MISTAKE!!!**

Biconditional Elimination



$$D\ ,\ A \to B\ ,\ C \longleftrightarrow A\ ,\ C \longleftrightarrow D \quad \vdash \quad B$$

1

$$D\ ,\ A \to B\ ,\ C \leftrightarrow A\ ,\ C \leftrightarrow D \quad \vdash \quad B$$

1	D
2	$A \to B$
3	$C \leftrightarrow A$
4	$C \leftrightarrow D$

$$D\,,\ A \to B\,,\ C \leftrightarrow A\,,\ C \leftrightarrow D \quad \vdash \quad B$$

1	D	
2	$A \to B$	
3	$C \leftrightarrow A$	
4	$C \leftrightarrow D$	
5	C	$\leftrightarrow E$ 1, 4

$$D\,,\ A \to B\,,\ C \leftrightarrow A\,,\ C \leftrightarrow D \quad \vdash \quad B$$

1	D	
2	$A \to B$	
3	$C \leftrightarrow A$	
4	$C \leftrightarrow D$	
5	C	$\leftrightarrow E$ 1, 4
6	A	$\leftrightarrow E$ 3, 5

$$D\,,\ A \to B\,,\ C \leftrightarrow A\,,\ C \leftrightarrow D \quad \vdash \quad B$$

1	D	
2	$A \to B$	
3	$C \leftrightarrow A$	
4	$C \leftrightarrow D$	
5	C	$\leftrightarrow E$ 1, 4
6	A	$\leftrightarrow E$ 3, 5
7	B	$\to E$ 2, 6

$$D\,,\ A \to B\,,\ C \leftrightarrow A\,,\ C \leftrightarrow D \quad \vdash \quad B$$

1	D	
2	$A \to B$	
3	$C \leftrightarrow A$	
4	$C \leftrightarrow D$	
5	C	$\leftrightarrow E$ 1, 4
6	A	$\leftrightarrow E$ 3, 5
7	B	$\to E$ 2, 6

$P \land (P \to Q) \quad \vdash \quad Q$

1 $\quad$ $P \land (P \to Q)$

$P \wedge (P \to Q) \quad \vdash \quad Q$

1	$P \wedge (P \to Q)$		
2	P		$\wedge E$ 1

$P \wedge (P \rightarrow Q) \quad \vdash \quad Q$

1	$P \wedge (P \rightarrow Q)$	
2	P	$\wedge E$ 1
3	$P \rightarrow Q$	$\wedge E$ 1

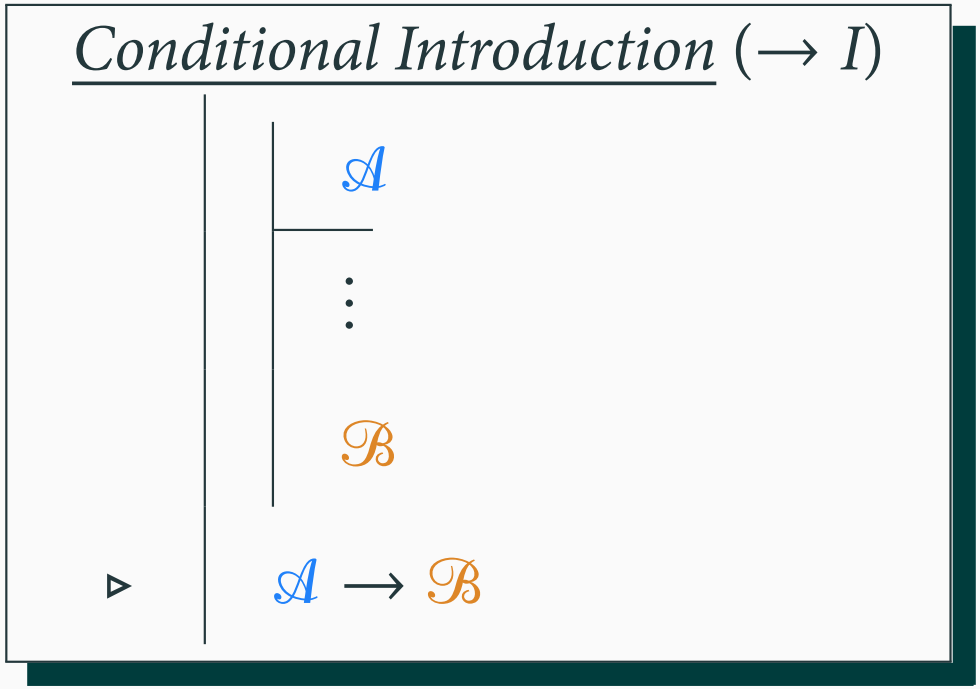
$P \wedge (P \rightarrow Q) \quad \vdash \quad Q$

1	$P \wedge (P \rightarrow Q)$	
2	P	$\wedge E$ 1
3	$P \rightarrow Q$	$\wedge E$ 1
4	Q	$\rightarrow E$ 2, 3

$P \wedge (P \rightarrow Q) \quad \vdash \quad Q$

1	$P \wedge (P \rightarrow Q)$	
2	P	$\wedge E$ 1
3	$P \rightarrow Q$	$\wedge E$ 1
4	Q	$\rightarrow E$ 2, 3

Conditional Introduction



$\vdash \quad (P \land (P \to Q)) \to Q$

1	$P \land (P \to Q)$	
2	P	$\land E$ 1
3	$P \to Q$	$\land E$ 1
4	Q	$\to E$ 2, 3

$\vdash \quad (P \land (P \to Q)) \to Q$

1	$P \land (P \to Q)$	
2	P	$\land E$ 1
3	$P \to Q$	$\land E$ 1
4	Q	$\to E$ 2, 3
5	$(P \land (P \to Q)) \to Q$	$\to I$ 1–4

Notation

- If there is a legal SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its final line*, then I will write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$$

Notation

- If there is a legal SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its main scope line, then I will write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$$

Notation

- If there is a legal SL natural deduction proof which has the sentences $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N$ as assumptions and has $\mathscr{C}$ appearing on its main scope line, then I will write:

$$\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_N \vdash \mathscr{C}$$

- If there is a legal SL natural deduction proof which has $\mathscr{C}$ appearing *outside the scope line of any assumptions*, then I will write:

$$\vdash \mathscr{C}$$

Tautologies

Tautologies

Tautologies

1	P	Ass.
2	$P \vee Q$	$\vee I$ 1
3	$P \rightarrow (P \vee Q)$	$\rightarrow I$ 1–2

Tautologies

1	P	Ass.
2	$P \vee Q$	$\vee I$ 1
3	$P \rightarrow (P \vee Q)$	$\rightarrow I$ 1–2

$$\vdash P \rightarrow (P \vee Q)$$

Tautologies

P	Q	$P \rightarrow (P \vee Q)$
T	T	
T	F	
F	T	
F	F	

Tautologies

P	Q	$P \rightarrow (P \vee Q)$
T	T	T
T	F	T
F	T	F
F	F	F

Tautologies

P	Q	P	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T			T		T
T	F			T		F
F	T			F		T
F	F			F		F

Tautologies

P	Q	P	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T			T	T	T
T	F			T	T	F
F	T			F	T	T
F	F			F	F	F

Tautologies

P	Q	P	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T		T	T	T
T	F	T		T	T	F
F	T	F		F	T	T
F	F	F		F	F	F

Tautologies

P	Q	P	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F

Tautologies

P	Q	P	$\rightarrow$	$(P$	$\vee$	$Q)$
T	T	T	T	T	T	T
T	F	T	T	T	T	F
F	T	F	T	F	T	T
F	F	F	T	F	F	F

So:

$$\models P \rightarrow (P \vee Q)$$

$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1

$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1	P
2	$P \to Q$
3	$Q \to R$

$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	Q	$\to E$ 1, 2

$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	Q	$\to E$ 1, 2
5	R	$\to E$ 3, 4

$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	Q	$\to E$ 1, 2
5	R	$\to E$ 3, 4

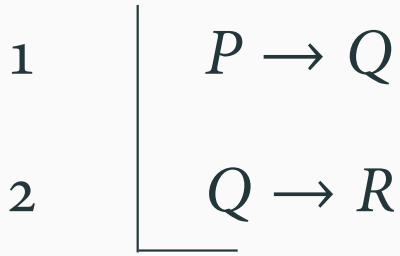
$P\,,\ P \to Q\,,\ Q \to R \quad \vdash \quad R$

