Natural Deduction Proofs for SL

Derived Rules

1	A	Ass. (LEM)
2	$A \wedge A$	$\wedge I$ 1
3	$\neg A$	Ass. (LEM)
4	$\neg A \vee A$	$\vee I$ 3
5	$(A \wedge A) \vee (\neg A \vee A)$	LEM 1–2, 3–4

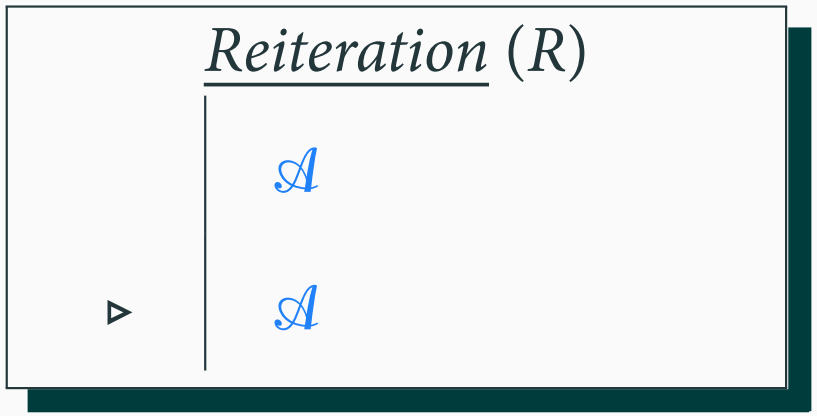
Outline

Derived Rules

Derived Rules

- The plan: use the basic rules to establish additional rules

Reiteration



Reiteration

$$\mathrm{m} \mid \mathcal{A}$$

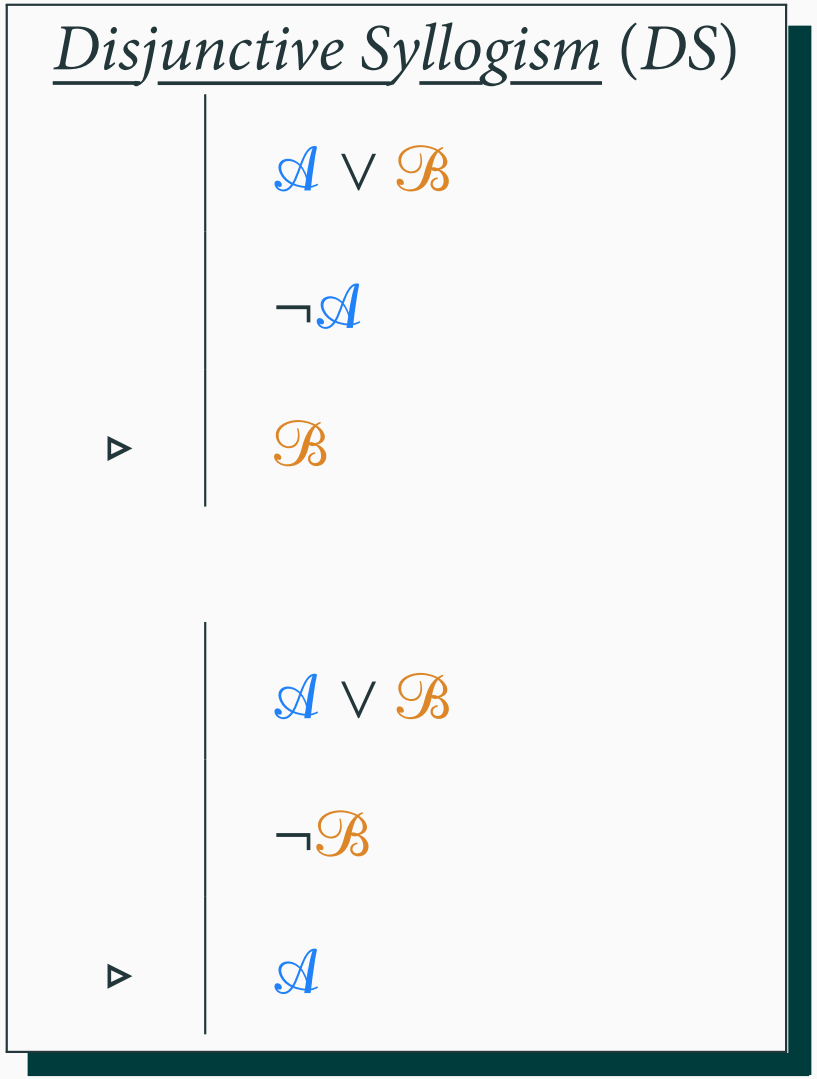
Reiteration

m	$\mathscr{A}$	
k	$\mathscr{A} \wedge \mathscr{A}$	$\wedge I$ m

Reiteration

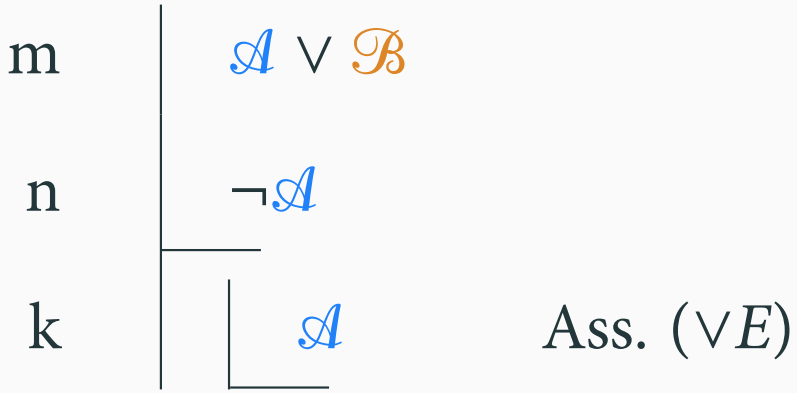
m	$\mathscr{A}$	
k	$\mathscr{A} \wedge \mathscr{A}$	$\wedge I$ m
k+1	$\mathscr{A}$	$\wedge E$ k

Disjunctive Syllogism

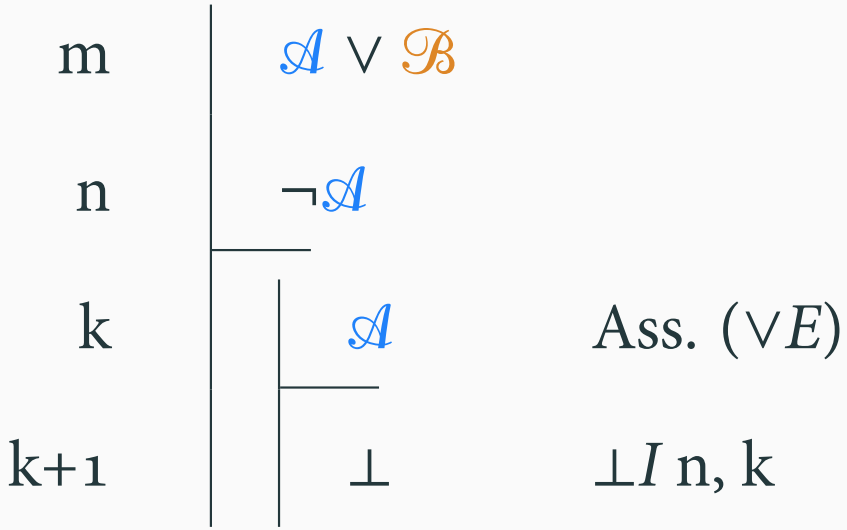


Disjunctive Syllogism

Disjunctive Syllogism



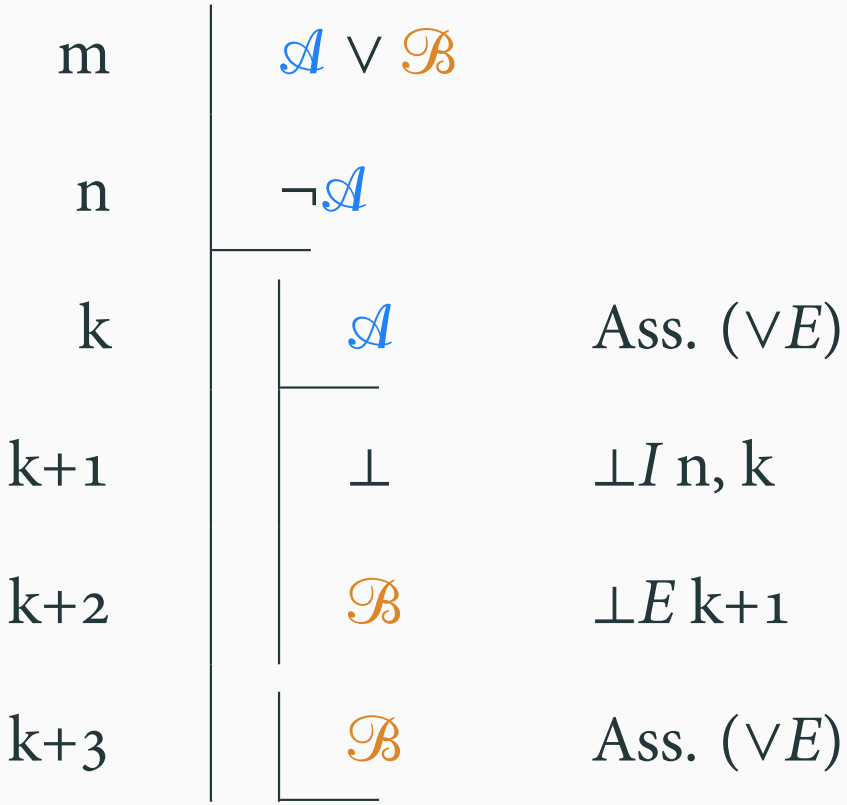
Disjunctive Syllogism



Disjunctive Syllogism

m	$\mathscr{A} \vee \mathscr{B}$	
n	$\neg\mathscr{A}$	
k	$\mathscr{A}$	Ass. $(\vee E)$
k+1	$\bot$	$\bot I$ n, k
k+2	$\mathscr{B}$	$\bot E$ k+1

