

Lecture Notes for *Logic*

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Contents

1	Basic Concepts of Logic	4
1.1	Finding Argumentative Structure	5
1.2	Conditionals	7
1.2.1	Necessary and Sufficient Conditions	7
1.3	Deductive Validity	8
1.4	Inductive Strength	9
2	Basic Concepts of Logic, Day 2	10
2.0.1	Review	10
2.1	Proving Invalidity, take 1	10
2.1.1	Venn Diagrams	11
2.1.2	Venn Diagrams, Counterexamples, and Validity	12
3	Basic Concepts of Logic, Day 3	15
3.1	Review	15
3.2	Formal Deductive Validity	15
3.3	Proving Invalidity, take 2	17
3.4	Formal Deductive Invalidity	18
4	Dialectics	21
4.1	Rules for Dialectics	23
4.1.1	Example: A Failed Dialectic	25
5	Informal Fallacies	27
5.1	Fallacies of Irrelevance	28
5.1.1	Argument Against the Person (Ad Hominem)	28
5.1.2	Straw Man	29
5.1.3	Appeal to Force (Ad Baculum)	29
5.1.4	Appeal to the People (Ad Populum)	30
5.1.5	Appeal to Ignorance (Ad Ignorantiam)	30
5.1.6	Red Herring (Ignoratio Elenchi)	31
6	Informal Fallacies, Day 2	32
6.1	Fallacies Involving Ambiguity	32
6.1.1	Equivocation	32
6.1.2	Amphiboly	33
6.1.3	Composition/Division	33
6.2	Fallacies Involving Unwarranted Assumptions	34
6.2.1	Begging the Question (Petitio Principii)	34
6.2.2	False Dilemma	34
6.2.3	False Cause Fallacy	35

7	Propositional Logic	37
7.1	Syntax for PL	37
7.1.1	Vocabulary	37
7.1.2	Grammar	38
7.1.3	Main Operators and Subformulae	39
7.2	Semantics for PL	40
7.2.1	The Meaning of the Statement Letters	41
7.2.2	The Meaning of ' \sim '	41
7.2.3	The Meaning of ' \bullet '	41
7.2.4	The Meaning of ' \vee '	41
7.2.5	The Meaning of ' \supset '	42
7.2.6	The Meaning of ' \equiv '	42
7.2.7	Determining the Truth-value of a wff of PL	42
7.3	Translation from PL to English	44
7.4	Translation from English to PL	46
7.4.1	Negation	46
7.4.2	Conjunction	47
7.4.3	Disjunction	47
7.4.4	The Material Conditional and Biconditional	48
8	Logical Notions of PL: Validity	49
8.1	How to Construct a Truth-Table	49
8.2	What a Truth-Table Represents	51
8.3	PL -Validity	52
9	Propositional Logic Derivations, day 1	56
9.1	The Basics	56
9.2	Rules of Implication	57
9.2.1	Modus Ponens	57
9.2.2	Modus Tollens	57
9.2.3	Hypothetical Syllogism	58
9.2.4	Disjunctive Syllogism	58
9.2.5	Simplification	60
9.2.6	Conjunction	60
9.2.7	Addition	61
9.2.8	Constructive Dilemma	61
9.3	A Mistake to Avoid	63
9.4	Rules of Replacement	63
9.4.1	De Morgan's	64
9.4.2	Commutativity	64
9.4.3	Associativity	65
9.4.4	Distribution	66
9.4.5	Double Negation	67
9.4.6	Transposition	68
9.4.7	Material Implication	69
9.4.8	Material Equivalence	69
9.4.9	Exportation	70
10	Propositional Logic Derivations, Day 2	72
10.1	Four Final Rules of Inference	72
10.1.1	Subderivations	72
10.1.2	Conditional Proof	75
10.1.3	Indirect Proof	76
10.2	PL -Derivability and the Logical Notions of PL	78

10.2.1	Some New Notation	78
10.2.2	PL -Validity	78
10.2.3	PL -Tautologies and PL -Self-Contradictions	79
10.2.4	PL -Equivalence and PL -Contradiction	81
10.2.5	PL -Inconsistency	83
11	Categorical Propositions	85
11.1	Categorical Propositions	85
11.1.1	The Components of a Categorical Proposition	86
11.1.2	Quality, Quantity, and Distribution	87
11.2	The Square of Opposition	89
11.3	Categorical Syllogisms	93
11.3.1	Major Terms, Minor Terms, Middle Terms	94
11.3.2	Mood and Figure	94
11.4	Venn Diagrams	96
11.4.1	Representing Standard Form Categorical Propositions with Venn Diagrams (Modern Understanding)	97
11.4.2	Testing the Validity of Categorical Syllogisms with Venn Diagrams (Modern Understanding)	98
12	Quantificational Logic: Syntax and Semantics	101
12.1	The Language QL	101
12.1.1	The Syntax of QL	102
12.1.2	Semantics for QL	107
12.2	Translations from QL into English	110
12.2.1	Translating Simple Quantified wffs of QL	110
12.2.2	Translating More Complicated Quantified wffs of QL	110
12.3	Translations from English into QL	112
13	Quantificational Logic: Validity, Equivalence, and the Rest	114
13.1	Notation	114
13.2	QL -Validity	114
13.3	QL -tautologies, QL -self-contradictions, & QL -contingencies	116
13.4	QL -Equivalence & QL -Contradiction	117
13.5	QL -Consistency & QL -Inconsistency	119
14	Quantificational Logic: Derivations	121
14.1	Substitution Instances	121
14.2	QL -Derivations	122
14.2.1	New Rules of Replacement	122
14.2.2	New Rules of Implication	123
14.2.3	Final Thoughts	131
14.3	QL -Derivability and the Logical Notions of QL	132
14.3.1	QL -Validity	133
14.3.2	QL -Tautologies and QL -Self-Contradictions	133
14.3.3	QL -Equivalence and QL -Contradiction	134
14.3.4	QL -Inconsistency	135
15	Quantificational Logic with Identity	137
15.1	The Language QLI	137
15.1.1	Syntax for QLI	137
15.1.2	Semantics for QLI	138
15.2	QLI Derivations	139
15.3	Translations from English to QLI	143
15.3.1	Number Claims	143
15.4	The Only	147