1	P	
2	$P \to Q$	
3	$Q \to R$	
4	Q	$\to E\ 1, 2$
5	R	$\to E\ 3, 4$

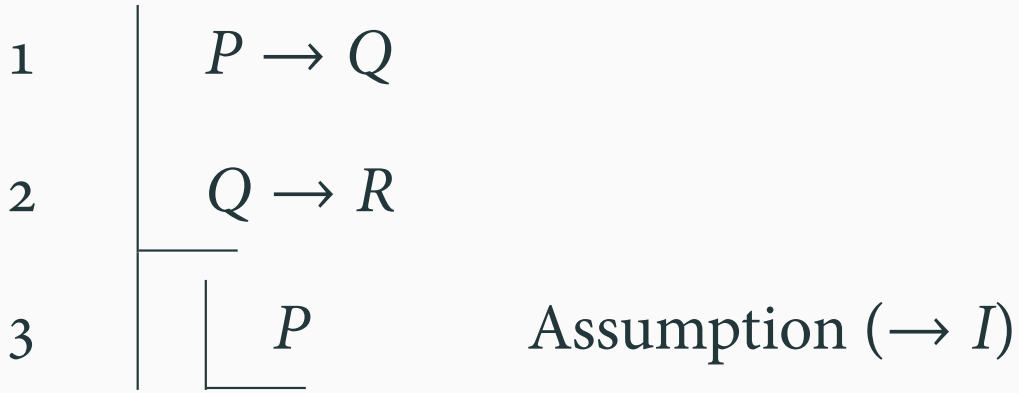
How to show:

$$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R \quad ?$$

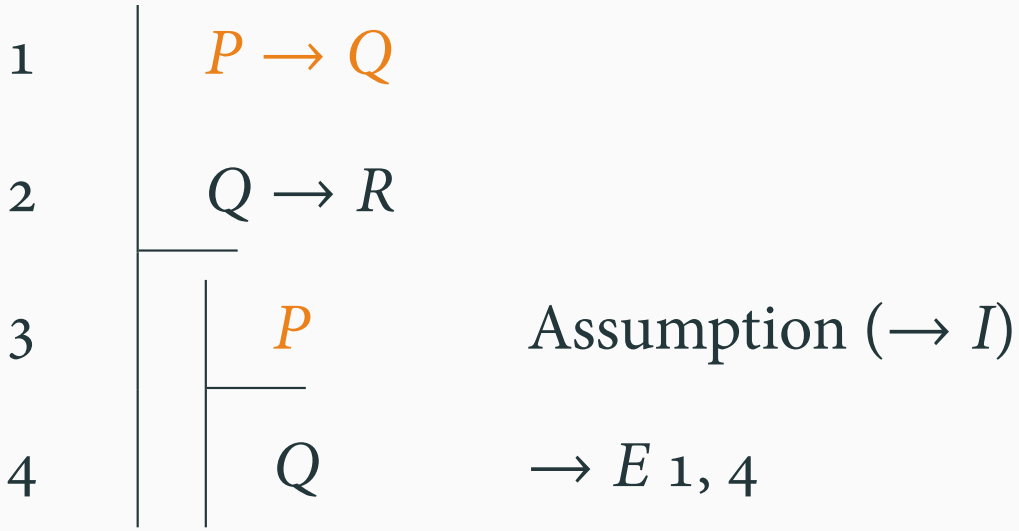
$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$



$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$



$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$



$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$

1	$P \to Q$	
2	$Q \to R$	
3	$\quad P$	Assumption ($\to I$)
4	$\quad Q$	$\to E$ 1, 4
5	$\quad R$	$\to E$ 2, 5

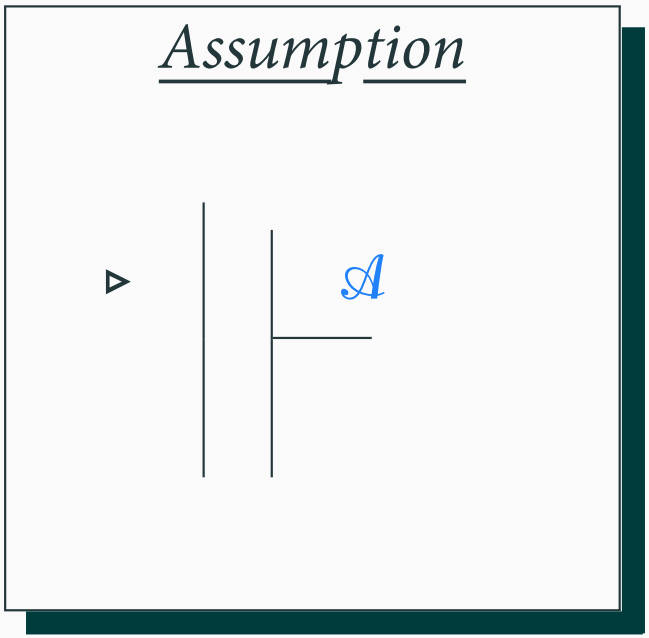
$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$

1	$P \to Q$	
2	$Q \to R$	
3	$\quad P$	Assumption ($\to I$)
4	$\quad Q$	$\to E$ 1, 4
5	$\quad R$	$\to E$ 2, 5
6	$P \to R$	$\to I$ 3–6

$P \to Q\,,\ Q \to R \quad \vdash \quad P \to R$

1	$P \to Q$	
2	$Q \to R$	
3	$\quad P$	Assumption ($\to I$)
4	$\quad Q$	$\to E$ 1, 4
5	$\quad R$	$\to E$ 2, 5
6	$P \to R$	$\to I$ 3–6

Assumption



Accessibility

- You may decide to *end* a subderivation whenever you wish.

Accessibility

- You may decide to *end* a subderivation whenever you wish.
- This threatens to make it too easy to derive anything at all:

1	$A \to B$		
2	$\quad A$	Assumption	
3	B	$\to E$ 1, 2	⟵ **MISTAKE!!!**

Accessibility

A sentence is ACCESSIBLE at your line if and only if 1) it appears on a line *before* yours, and 2) it lies within a subproof whose vertical scope line extends to your line of the proof (or it lies outside the scope of any subproof).

Accessibility

A sentence is ACCESSIBLE at your line iff it comes before your line and does not lie inside of a completed subproof.

Accessibility

An *entire subproof* is accessible at your line if and only if 1) it appears on lines *before* yours, and 2) it lies within a subproof which extends to your line of the proof (or it lies outside the scope of any subproof).

Accessibility

An *entire subproof* is accessible at your line iff it comes before your line and does not lie inside of another completed subproof.

Accessibility

An *entire subproof* is accessible at your line iff it comes before your line and does not lie inside of another completed subproof.

- while you may end a subproof whenever you wish, once you do so, none of the lines or subproofs appearing within the scope of that subproof are accessible any longer.

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \rightarrow C$	$\rightarrow I$ 3–4
6	$A \rightarrow (B \rightarrow C)$	$\rightarrow I$ 2–5
7	$\quad B \rightarrow A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \rightarrow A) \rightarrow C$	$\rightarrow I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

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3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
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7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
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Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

Accessibility

1	$C \wedge Z$	
2	$\quad A$	Ass.
3	$\quad\quad B$	Ass.
4	$\quad\quad C$	$\wedge E$ 1
5	$\quad B \to C$	$\to I$ 3–4
6	$A \to (B \to C)$	$\to I$ 2–5
7	$\quad B \to A$	Ass.
8	$\quad C$	$\wedge E$ 1
9	$(B \to A) \to C$	$\to I$ 7–8

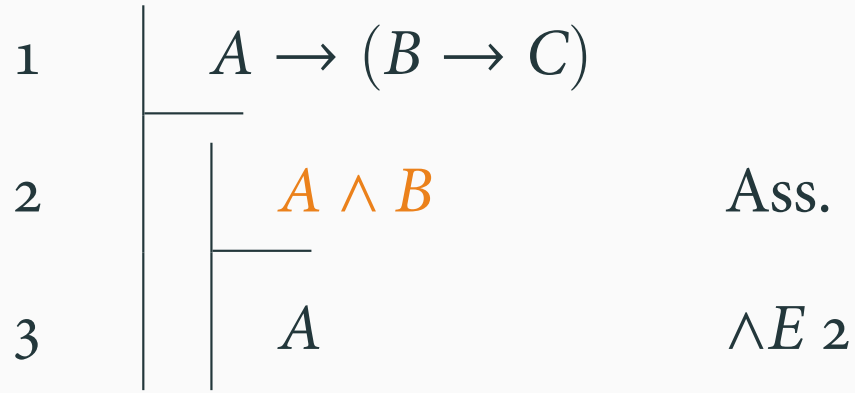
$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1 $\quad$ $A \to (B \to C)$

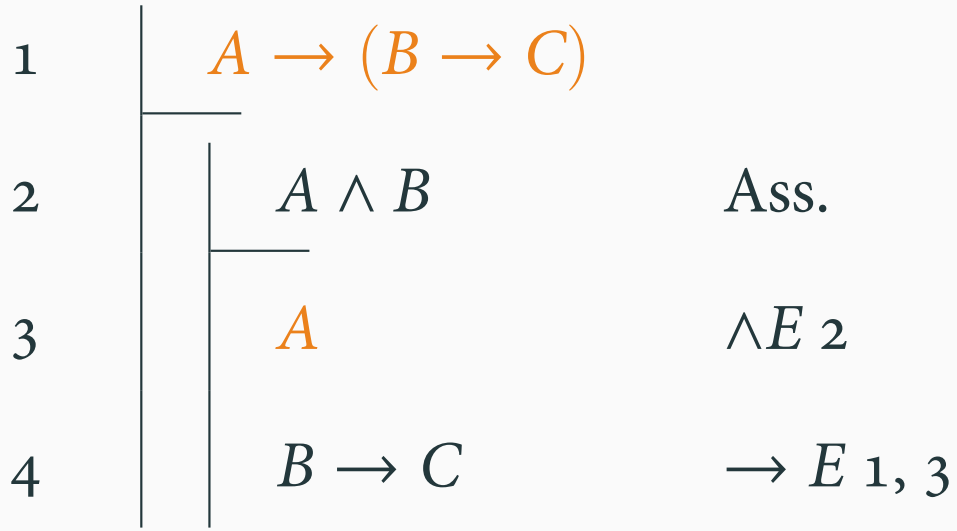
$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$A \wedge B$	Ass.