Disjunctive Syllogism



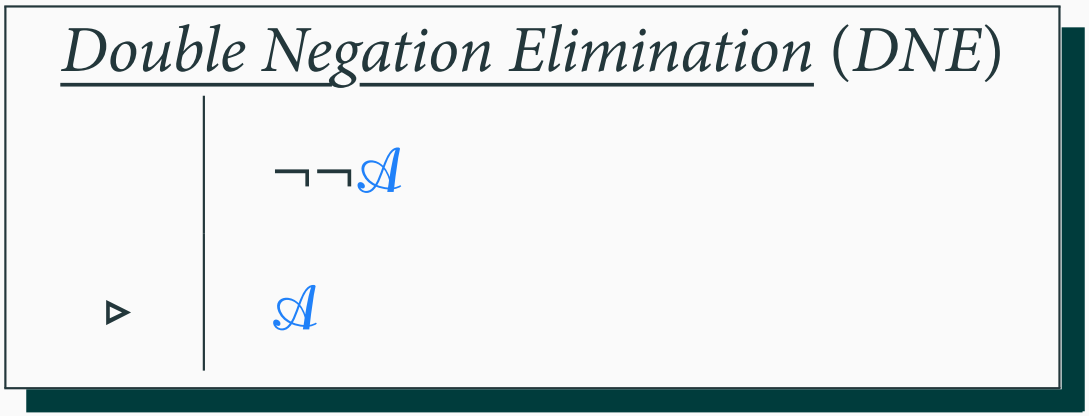
Disjunctive Syllogism

m		$\mathscr{A} \vee \mathscr{B}$	
n		$\neg\mathscr{A}$	
k		$\mathscr{A}$	Ass. $(\vee E)$
k+1		$\bot$	$\bot I$ n, k
k+2		$\mathscr{B}$	$\bot E$ k+1
k+3		$\mathscr{B}$	Ass. $(\vee E)$
k+4		$\mathscr{B}$	R k+3

Disjunctive Syllogism

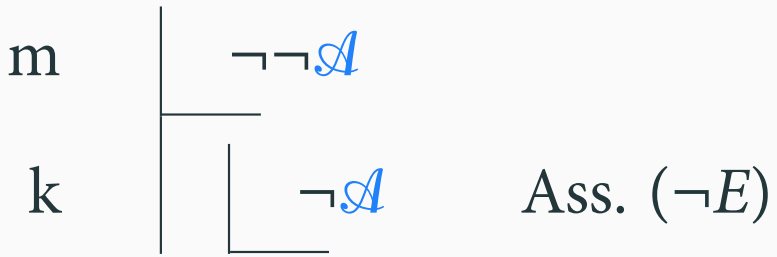
m	$\mathscr{A} \vee \mathscr{B}$	
n	$\neg\mathscr{A}$	
k	$\mathscr{A}$	Ass. ($\vee E$)
k+1	$\bot$	$\bot I$ n, k
k+2	$\mathscr{B}$	$\bot E$ k+1
k+3	$\mathscr{B}$	Ass. ($\vee E$)
k+4	$\mathscr{B}$	R k+3
k+5	$\mathscr{B}$	$\vee E$ m, k–k+2, k+3–k+4

Double Negation Elimination

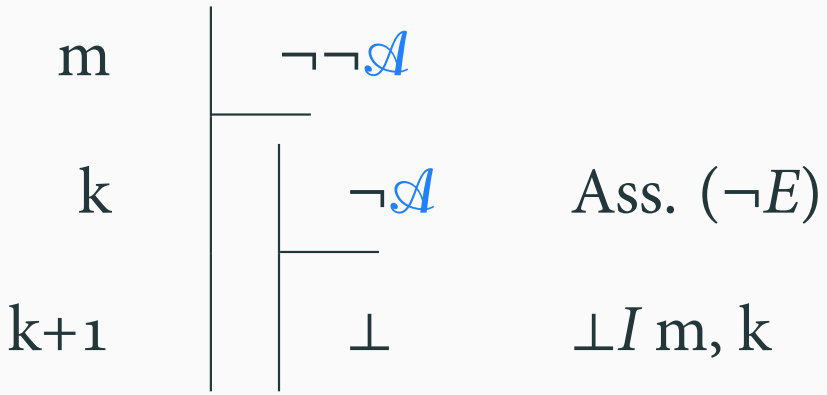


Double Negation Elimination

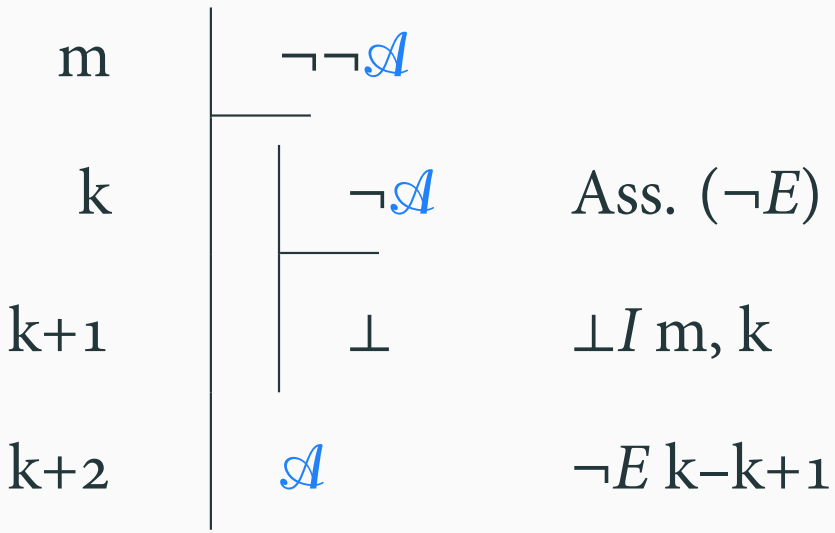
Double Negation Elimination



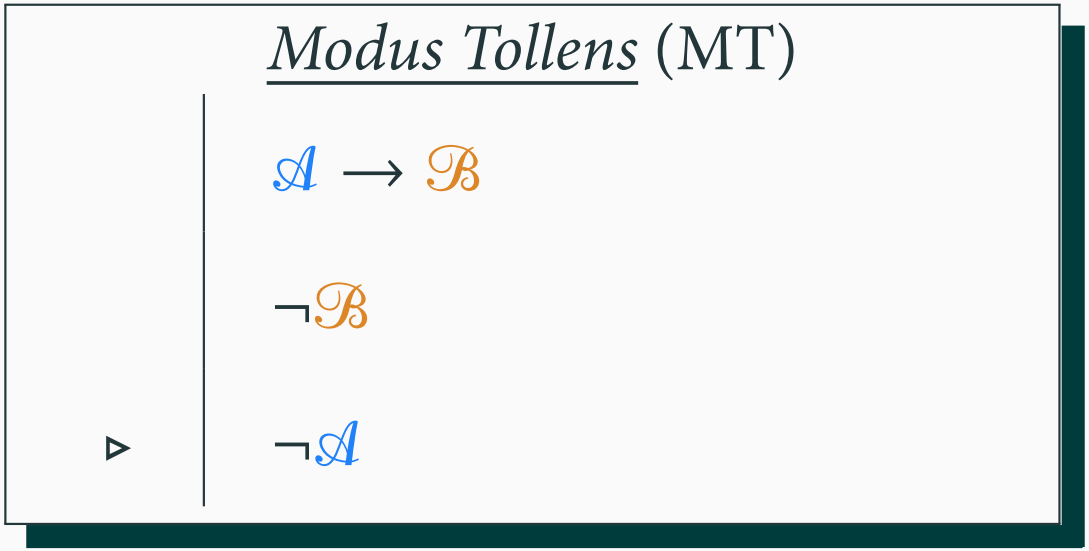
Double Negation Elimination



Double Negation Elimination



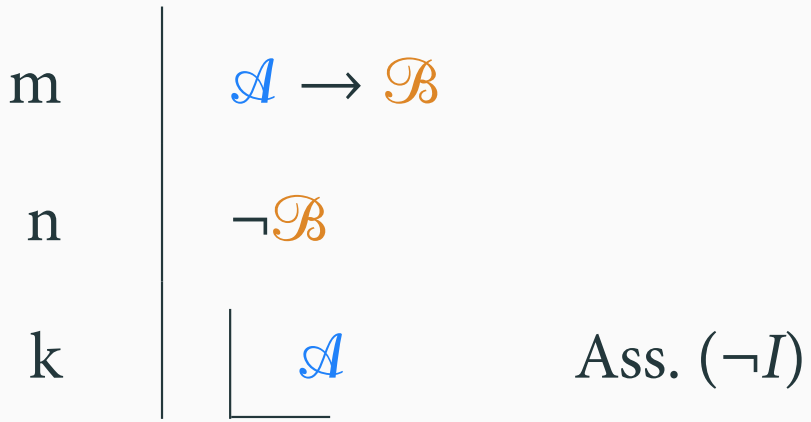
Modus Tollens



Modus Tollens

m	$\mathscr{A} \longrightarrow \mathscr{B}$
n	$\neg\mathscr{B}$

Modus Tollens



Modus Tollens

m		$\mathscr{A} \to \mathscr{B}$	
n		$\neg\mathscr{B}$	
k	|	$\mathscr{A}$	Ass. $(\neg I)$
k+1	|	$\mathscr{B}$	$\to E$ m, k