$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$



$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$



$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$A \wedge B$	Ass.
3	A	$\wedge E$ 2
4	$B \to C$	$\to E$ 1, 3
5	B	$\wedge E$ 2

$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$A \wedge B$	Ass.
3	A	$\wedge E$ 2
4	$B \to C$	$\to E$ 1, 3
5	B	$\wedge E$ 2
6	C	$\to E$ 4, 5

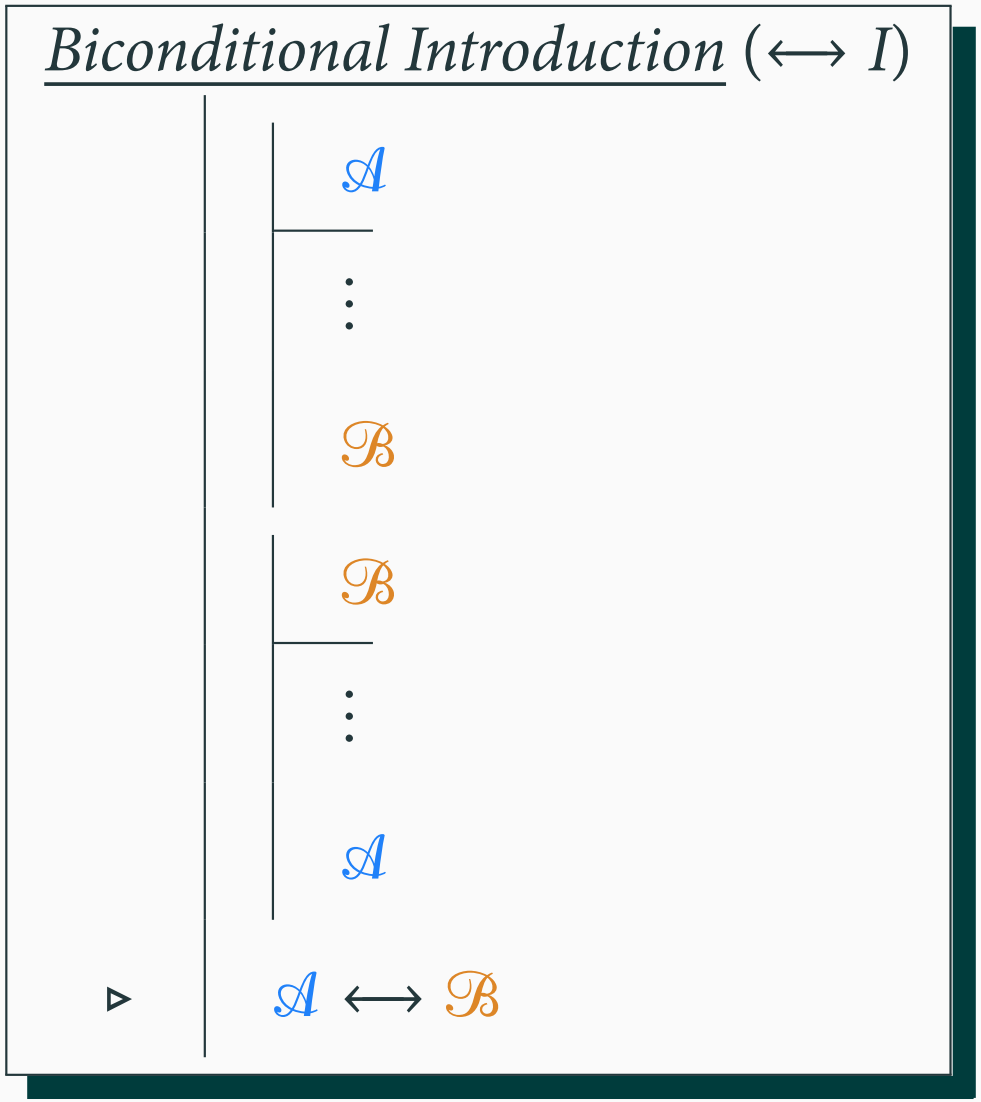
$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$A \wedge B$	Ass.
3	A	$\wedge E$ 2
4	$B \to C$	$\to E$ 1, 3
5	B	$\wedge E$ 2
6	C	$\to E$ 4, 5
7	$(A \wedge B) \to C$	$\to I$ 2–6

$A \to (B \to C) \quad \vdash \quad (A \wedge B) \to C$

1	$A \to (B \to C)$	
2	$\quad A \wedge B$	Ass.
3	$\quad A$	$\wedge E$ 2
4	$\quad B \to C$	$\to E$ 1, 3
5	$\quad B$	$\wedge E$ 2
6	$\quad C$	$\to E$ 4, 5
7	$(A \wedge B) \to C$	$\to I$ 2–6

Biconditional Introduction

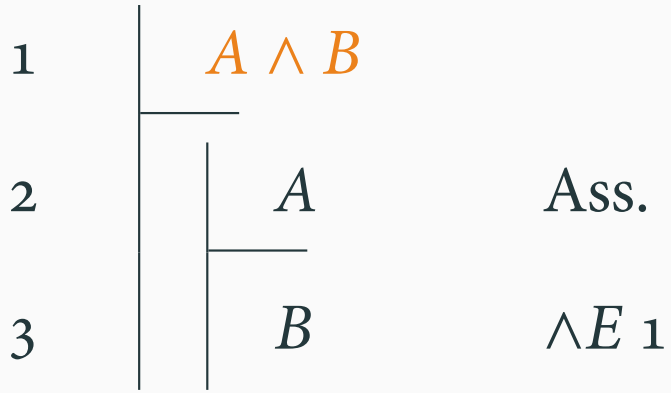


$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

1	$A \wedge B$	
2	A	Ass.

$A \wedge B \quad \vdash \quad A \longleftrightarrow B$



$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

1	$A \wedge B$	
2	A	Ass.
3	B	$\wedge E$ 1
4	B	Ass.

$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

1	$A \wedge B$	
2	A	Ass.
3	B	$\wedge E$ 1
4	B	Ass.
5	A	$\wedge E$ 1

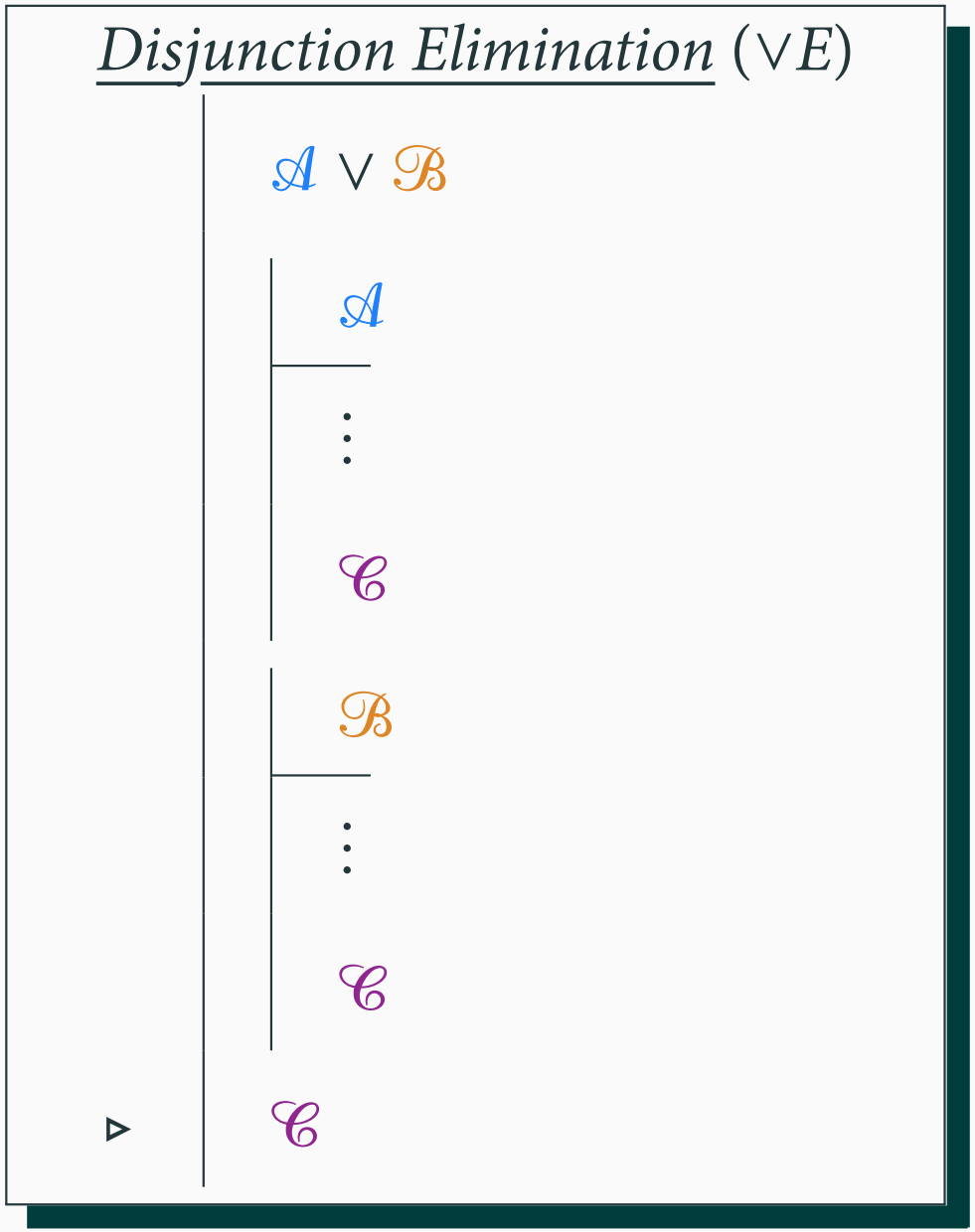
$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

1	$A \wedge B$	
2	A	Ass.
3	B	$\wedge E$ 1
4	B	Ass.
5	A	$\wedge E$ 1
6	$A \longleftrightarrow B$	$\longleftrightarrow I$ 2–3, 4–5

$A \wedge B \quad \vdash \quad A \longleftrightarrow B$

1	$A \wedge B$	
2	A	Ass.
3	B	$\wedge E$ 1
4	B	Ass.
5	A	$\wedge E$ 1
6	$A \longleftrightarrow B$	$\longleftrightarrow I$ 2–3, 4–5

Disjunction Elimination



$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$
2	$A \to C$
3	$B \to D$

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	$\quad A$	Ass. $(\vee E)$

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	$\quad A$	Ass. $(\vee E)$
5	$\quad C$	$\to E$ 2, 4

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	A	Ass. ($\vee E$)
5	C	$\to E$ 2, 4
6	$C \vee D$	$\vee I$ 5

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	A	Ass. ($\vee E$)
5	C	$\to E$ 2, 4
6	$C \vee D$	$\vee I$ 5
7	B	Ass. ($\vee E$)

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	A	Ass. ($\vee E$)
5	C	$\to E$ 2, 4
6	$C \vee D$	$\vee I$ 5
7	B	Ass. ($\vee E$)
8	D	$\to E$ 3, 7

$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	A	Ass. ($\vee E$)
5	C	$\to E$ 2, 4
6	$C \vee D$	$\vee I$ 5
7	B	Ass. ($\vee E$)
8	D	$\to E$ 3, 7
9	$C \vee D$	$\vee I$ 8

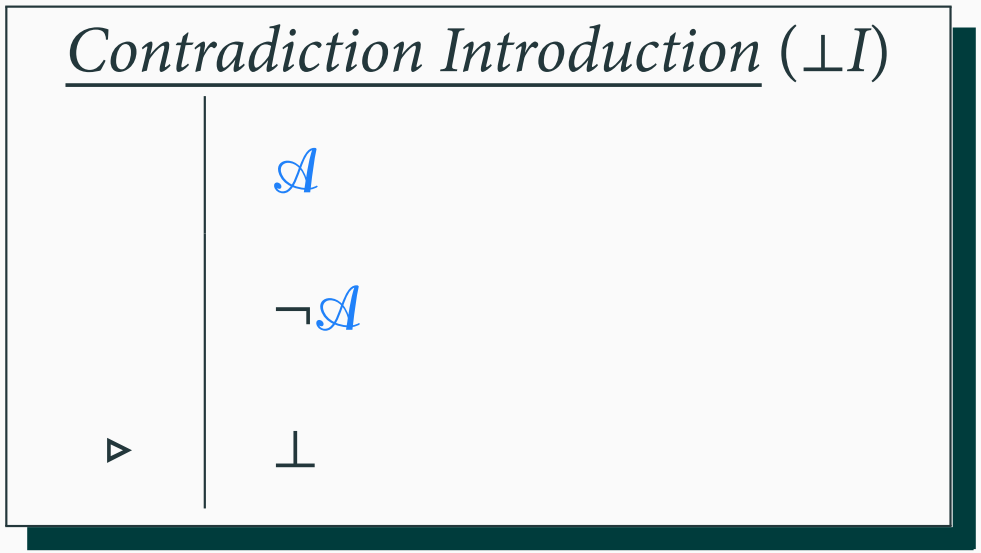
$A \vee B\,,\ A \to C\,,\ B \to D \ \vdash\ C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	$\quad A$	Ass. ($\vee E$)
5	$\quad C$	$\to E$ 2, 4
6	$\quad C \vee D$	$\vee I$ 5
7	$\quad B$	Ass. ($\vee E$)
8	$\quad D$	$\to E$ 3, 7
9	$\quad C \vee D$	$\vee I$ 8
10	$C \vee D$	$\vee E$ 1, 4–6, 7–9

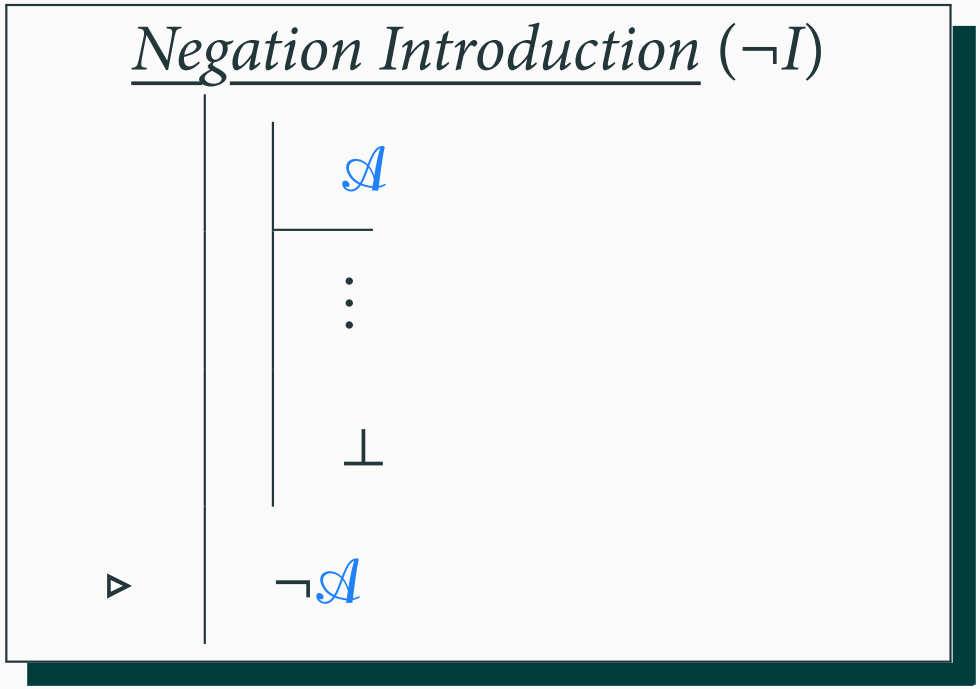
$A \vee B\,,\ A \to C\,,\ B \to D \quad \vdash \quad C \vee D$

1	$A \vee B$	
2	$A \to C$	
3	$B \to D$	
4	A	Ass. ($\vee E$)
5	C	$\to E$ 2, 4
6	$C \vee D$	$\vee I$ 5
7	B	Ass. ($\vee E$)
8	D	$\to E$ 3, 7
9	$C \vee D$	$\vee I$ 8
10	$C \vee D$	$\vee E$ 1, 4–6, 7–9

Contradiction Introduction

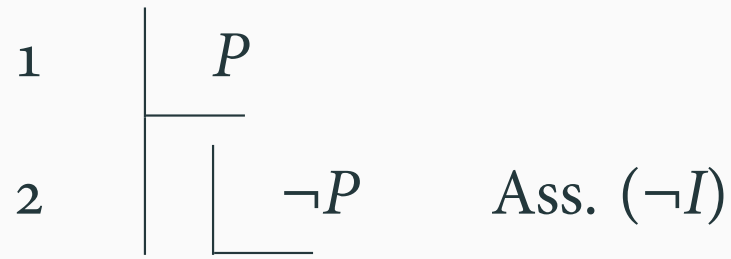


Negation Introduction

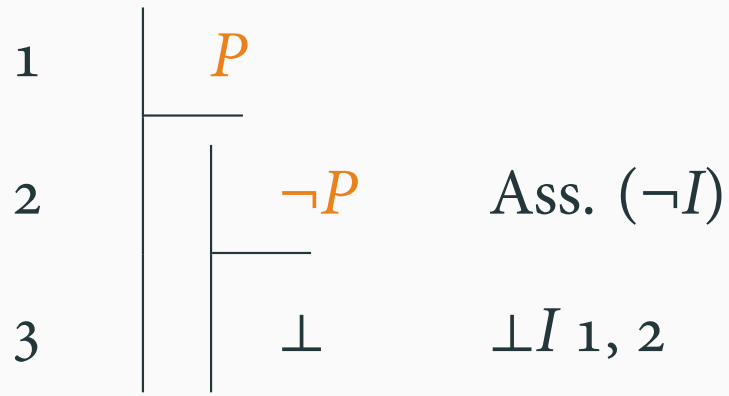


$P \vdash \neg\neg P$

$P \vdash \neg\neg P$



$P \vdash \neg\neg P$



$P \vdash \neg\neg P$

1	P		
2		$\neg P$	Ass. $(\neg I)$
3		$\bot$	$\bot I$ 1, 2
4		$\neg\neg P$	$\neg I$ 2–3

$P \vdash \neg\neg P$

1	P	
2	$\neg P$	Ass. ($\neg I$)
3	$\bot$	$\bot I$ 1, 2
4	$\neg\neg P$	$\neg I$ 2–3

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

1 $X \wedge Y$

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\neg Y \wedge X$	Ass. ($\neg I$)