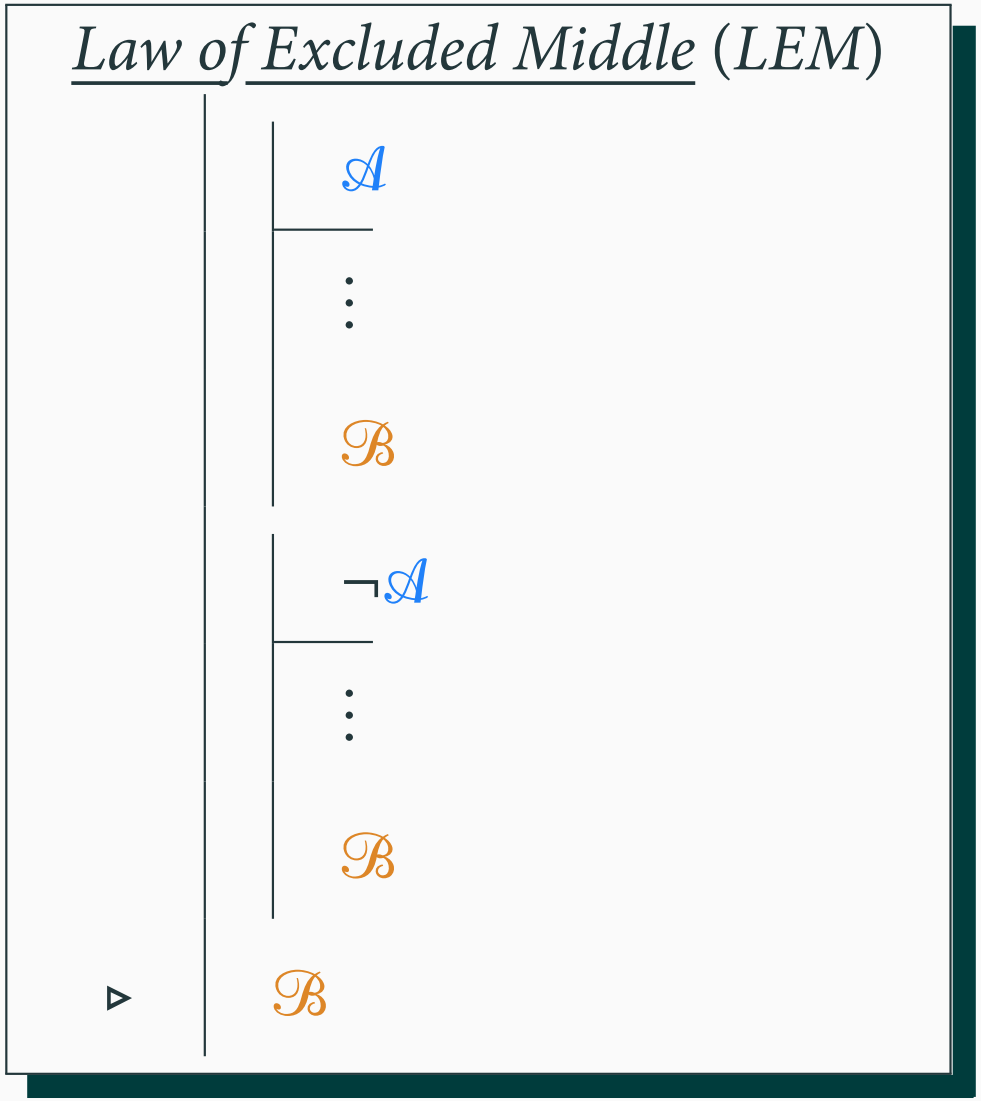
Modus Tollens

m	$\mathcal{A} \to \mathcal{B}$	
n	$\neg\mathcal{B}$	
k	$\mathcal{A}$	Ass. ($\neg I$)
k+1	$\mathcal{B}$	$\to E$ m, k
k+2	$\bot$	$\bot I$ n, k+1

Modus Tollens

m	$\mathcal{A} \to \mathcal{B}$	
n	$\neg\mathcal{B}$	
k	$\mathcal{A}$	Ass. ($\neg I$)
k+1	$\mathcal{B}$	$\to E$ m, k
k+2	$\bot$	$\bot I$ n, k+1
k+3	$\neg\mathcal{A}$	$\neg I$ k–k+2

Law of Excluded Middle



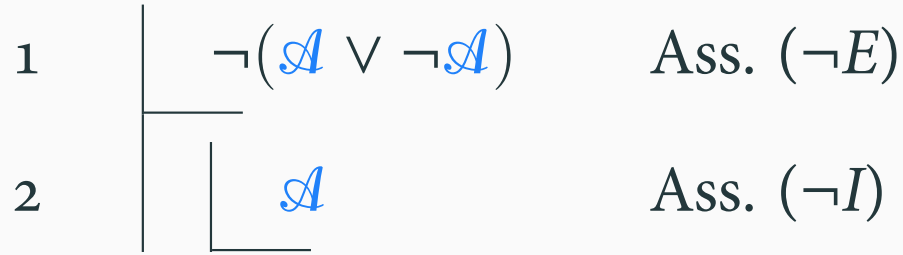
Law of Excluded Middle

1

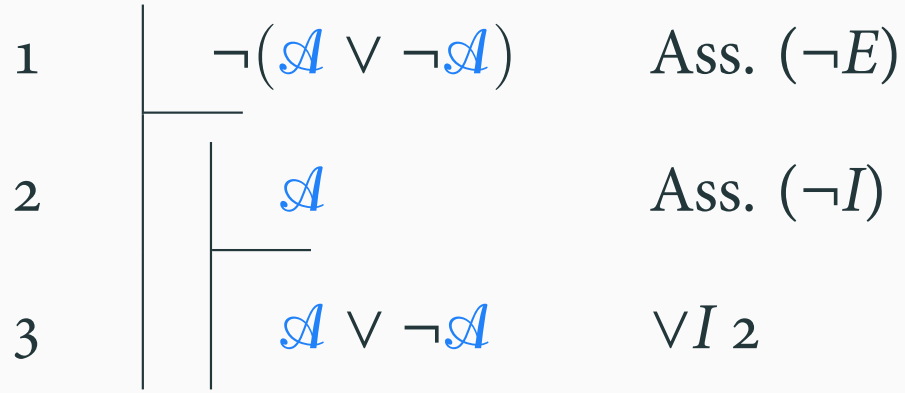
Law of Excluded Middle

1		$\neg(\mathscr{A} \vee \neg\mathscr{A})$	Ass. $(\neg E)$

Law of Excluded Middle



Law of Excluded Middle



Law of Excluded Middle

1	$\neg(\mathscr{A} \vee \neg\mathscr{A})$	Ass. $(\neg E)$
2	$\mathscr{A}$	Ass. $(\neg I)$
3	$\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3

Law of Excluded Middle

1	$\neg(\mathscr{A} \vee \neg\mathscr{A})$	Ass. $(\neg E)$
2	$\mathscr{A}$	Ass. $(\neg I)$
3	$\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A}$	$\neg I$ 2–4

Law of Excluded Middle

1	$\neg(\mathscr{A} \vee \neg\mathscr{A})$	Ass. $(\neg E)$
2	$\quad\mathscr{A}$	Ass. $(\neg I)$
3	$\quad\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 2
4	$\quad\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A}$	$\neg I$ 2–4
6	$\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 5

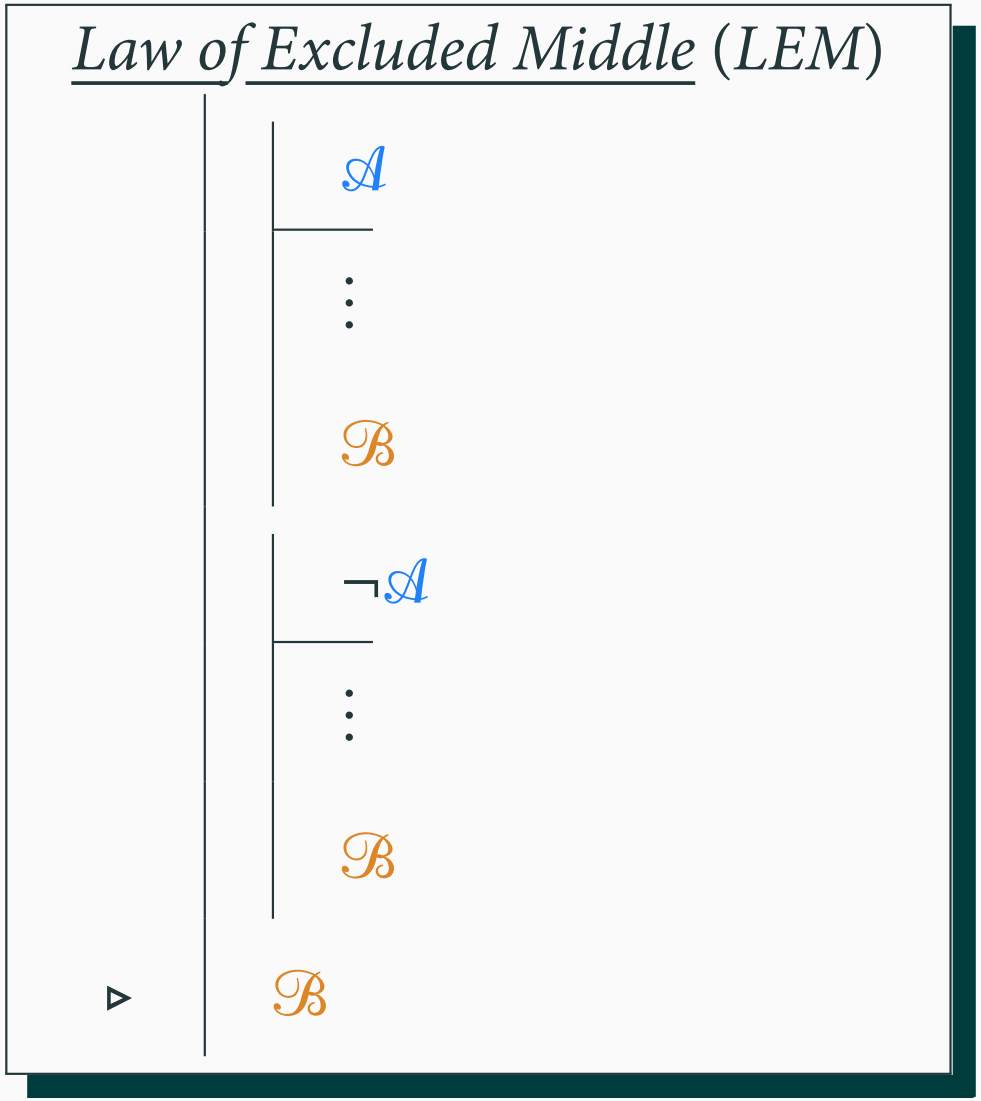
Law of Excluded Middle

1	$\neg(\mathscr{A} \vee \neg\mathscr{A})$	Ass. $(\neg E)$
2	$\quad\mathscr{A}$	Ass. $(\neg I)$
3	$\quad\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 2
4	$\quad\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A}$	$\neg I$ 2–4
6	$\mathscr{A} \vee \neg\mathscr{A}$	$\vee I$ 5
7	$\bot$	$\bot I$ 1, 6

Law of Excluded Middle

1	$\neg(\mathcal{A} \vee \neg\mathcal{A})$	Ass. $(\neg E)$
2	$\quad\mathcal{A}$	Ass. $(\neg I)$
3	$\quad\mathcal{A} \vee \neg\mathcal{A}$	$\vee I$ 2
4	$\quad\bot$	$\bot I$ 1, 3
5	$\neg\mathcal{A}$	$\neg I$ 2–4
6	$\mathcal{A} \vee \neg\mathcal{A}$	$\vee I$ 5
7	$\bot$	$\bot I$ 1, 6
8	$\mathcal{A} \vee \neg\mathcal{A}$	$\neg E$ 1–7

Law of Excluded Middle

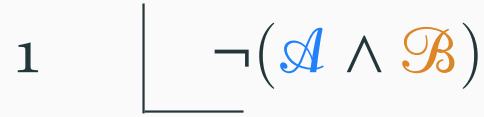


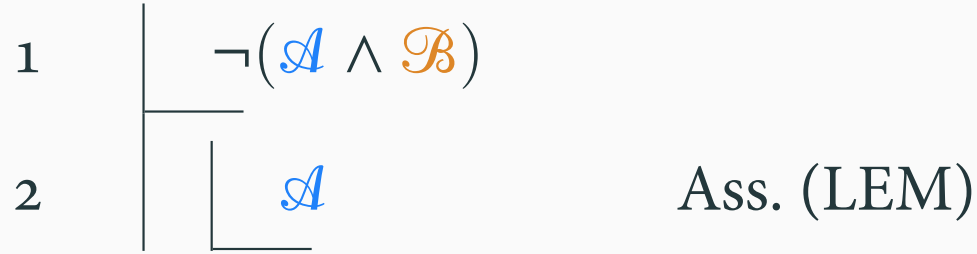
DeMorgan's Rules

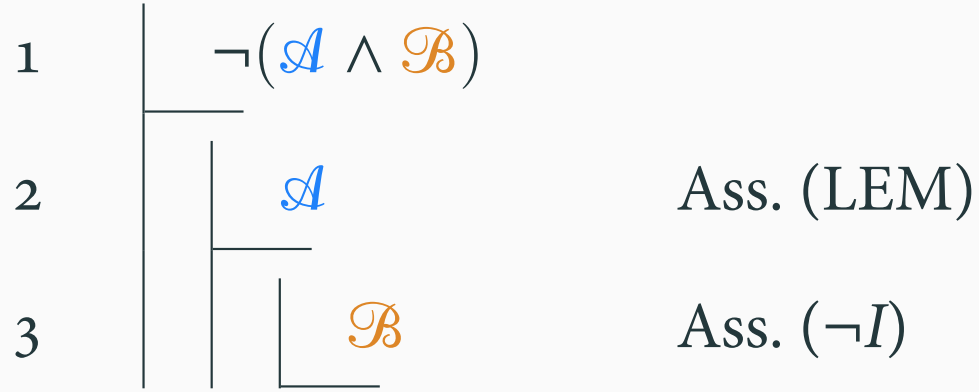
DeMorgan's Rules (DeM)

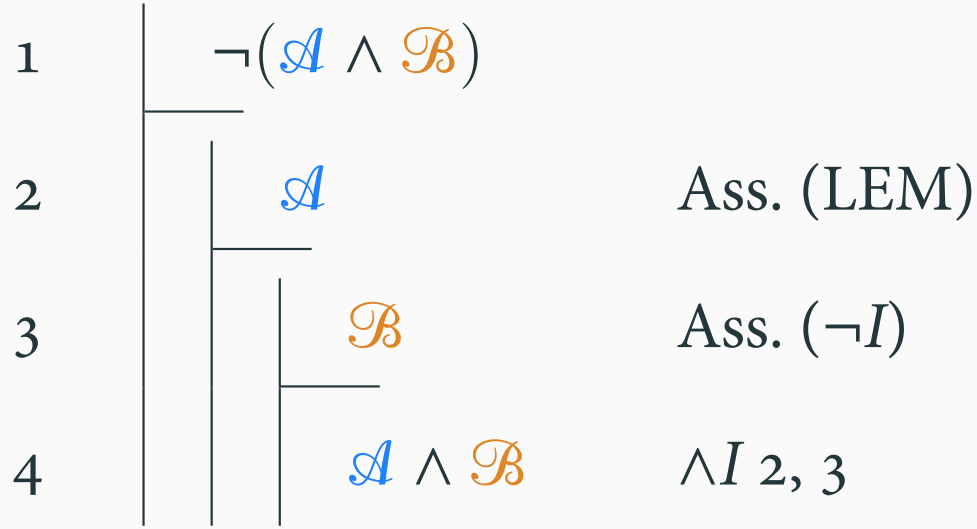
$$\neg(\mathscr{A} \wedge \mathscr{B}) \quad \blacktriangleleft\blacktriangleright \quad \neg\mathscr{A} \vee \neg\mathscr{B}$$

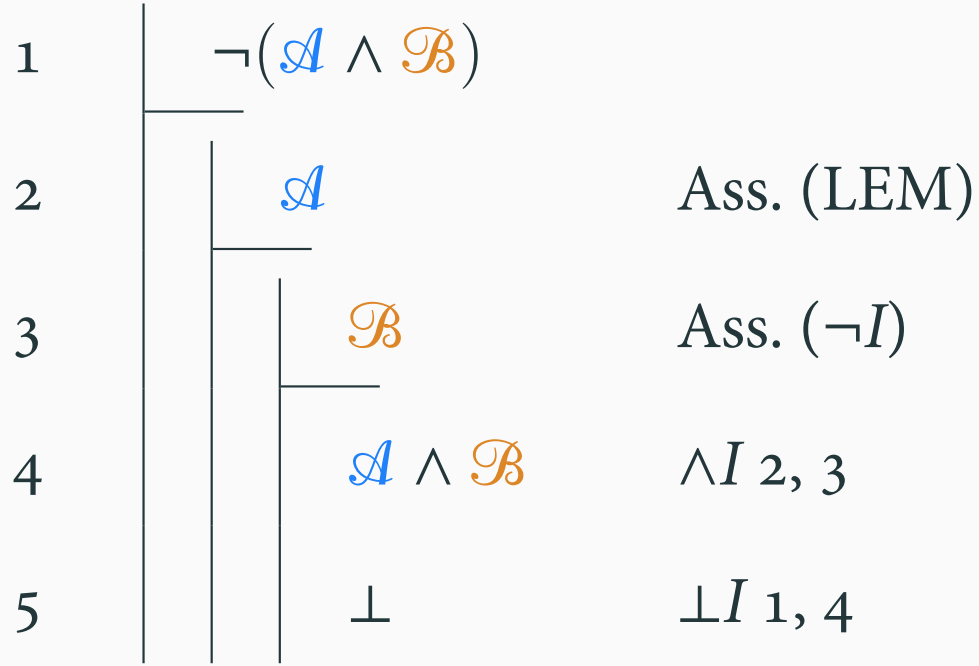
$$\neg(\mathscr{A} \vee \mathscr{B}) \quad \blacktriangleleft\blacktriangleright \quad \neg\mathscr{A} \wedge \neg\mathscr{B}$$

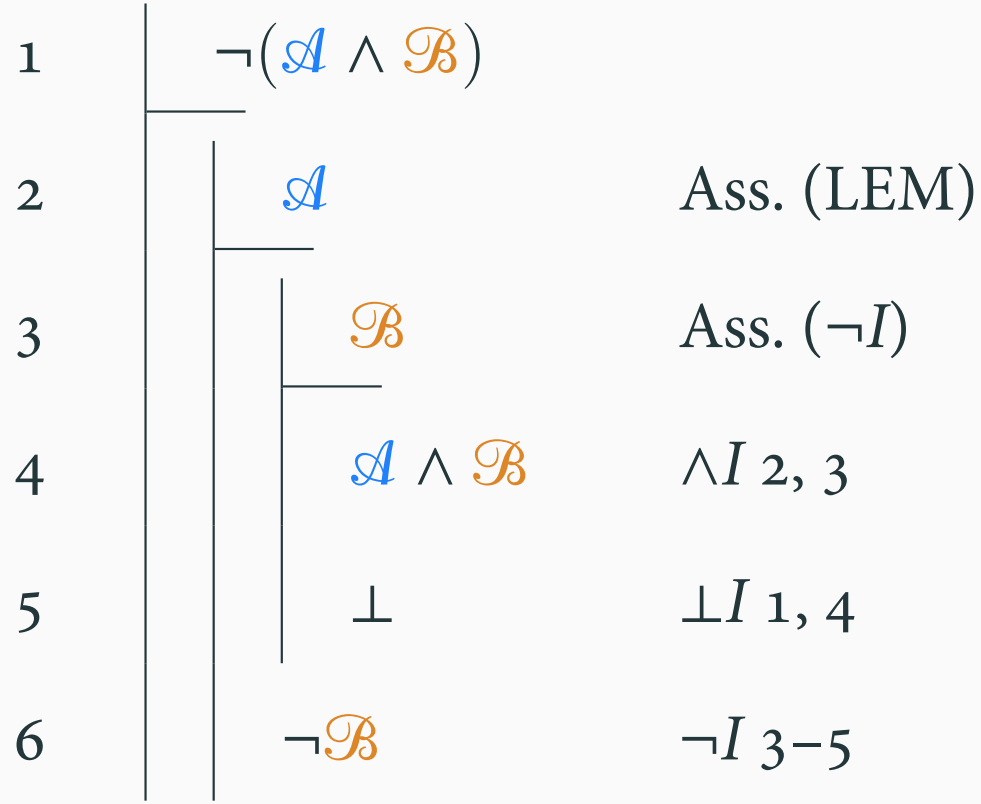

1
¬(𝒜 ∧ ℬ)