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\neg Y \wedge X$	Ass. ($\neg I$)
4	$\neg Y$	$\wedge E$ 3

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\neg Y \wedge X$	Ass. ($\neg I$)
4	$\neg Y$	$\wedge E$ 3
5	$\bot$	$\bot I$ 2, 4

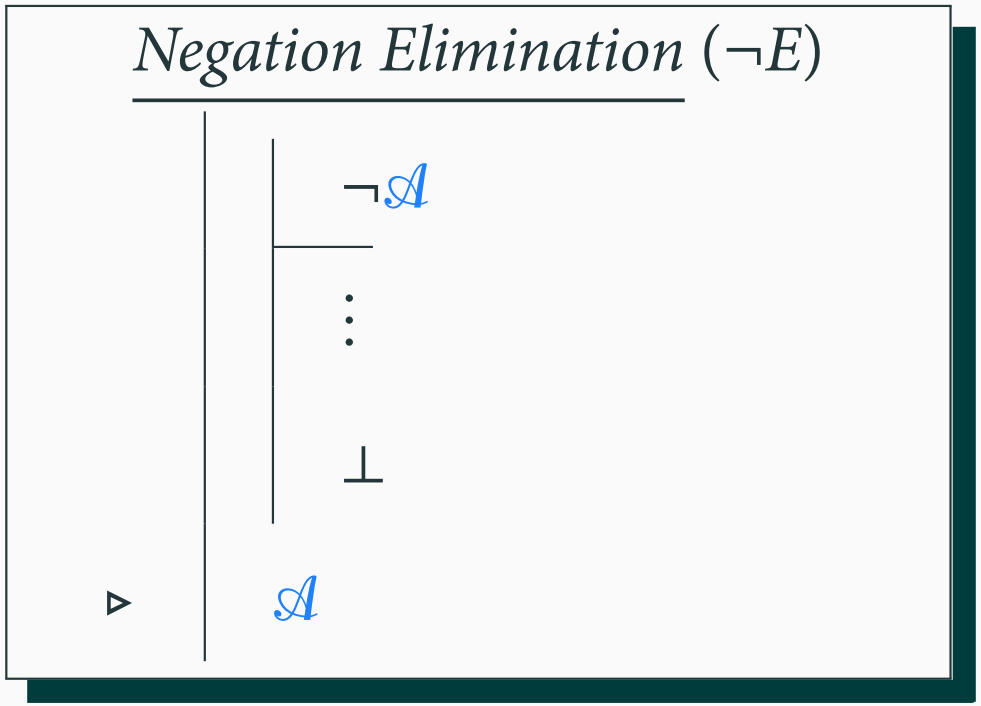
$X \wedge Y \vdash \neg(\neg Y \wedge X)$

1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\neg Y \wedge X$	Ass. ($\neg I$)
4	$\neg Y$	$\wedge E$ 3
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\neg Y \wedge X)$	$\neg I$ 3–5

$X \wedge Y \vdash \neg(\neg Y \wedge X)$

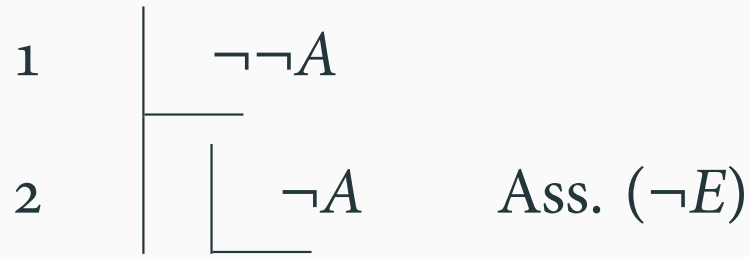
1	$X \wedge Y$	
2	Y	$\wedge E$ 1
3	$\neg Y \wedge X$	Ass. ($\neg I$)
4	$\neg Y$	$\wedge E$ 3
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\neg Y \wedge X)$	$\neg I$ 3–5

Negation Elimination

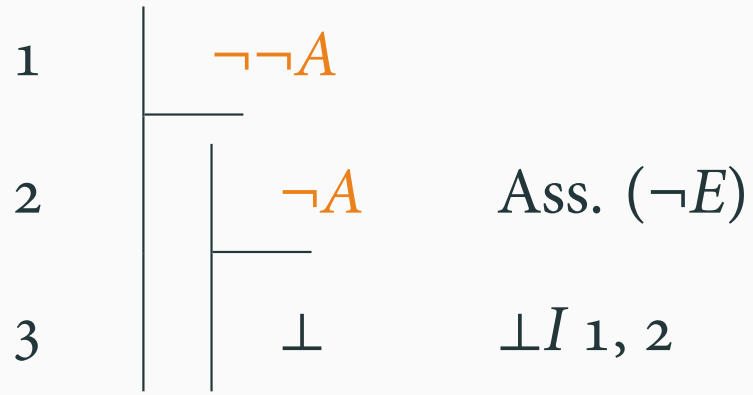


$\neg\neg A \quad \vdash \quad A$

$\neg\neg A \vdash A$



$\neg\neg A \vdash A$



$\neg\neg A \quad \vdash \quad A$

1	$\neg\neg A$	
2	$\neg A$	Ass. $(\neg E)$
3	$\bot$	$\bot I$ 1, 2
4	A	$\neg E$ 2–3

$\neg\neg A \vdash A$

1		$\neg\neg A$	
2	|	$\neg A$	Ass. ($\neg E$)
3	|	$\bot$	$\bot I$ 1, 2
4		A	$\neg E$ 2–3

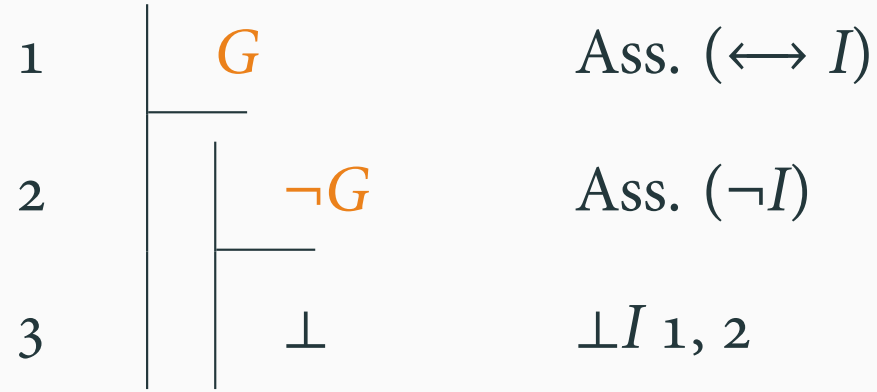
$\vdash \quad G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)

$\vdash \quad G \longleftrightarrow \neg\neg G$

1	G	Ass. $(\longleftrightarrow I)$
2	$\neg G$	Ass. $(\neg I)$

$\vdash \quad G \longleftrightarrow \neg\neg G$



$\vdash \quad G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3

$\vdash \quad G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)

$\vdash \quad G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)
6	$\quad \neg G$	Ass. ($\neg E$)

$\vdash G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)
6	$\quad \neg G$	Ass. ($\neg E$)
7	$\quad \bot$	$\bot I$ 5, 6

$\vdash G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)
6	$\quad \neg G$	Ass. ($\neg E$)
7	$\quad \bot$	$\bot I$ 5, 6
8	G	$\neg E$ 6–7

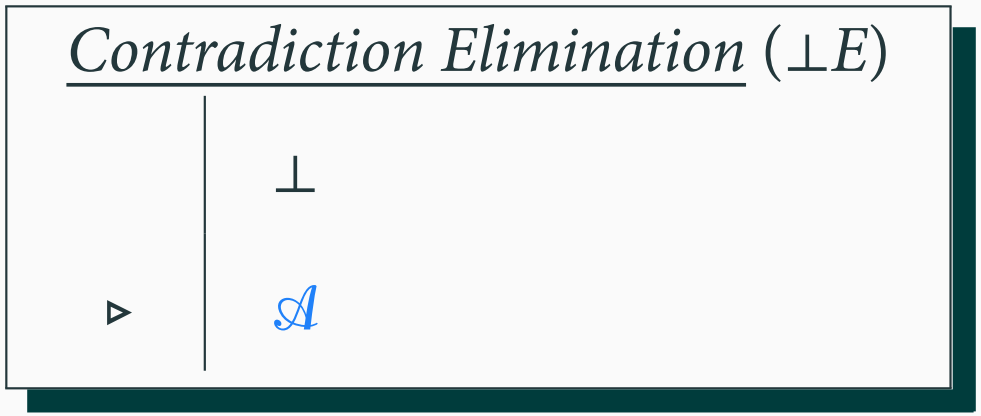
$\vdash G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)
6	$\quad \neg G$	Ass. ($\neg E$)
7	$\quad \bot$	$\bot I$ 5, 6
8	G	$\neg E$ 6–7
9	$G \longleftrightarrow \neg\neg G$	$\longleftrightarrow I$ 1–4, 5–8

$\vdash \; G \longleftrightarrow \neg\neg G$

1	G	Ass. ($\longleftrightarrow I$)
2	$\quad \neg G$	Ass. ($\neg I$)
3	$\quad \bot$	$\bot I$ 1, 2
4	$\neg\neg G$	$\neg I$ 2, 3
5	$\neg\neg G$	Ass. ($\longleftrightarrow I$)
6	$\quad \neg G$	Ass. ($\neg E$)
7	$\quad \bot$	$\bot I$ 5, 6
8	G	$\neg E$ 6–7
9	$G \longleftrightarrow \neg\neg G$	$\longleftrightarrow I$ 1–4, 5–8