1
¬(𝒜 ∧ ℬ)
2
𝒜
Ass. (LEM)

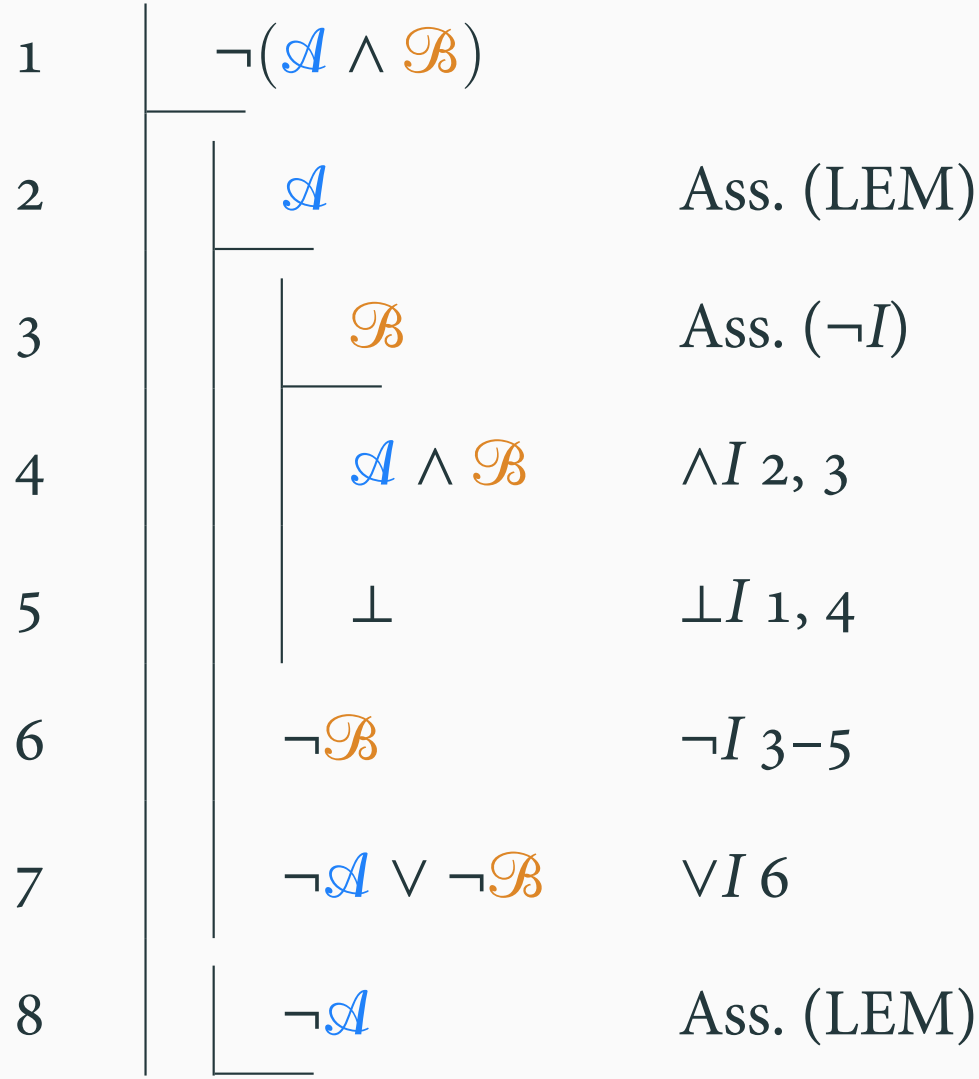

1
¬(𝒜 ∧ ℬ)
2
𝒜
Ass. (LEM)
3
ℬ
Ass. (¬I)


1
¬(𝒜 ∧ ℬ)
2
𝒜
Ass. (LEM)
3
ℬ
Ass. (¬I)
4
𝒜 ∧ ℬ
∧I 2, 3

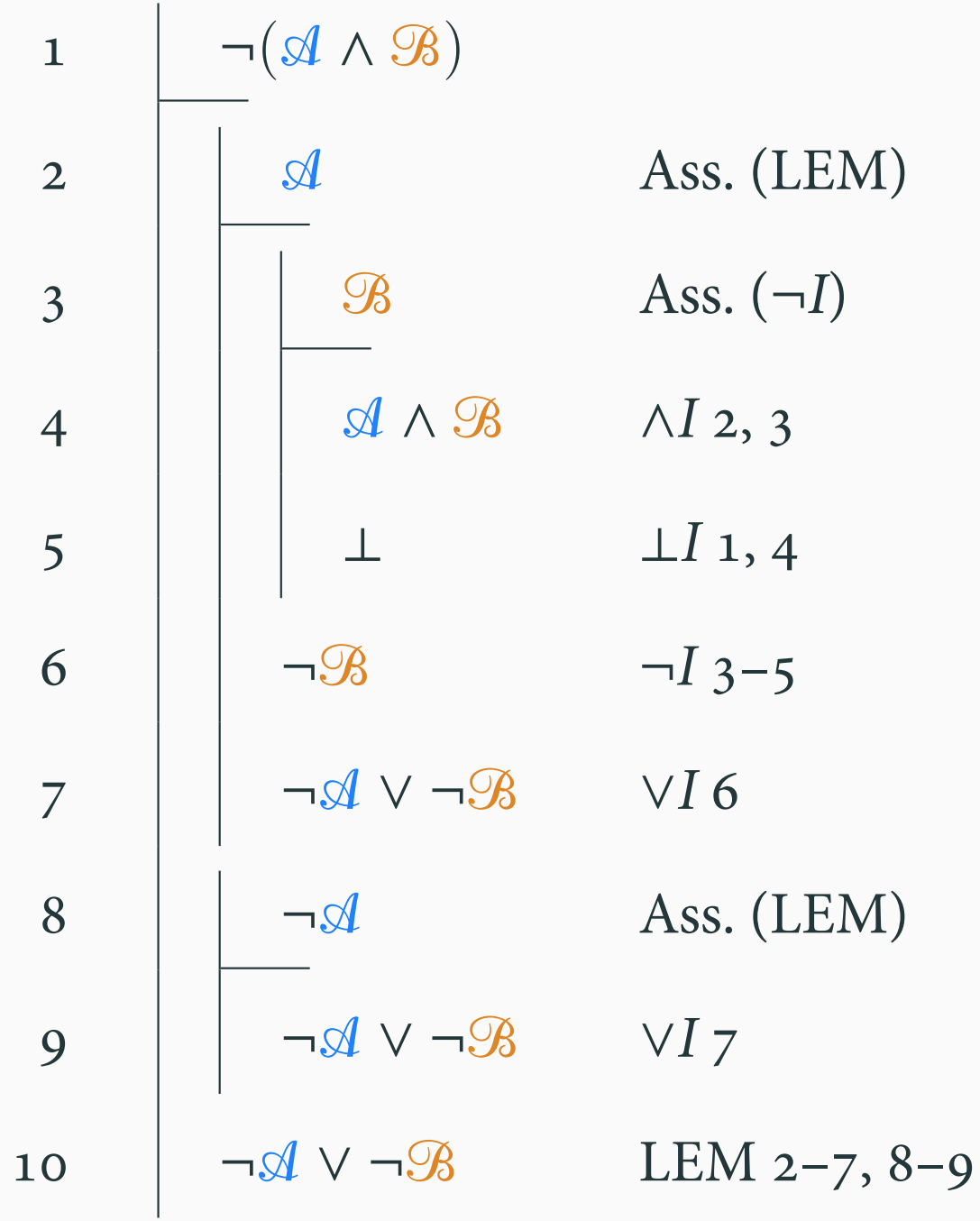

1
¬(𝒜 ∧ ℬ)
2
𝒜
Ass. (LEM)
3
ℬ
Ass. (¬I)
4
𝒜 ∧ ℬ
∧I 2, 3
5
⊥
⊥I 1, 4


1
¬(𝒜 ∧ ℬ)
2
𝒜
Ass. (LEM)
3
ℬ
Ass. (¬I)
4
𝒜 ∧ ℬ
∧I 2, 3
5
⊥
⊥I 1, 4
6
¬ℬ
¬I 3–5


1 ¬(𝒜 ∧ ℬ)
2 𝒜 Ass. (LEM)
3 ℬ Ass. (¬I)
4 𝒜 ∧ ℬ ∧I 2, 3
5 ⊥ ⊥I 1, 4
6 ¬ℬ ¬I 3–5
7 ¬𝒜 ∨ ¬ℬ ∨I 6

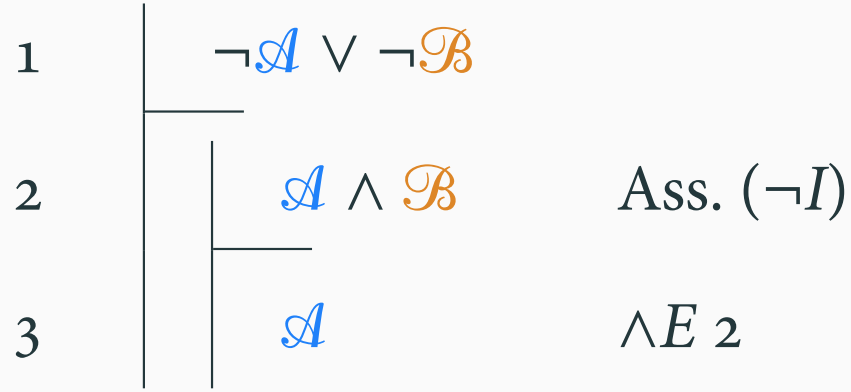

1 ¬(𝒜 ∧ ℬ)
2 𝒜 Ass. (LEM)
3 ℬ Ass. (¬I)
4 𝒜 ∧ ℬ ∧I 2, 3
5 ⊥ ⊥I 1, 4
6 ¬ℬ ¬I 3–5
7 ¬𝒜 ∨ ¬ℬ ∨I 6
8 ¬𝒜 Ass. (LEM)

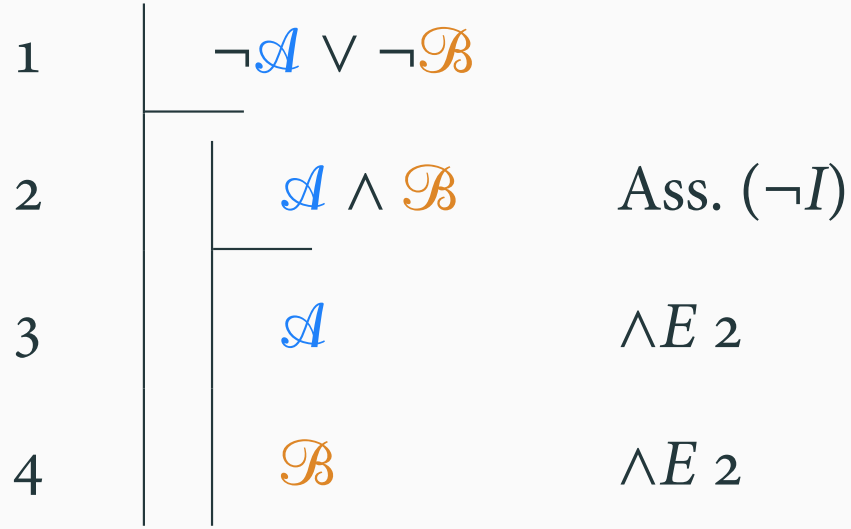

1 ¬(𝒜 ∧ ℬ)
2 𝒜 Ass. (LEM)
3 ℬ Ass. (¬I)
4 𝒜 ∧ ℬ ∧I 2, 3
5 ⊥ ⊥I 1, 4
6 ¬ℬ ¬I 3–5
7 ¬𝒜 ∨ ¬ℬ ∨I 6
8 ¬𝒜 Ass. (LEM)
9 ¬𝒜 ∨ ¬ℬ ∨I 7

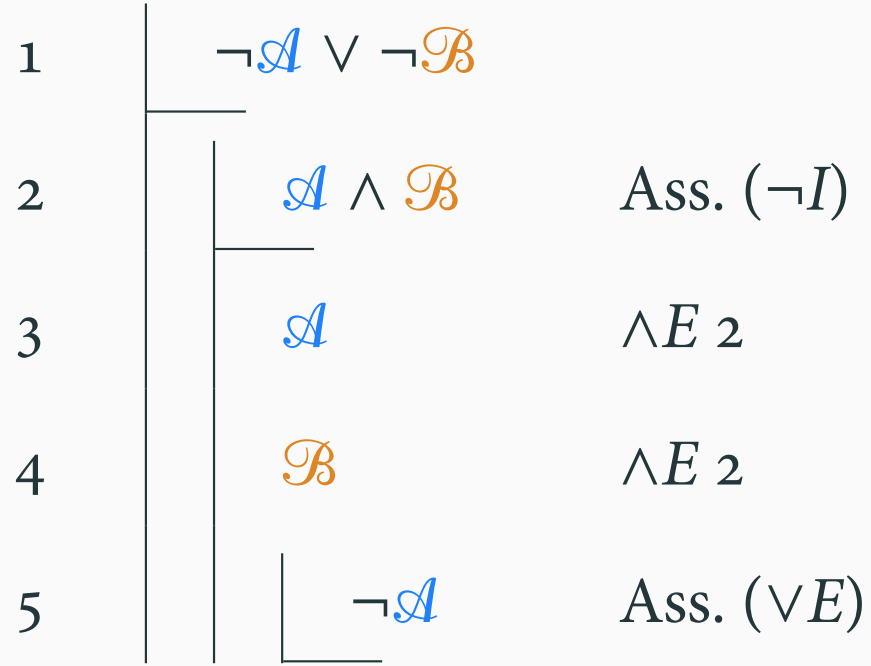

1 ¬(𝒜 ∧ ℬ)
2 𝒜 Ass. (LEM)
3 ℬ Ass. (¬I)
4 𝒜 ∧ ℬ ∧I 2, 3
5 ⊥ ⊥I 1, 4
6 ¬ℬ ¬I 3–5
7 ¬𝒜 ∨ ¬ℬ ∨I 6
8 ¬𝒜 Ass. (LEM)
9 ¬𝒜 ∨ ¬ℬ ∨I 7
10 ¬𝒜 ∨ ¬ℬ LEM 2–7, 8–9

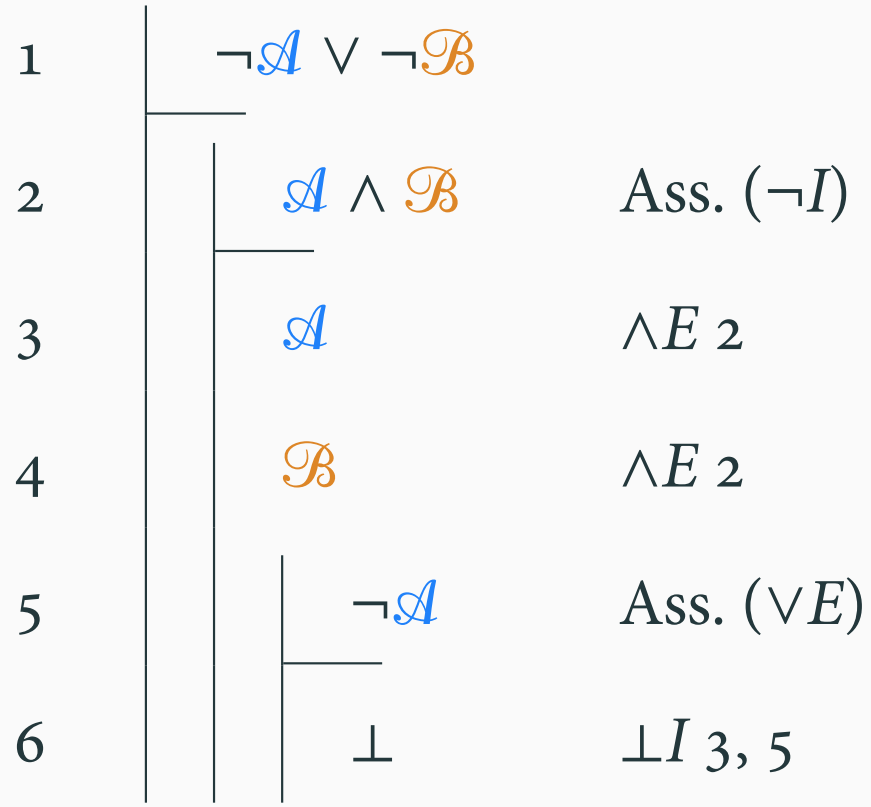

1
¬𝒜 ∨ ¬ℬ

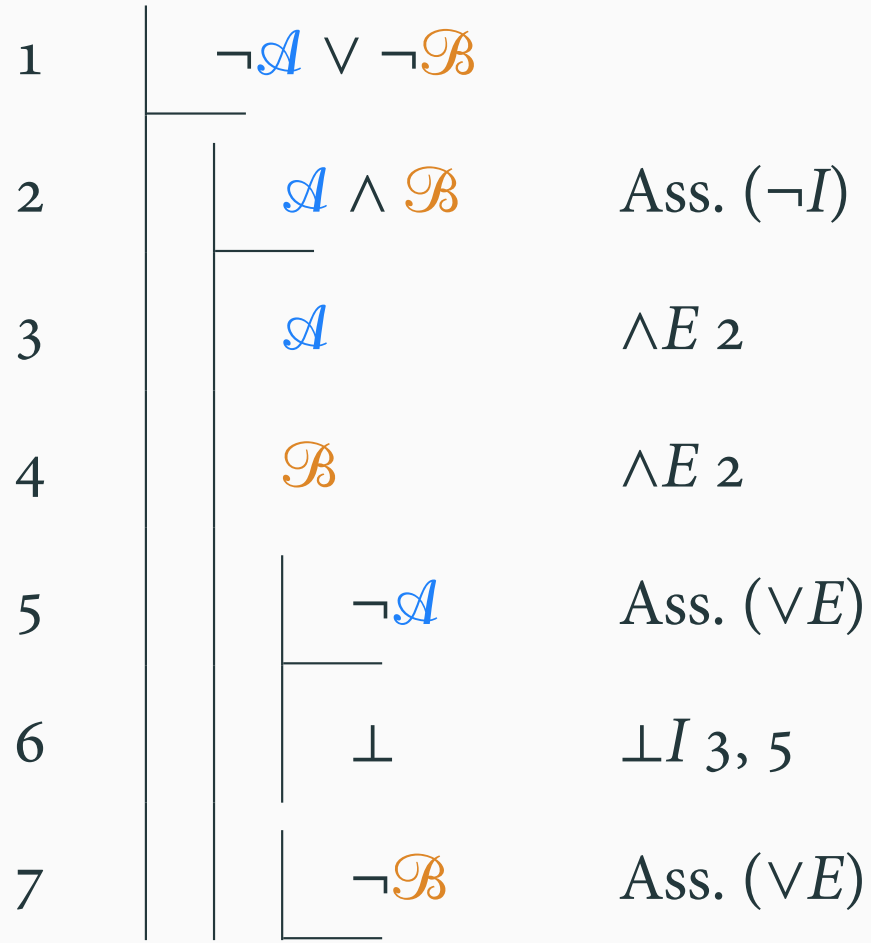

1
¬𝒜 ∨ ¬ℬ
2
𝒜 ∧ ℬ
Ass. (¬I)