Contradiction Elimination



$P \rightarrow \neg P \ \therefore \ P \rightarrow Q$

1 | $P \rightarrow \neg P$

$P \rightarrow \neg P \ \therefore \ P \rightarrow Q$

1	$P \rightarrow \neg P$	
2	P	Ass. ($\rightarrow I$)

$P \to \neg P \ \therefore \ P \to Q$

1	$P \to \neg P$	
2	P	Ass. ($\to I$)
3	$\neg P$	$\to E$ 1, 2

$P \to \neg P \ \therefore \ P \to Q$

1	$P \to \neg P$	
2	P	Ass. ($\to I$)
3	$\neg P$	$\to E$ 1, 2
4	$\bot$	$\bot I$ 2, 3

$P \to \neg P \ \therefore \ P \to Q$

1	$P \to \neg P$	
2	P	Ass. ($\to I$)
3	$\neg P$	$\to E$ 1, 2
4	$\bot$	$\bot I$ 2, 3
5	Q	$\bot E$ 4

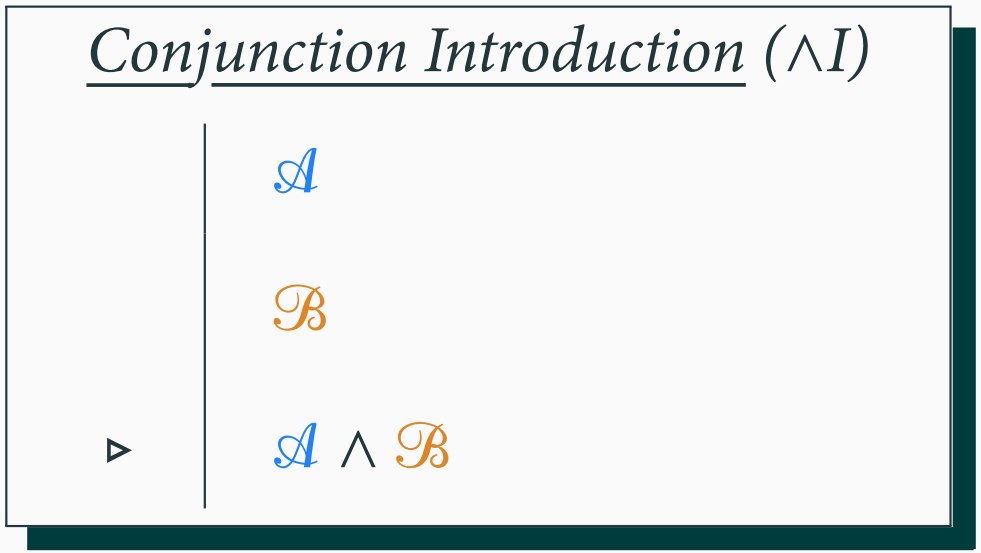
$P \to \neg P \ \therefore \ P \to Q$

1	$P \to \neg P$	
2	P	Ass. ($\to I$)
3	$\neg P$	$\to E$ 1, 2
4	$\bot$	$\bot I$ 2, 3
5	Q	$\bot E$ 4
6	$P \to Q$	$\to I$ 2–5

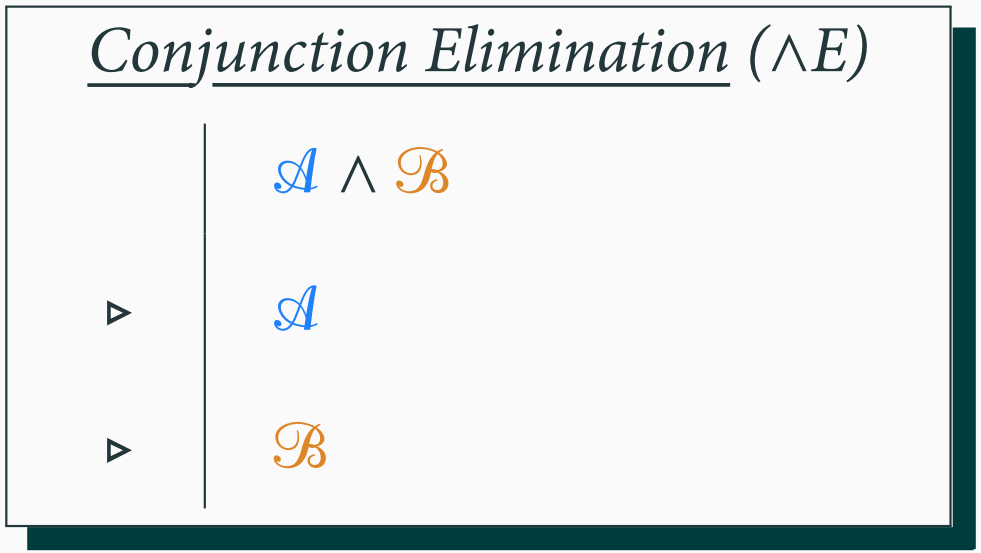
$P \to \neg P \ \therefore \ P \to Q$

1	$P \to \neg P$	
2	$\quad P$	Ass. ($\to I$)
3	$\quad \neg P$	$\to E$ 1, 2
4	$\quad \bot$	$\bot I$ 2, 3
5	$\quad Q$	$\bot E$ 4
6	$P \to Q$	$\to I$ 2–5

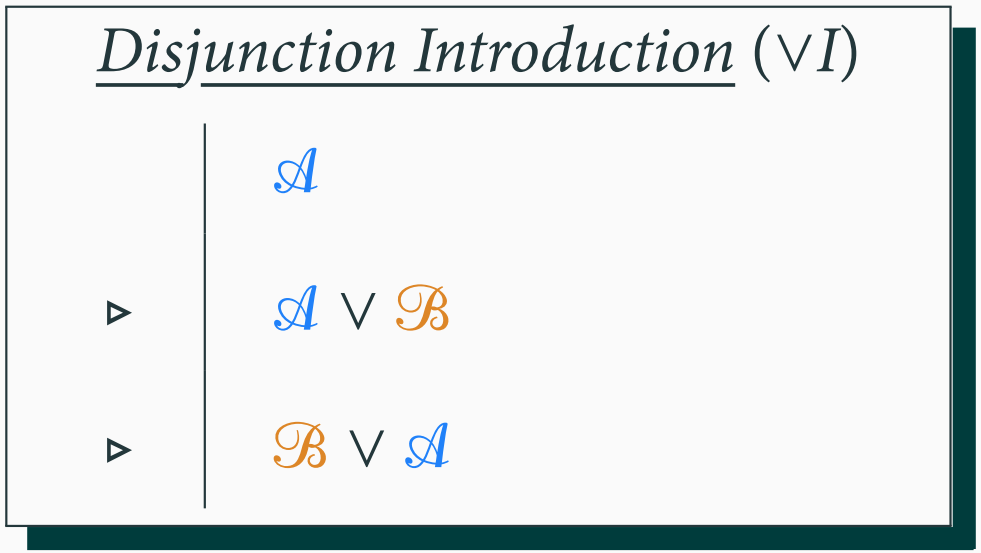
Conjunction Introduction



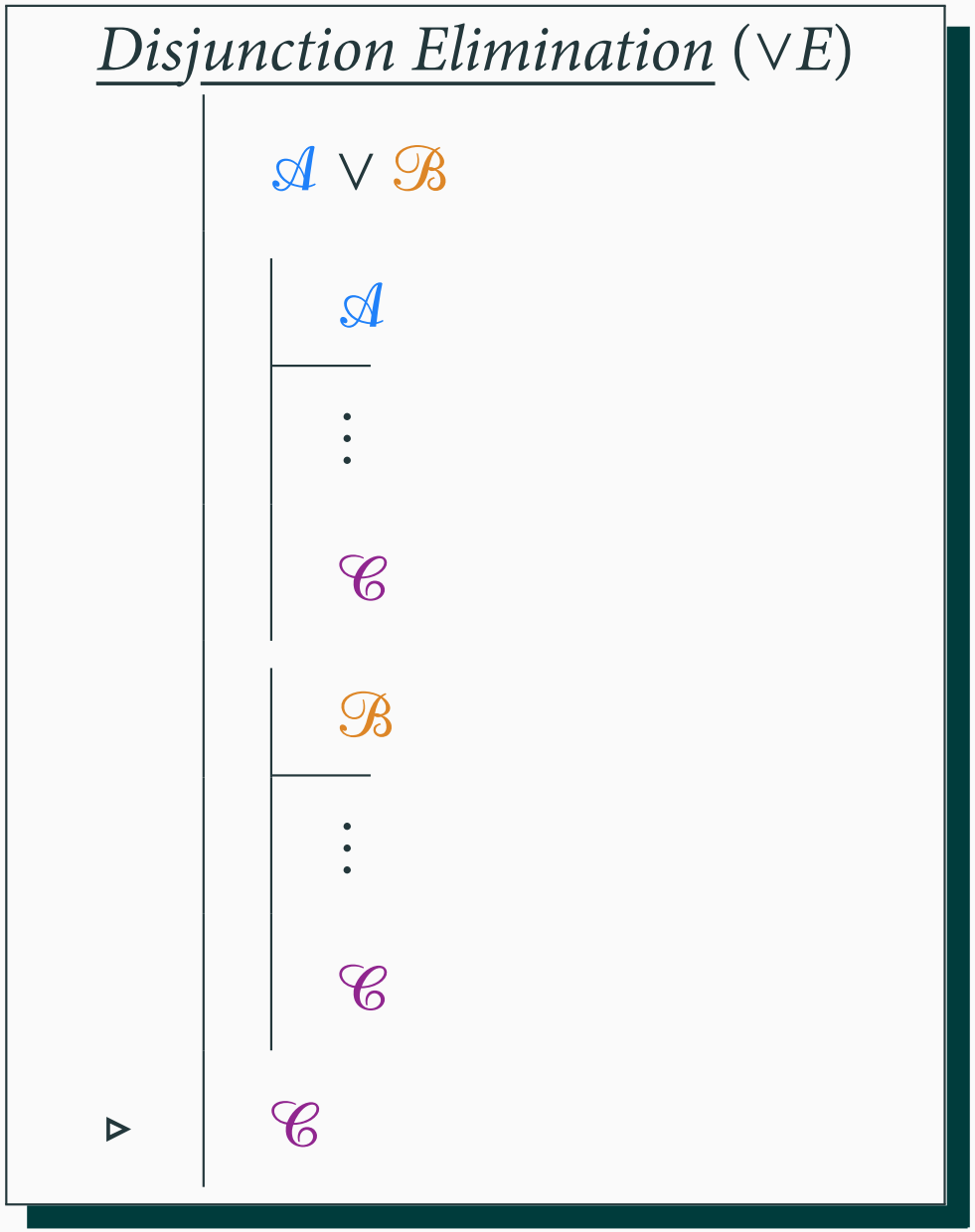
Conjunction Elimination



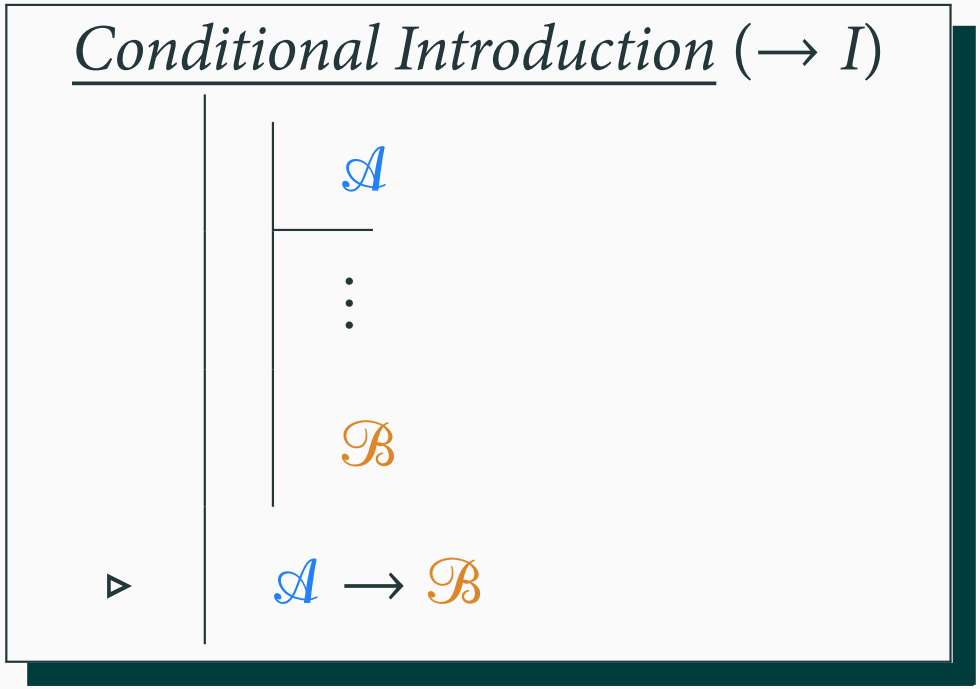
Disjunction Introduction



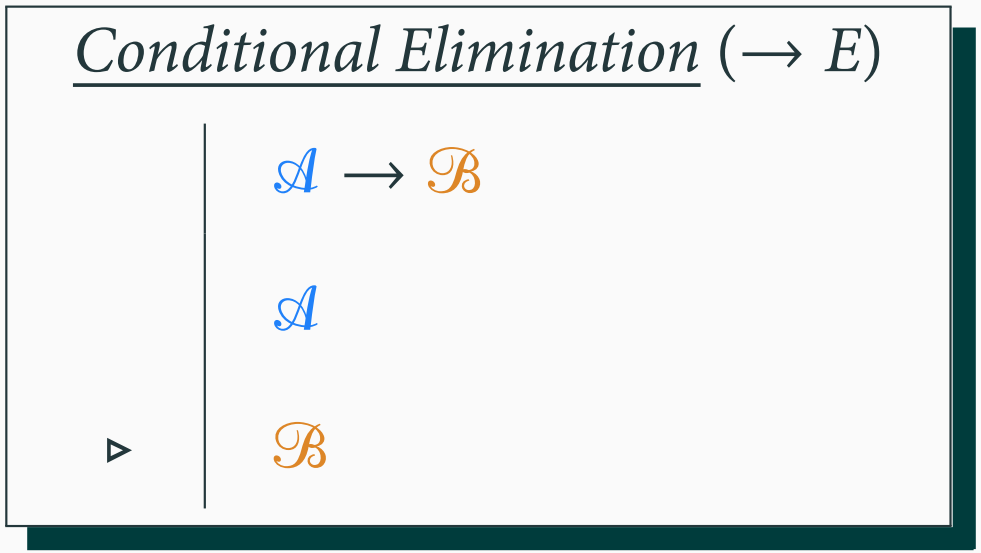
Disjunction Elimination



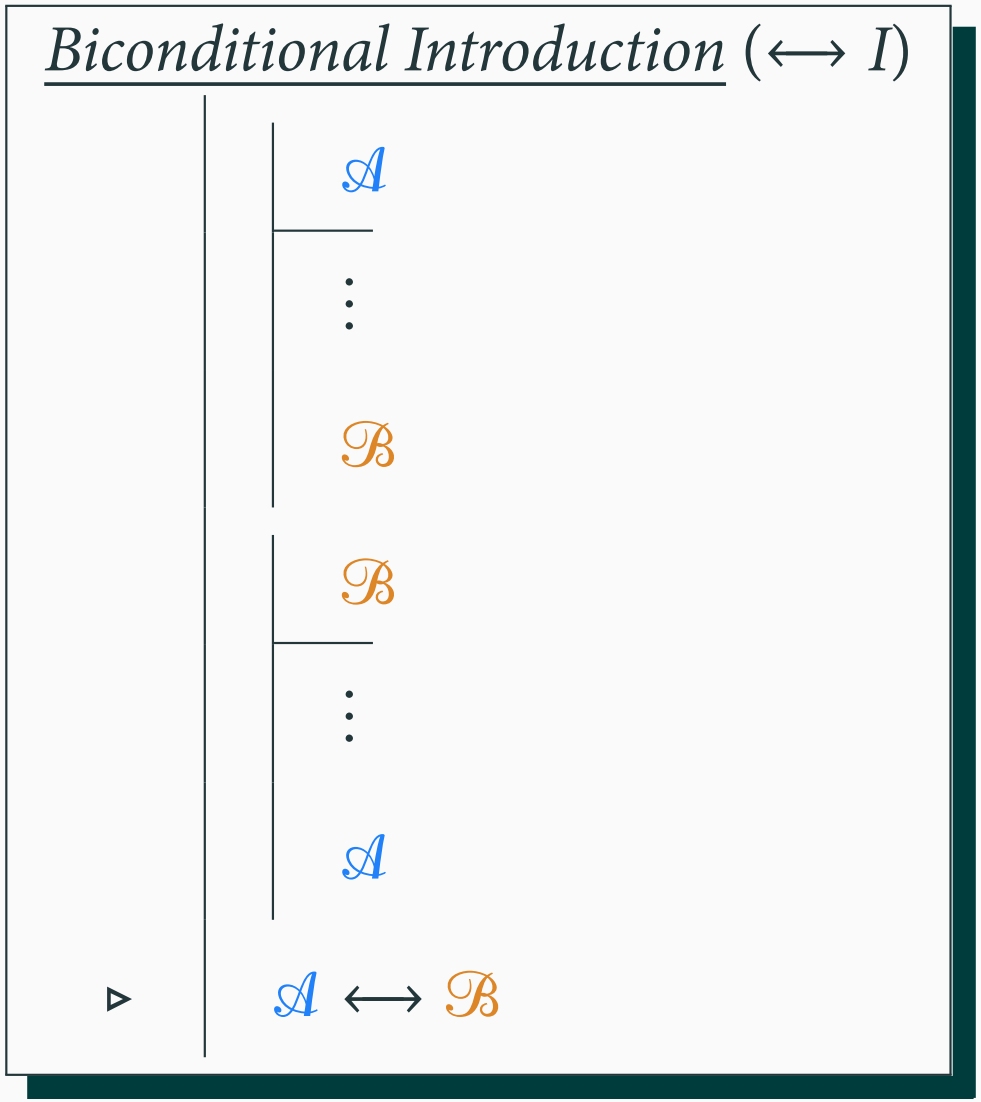
Conditional Introduction