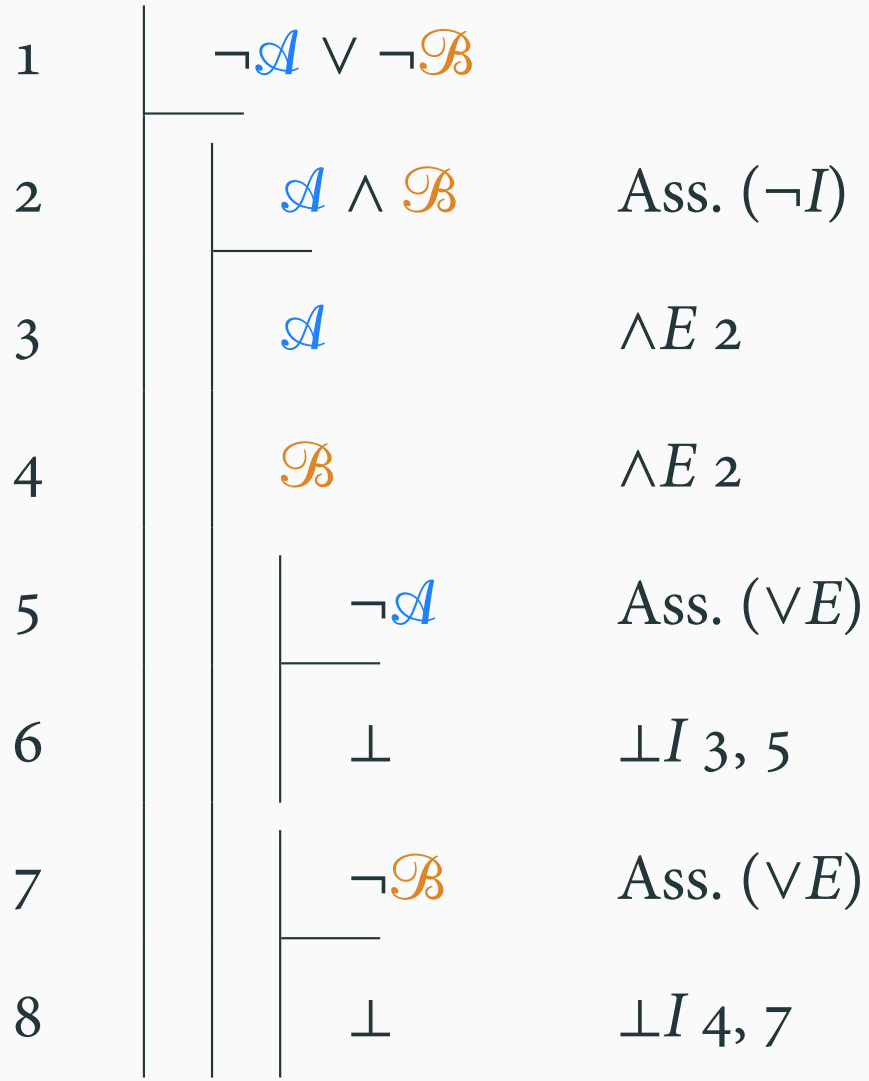

1
¬𝒜 ∨ ¬ℬ
2
𝒜 ∧ ℬ
Ass. (¬I)
3
𝒜
∧E 2


1
$\neg\mathcal{A} \vee \neg\mathcal{B}$
2
$\mathcal{A} \wedge \mathcal{B}$
Ass. $(\neg I)$
3
$\mathcal{A}$
$\wedge E$ 2
4
$\mathcal{B}$
$\wedge E$ 2


1
¬𝒜 ∨ ¬ℬ
2
𝒜 ∧ ℬ
Ass. (¬I)
3
𝒜
∧E 2
4
ℬ
∧E 2
5
¬𝒜
Ass. (∨E)


1 ¬𝒜 ∨ ¬ℬ
2 𝒜 ∧ ℬ Ass. (¬I)
3 𝒜 ∧E 2
4 ℬ ∧E 2
5 ¬𝒜 Ass. (∨E)
6 ⊥ ⊥I 3, 5


1 ¬𝒜 ∨ ¬ℬ
2 𝒜 ∧ ℬ Ass. (¬I)
3 𝒜 ∧E 2
4 ℬ ∧E 2
5 ¬𝒜 Ass. (∨E)
6 ⊥ ⊥I 3, 5
7 ¬ℬ Ass. (∨E)

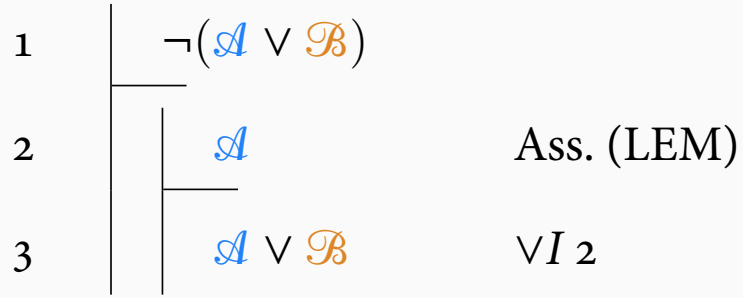

1 ¬𝒜 ∨ ¬ℬ
2 𝒜 ∧ ℬ Ass. (¬I)
3 𝒜 ∧E 2
4 ℬ ∧E 2
5 ¬𝒜 Ass. (∨E)
6 ⊥ ⊥I 3, 5
7 ¬ℬ Ass. (∨E)
8 ⊥ ⊥I 4, 7

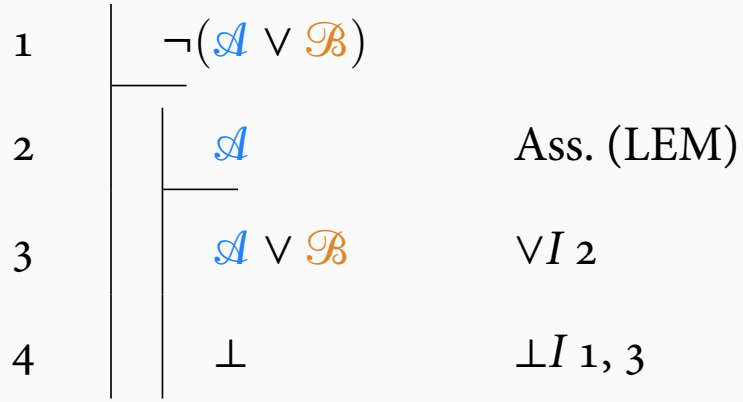

1 ¬𝒜 ∨ ¬ℬ
2 𝒜 ∧ ℬ Ass. (¬I)
3 𝒜 ∧E 2
4 ℬ ∧E 2
5 ¬𝒜 Ass. (∨E)
6 ⊥ ⊥I 3, 5
7 ¬ℬ Ass. (∨E)
8 ⊥ ⊥I 4, 7
9 ⊥ ∨E 1, 5–6, 7–8

1	$\neg\mathcal{A} \vee \neg\mathcal{B}$	
2	$\quad \mathcal{A} \wedge \mathcal{B}$	Ass. ($\neg I$)
3	$\quad \mathcal{A}$	$\wedge E$ 2
4	$\quad \mathcal{B}$	$\wedge E$ 2
5	$\quad\quad \neg\mathcal{A}$	Ass. ($\vee E$)
6	$\quad\quad \bot$	$\bot I$ 3, 5
7	$\quad\quad \neg\mathcal{B}$	Ass. ($\vee E$)
8	$\quad\quad \bot$	$\bot I$ 4, 7
9	$\quad \bot$	$\vee E$ 1, 5–6, 7–8
10	$\neg(\mathcal{A} \wedge \mathcal{B})$	$\neg I$ 2–9

1 $\neg(\mathscr{A} \vee \mathscr{B})$


1
¬(𝒜 ∨ ℬ)
2
𝒜
Ass. (LEM)


1
¬(𝒜 ∨ ℬ)
2
𝒜
Ass. (LEM)
3
𝒜 ∨ ℬ
∨I 2


1 ¬(𝒜 ∨ ℬ)
2 𝒜 Ass. (LEM)
3 𝒜 ∨ ℬ ∨I 2
4 ⊥ ⊥I 1, 3

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\mathscr{A}$	Ass. (LEM)
3	$\mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\mathscr{A}$	Ass. (LEM)
3	$\mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\neg\mathscr{A}$	Ass. (LEM)

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\mathscr{A}$	Ass. (LEM)
3	$\mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\neg\mathscr{A}$	Ass. (LEM)
7	$\mathscr{B}$	Ass. ($\neg I$)

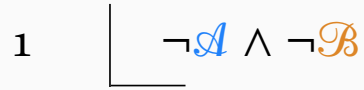
1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\mathscr{A}$	Ass. (LEM)
3	$\mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\bot$	$\bot I$ 1, 3
5	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\neg\mathscr{A}$	Ass. (LEM)
7	$\mathscr{B}$	Ass. ($\neg I$)
8	$\mathscr{A} \vee \mathscr{B}$	$\vee I$ 7

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\quad \mathscr{A}$	Ass. (LEM)
3	$\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\quad \neg\mathscr{A}$	Ass. (LEM)
7	$\quad\quad \mathscr{B}$	Ass. ($\neg I$)
8	$\quad\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 7
9	$\quad\quad \bot$	$\bot I$ 1, 8

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\quad \mathscr{A}$	Ass. (LEM)
3	$\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\quad \neg\mathscr{A}$	Ass. (LEM)
7	$\quad\quad \mathscr{B}$	Ass. ($\neg I$)
8	$\quad\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 7
9	$\quad\quad \bot$	$\bot I$ 1, 8
10	$\quad \neg\mathscr{B}$	$\neg I$ 7–9

1	$\neg(\mathcal{A} \vee \mathcal{B})$	
2	$\quad \mathcal{A}$	Ass. (LEM)
3	$\quad \mathcal{A} \vee \mathcal{B}$	$\vee I$ 2
4	$\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\mathcal{A} \wedge \neg\mathcal{B}$	$\bot E$ 4
6	$\quad \neg\mathcal{A}$	Ass. (LEM)
7	$\quad\quad \mathcal{B}$	Ass. ($\neg I$)
8	$\quad\quad \mathcal{A} \vee \mathcal{B}$	$\vee I$ 7
9	$\quad\quad \bot$	$\bot I$ 1, 8
10	$\quad \neg\mathcal{B}$	$\neg I$ 7–9
11	$\quad \neg\mathcal{A} \wedge \neg\mathcal{B}$	$\wedge I$ 6, 0

1	$\neg(\mathscr{A} \vee \mathscr{B})$	
2	$\quad \mathscr{A}$	Ass. (LEM)
3	$\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 2
4	$\quad \bot$	$\bot I$ 1, 3
5	$\quad \neg\mathscr{A} \wedge \neg\mathscr{B}$	$\bot E$ 4
6	$\quad \neg\mathscr{A}$	Ass. (LEM)
7	$\quad\quad \mathscr{B}$	Ass. ($\neg I$)
8	$\quad\quad \mathscr{A} \vee \mathscr{B}$	$\vee I$ 7
9	$\quad\quad \bot$	$\bot I$ 1, 8
10	$\quad \neg\mathscr{B}$	$\neg I$ 7–9
11	$\quad \neg\mathscr{A} \wedge \neg\mathscr{B}$	$\wedge I$ 6, 0
12	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	LEM 2–5, 6–11


1
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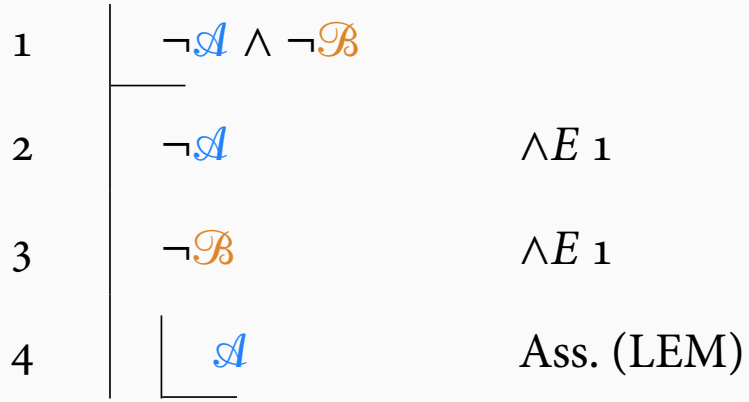
1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1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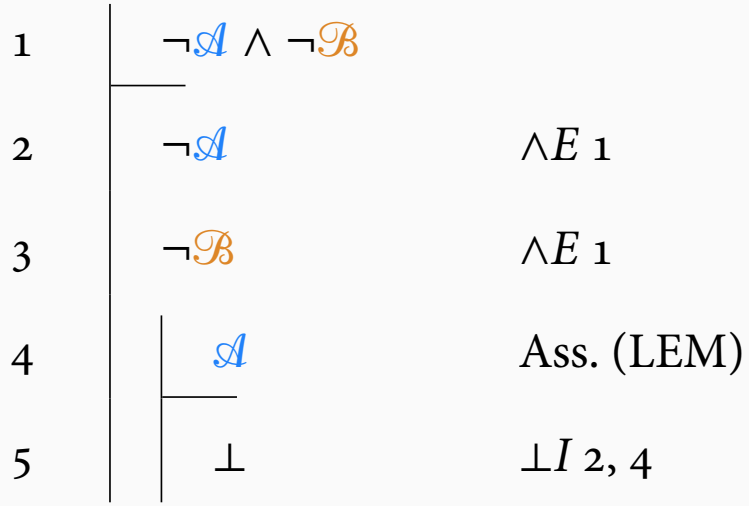
3 ¬ℬ ∧E 1
4 𝒜 Ass. (LEM)


1 ¬𝒜 ∧ ¬ℬ
2 ¬𝒜 ∧E 1
3 ¬ℬ ∧E 1
4 𝒜 Ass. (LEM)
5 ⊥ ⊥I 2, 4

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1
4	$\mathscr{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathscr{A} \vee \mathscr{B})$	$\bot E$ 5

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1
4	$\mathscr{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathscr{A} \vee \mathscr{B})$	$\bot E$ 5
7	$\neg\mathscr{A}$	Ass. (LEM)

1	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
2	$\neg\mathcal{A}$	$\wedge E$ 1
3	$\neg\mathcal{B}$	$\wedge E$ 1
4	$\mathcal{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathcal{A} \vee \mathcal{B})$	$\bot E$ 5
7	$\neg\mathcal{A}$	Ass. (LEM)
8	$\mathcal{A} \vee \mathcal{B}$	Ass. ($\neg I$)

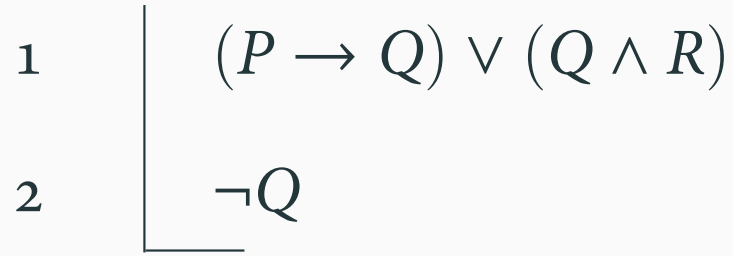
1	$\neg\mathcal{A} \wedge \neg\mathcal{B}$	
2	$\neg\mathcal{A}$	$\wedge E$ 1
3	$\neg\mathcal{B}$	$\wedge E$ 1
4	$\mathcal{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathcal{A} \vee \mathcal{B})$	$\bot E$ 5
7	$\neg\mathcal{A}$	Ass. (LEM)
8	$\mathcal{A} \vee \mathcal{B}$	Ass. ($\neg I$)
9	$\mathcal{A}$	DS 3, 8

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1
4	$\mathscr{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathscr{A} \vee \mathscr{B})$	$\bot E$ 5
7	$\neg\mathscr{A}$	Ass. (LEM)
8	$\mathscr{A} \vee \mathscr{B}$	Ass. ($\neg I$)
9	$\mathscr{A}$	DS 3, 8
10	$\bot$	$\bot I$ 7, 9

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1
4	$\mathscr{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathscr{A} \vee \mathscr{B})$	$\bot E$ 5
7	$\neg\mathscr{A}$	Ass. (LEM)
8	$\mathscr{A} \vee \mathscr{B}$	Ass. ($\neg I$)
9	$\mathscr{A}$	DS 3, 8
10	$\bot$	$\bot I$ 7, 9
11	$\neg(\mathscr{A} \vee \mathscr{B})$	$\neg I$ 8–10

1	$\neg\mathscr{A} \wedge \neg\mathscr{B}$	
2	$\neg\mathscr{A}$	$\wedge E$ 1
3	$\neg\mathscr{B}$	$\wedge E$ 1
4	$\mathscr{A}$	Ass. (LEM)
5	$\bot$	$\bot I$ 2, 4
6	$\neg(\mathscr{A} \vee \mathscr{B})$	$\bot E$ 5
7	$\neg\mathscr{A}$	Ass. (LEM)
8	$\mathscr{A} \vee \mathscr{B}$	Ass. ($\neg I$)
9	$\mathscr{A}$	DS 3, 8
10	$\bot$	$\bot I$ 7, 9
11	$\neg(\mathscr{A} \vee \mathscr{B})$	$\neg I$ 8–10
12	$\neg(\mathscr{A} \vee \mathscr{B})$	LEM 4–6, 7–11

A Sample Proof



A Sample Proof

1	$(P \to Q) \vee (Q \wedge R)$	
2	$\neg Q$	
3	$\neg Q \vee \neg R$	$\vee I$ 2

A Sample Proof

1	$(P \rightarrow Q) \vee (Q \wedge R)$	
2	$\neg Q$	
3	$\neg Q \vee \neg R$	$\vee I$ 2
4	$\neg(Q \wedge R)$	DeM 3

A Sample Proof

1	$(P \to Q) \lor (Q \land R)$	
2	$\neg Q$	
3	$\neg Q \lor \neg R$	$\lor I$ 2
4	$\neg(Q \land R)$	DeM 3
5	$P \to Q$	DS 1, 4

A Sample Proof

1	$(P \rightarrow Q) \vee (Q \wedge R)$	
2	$\neg Q$	
3	$\neg Q \vee \neg R$	$\vee I$ 2
4	$\neg(Q \wedge R)$	DeM 3
5	$P \rightarrow Q$	DS 1, 4
6	$\neg P$	MT 2, 5

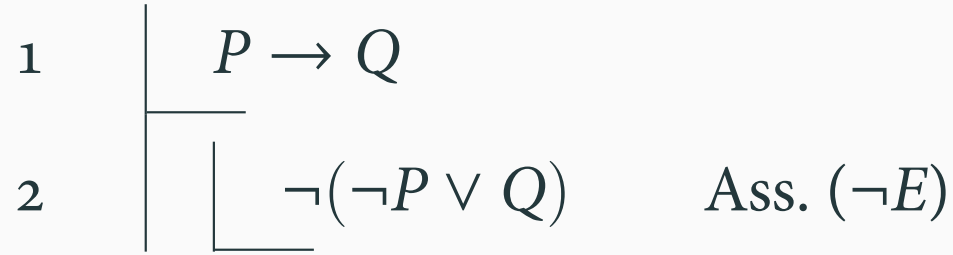
A Sample Proof

1	$(P \rightarrow Q) \vee (Q \wedge R)$	
2	$\neg Q$	
3	$\neg Q \vee \neg R$	$\vee I$ 2
4	$\neg(Q \wedge R)$	DeM 3
5	$P \rightarrow Q$	DS 1, 4
6	$\neg P$	MT 2, 5

A Sample Proof

1 $P \to Q$

A Sample Proof



A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3
5	P	DNE 4

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3
5	P	DNE 4
6	Q	$\to E$ 1, 5

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3
5	P	DNE 4
6	Q	$\to E$ 1, 5
7	$\neg Q$	$\wedge E$ 3

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3
5	P	DNE 4
6	Q	$\to E$ 1, 5
7	$\neg Q$	$\wedge E$ 3
8	$\bot$	$\bot I$ 6, 7

A Sample Proof

1	$P \to Q$	
2	$\neg(\neg P \vee Q)$	Ass. ($\neg E$)
3	$\neg\neg P \wedge \neg Q$	DeM 2
4	$\neg\neg P$	$\wedge E$ 3
5	P	DNE 4
6	Q	$\to E$ 1, 5
7	$\neg Q$	$\wedge E$ 3
8	$\bot$	$\bot I$ 6, 7
9	$\neg P \vee Q$	$\neg E$ 2–8