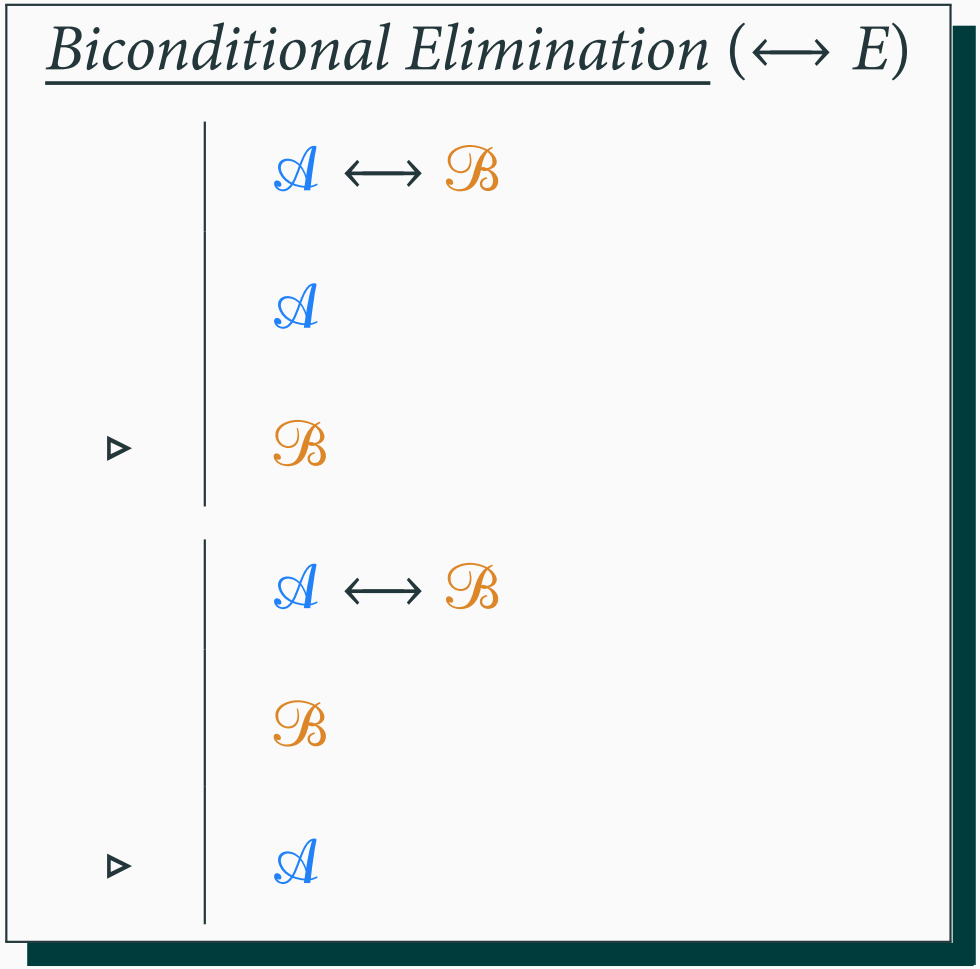
Conditional Elimination



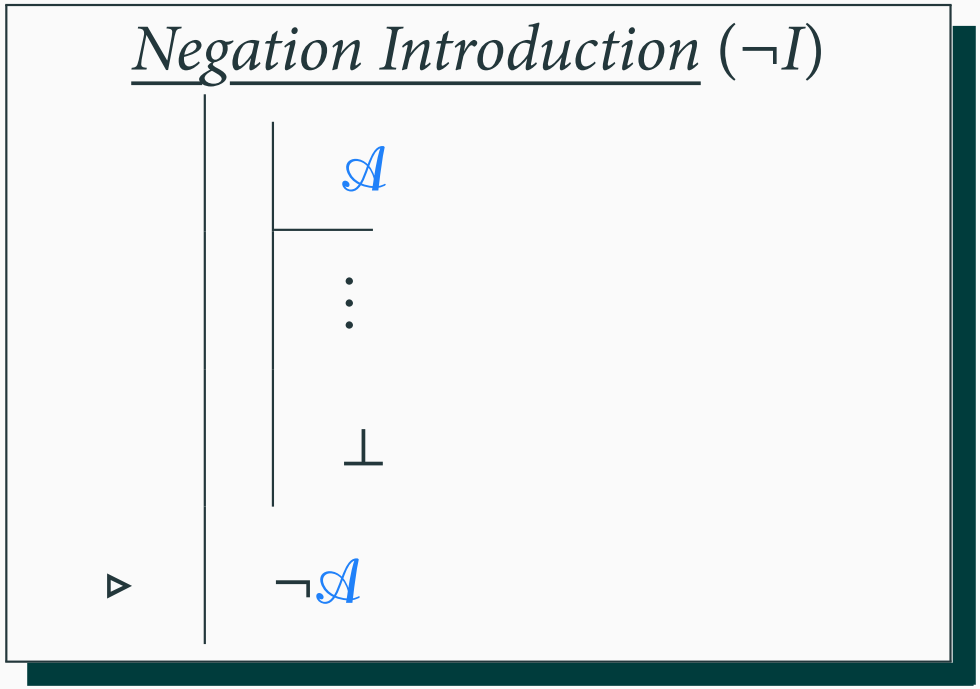
Biconditional Introduction



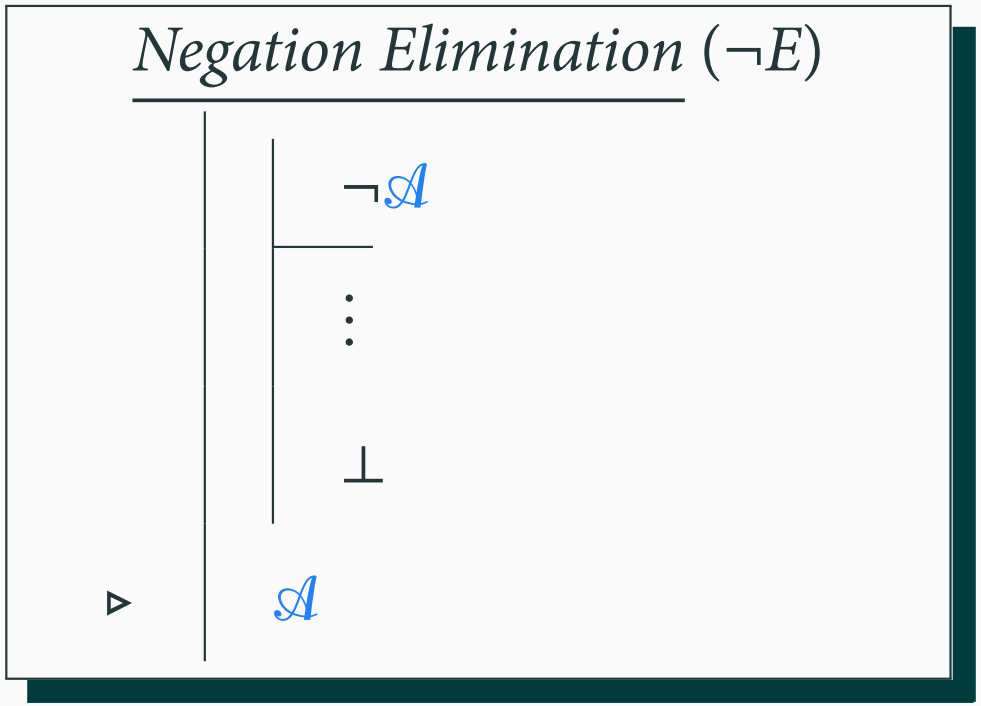
Biconditional Elimination



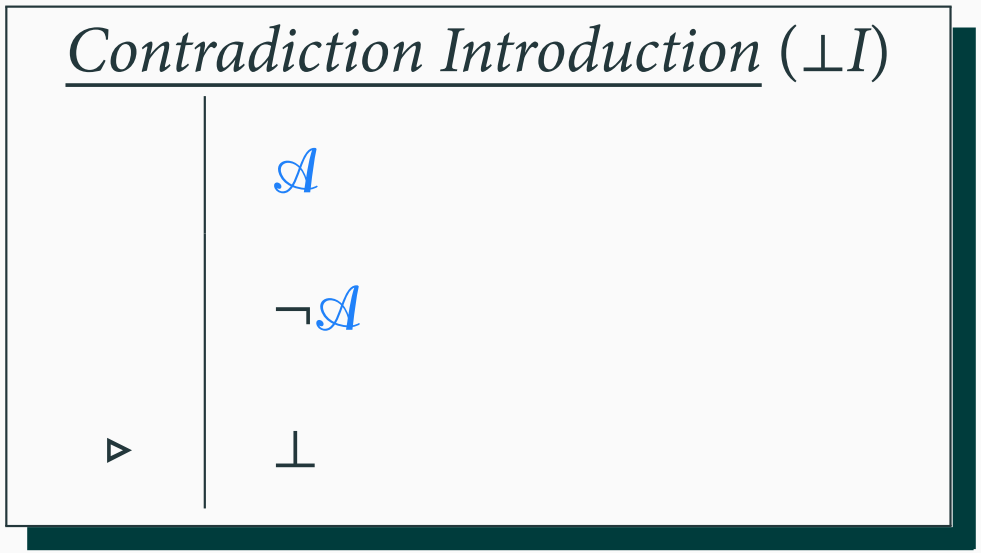
Negation Introduction



Negation Elimination



Contradiction Introduction



Contradiction Elimination