15.5 Definite Descriptions 148

Chapter 1

Basic Concepts of Logic

In many contexts, ‘argument’ can mean a fight, or a heated, vitriolic debate. In logic, we have a more technical understanding of what an argument is. In logic, we understand an ARGUMENT to be something that provides reasons to believe some claim. The claim that the argument is arguing *for* is called the CONCLUSION of the argument. The reasons that are adduced in the conclusion’s favor are known as the PREMISES of the argument. An argument attempts to persuade its audience to accept its conclusion by providing premises that the audience is expected to accept, and showing that they support the conclusion.

Our lives are filled with arguments. Each day we make and listen to myriad arguments. These arguments are on matters both personal and political, both mundane and profound. Our ability to rationally decide what we think about these matters depends upon our ability to evaluate these arguments well. Consider the following example. On MSNBC in 2013, there was the following exchange between Tony Perkins, the president of the Family Research Council, and MSNBC host Luke Russert:

Perkins: You say that people ought to be able to marry whoever they love. If love becomes the definition of what the boundaries of marriage are, how do we define that going forward? What if someone wants to immigrate to this country that lives in a country that allows multiple spouses? They come here—right now they can’t immigrate with those spouses—but if the criteria or the parameters are simply love, how do we prohibit them from coming into the country? So, if it’s all about just love, as it’s being used, where do we set the lines?

Russert: So you equate homosexuality with polygamy?

Perkins: No, that’s not the argument.

Russert: But you just said that, sir.

Perkins: No, the argument being made by those wanting to redefine marriage is saying that it’s all based on love. You ought to be able to marry who you love. Isn’t that the argument that they’re using? If that’s the case, where do you draw the boundaries? That’s all that I’m asking.

In this passage, Perkins asks many rhetorical questions. It’s not immediately obvious what the form of his argument is, what the conclusion might be, or even whether he is providing an argument at all. So, in order to evaluate what Perkins has to say, we must first decide whether he is making an argument, and, if so, what exactly that argument is. We might think, as Russert thought, that Perkins was making the following argument:

PREMISES	---	{	1. Gay marriage is morally tantamount to polygamy
			2. Polygamy is wrong.
CONCLUSION	---	_____	3. So, gay marriage is wrong.

However, Perkins contends that this isn’t the argument that he is making. What argument is he making? After the interview aired, some¹ took Perkins to be making an argument like the following.

¹ <http://thinkprogress.org/lgbt/2013/03/27/1783301/top-conservative-says-marriage-equality-will-lead-to-influx-of-immigrant-p>

1. Legalizing gay marriage will lead to the legalization of polygamy.
2. We ought not legalize polygamy.
3. So, we ought not legalize gay marriage.

But perhaps not. Perhaps this passage is best understood in some other way. Perhaps Perkins isn't making a claim about what *would* happen if we legalized gay marriage. Perhaps he is making a claim about what *follows from* the claim that gay marriage ought to be legalized. Perhaps, that is, he is saying that, if we think gay marriage should be legal, then we are committed to thinking that polygamy should be legal as well. That is, perhaps we should understand his argument along the following lines:

1. If we ought to legalize gay marriage, then we ought to legalize polygamy.
2. We ought not legalize polygamy.
3. So, we ought not legalize gay marriage.

Then again, perhaps, rather than providing an argument *against* gay marriage, Perkins is simply providing an *objection* to *somebody else's* argument *for* gay marriage. Perhaps he is objecting to another's premise that all loving relationships deserve the rights of marriage. That is, perhaps his argument is best understood along these lines:

1. If all loving relationships deserve the rights of marriage, then loving polygamous relationships deserve the rights of marriage.
2. Loving polygamous relationships don't deserve the rights of marriage.
3. So, not all loving relationships deserve the rights of marriage.

As we'll see later on, good objections to one of these arguments are not necessarily going to be good objections to any of the others. So, what we ought to say about Perkins' statements here will depend upon how we ought understand them—whether we ought to understand them as implicitly making the first, second, third, or fourth argument above (or whether we ought to understand them in some other way).

LOGIC is the study of arguments. The goal of logic is to give a theory of which arguments are good and which are bad, and to explain what it is that makes arguments good or bad. Since this is our goal, we ought not understand 'argument' in such a way that an argument has to be any good. So, in this class, we'll understand an argument to be any collection of STATEMENTS, one of which is presented as the *conclusion*, and the others of which are presented as the *premises*.

A STATEMENT is a sentence which is capable of being true or false. Questions, commands, suggestions, and exclamations are not statements, since they are not capable of being true or false. It doesn't make sense to say 'It's true that *Damn it!*' or 'It's false that when did you arrive?', so '*Damn it!*' and 'When did you arrive?' are not statements. It *does* make sense to say, *e.g.*, 'It's true that the store closes at eleven', so 'the store closes at eleven' is a statement.

A TEST: given some sentence, *P*, if 'It is true that *P*' makes sense, then *P* is a statement. If 'It is true that *P*' does not make sense, then *P* is not a statement.

1.1 Finding Argumentative Structure

As we saw with Tony Perkins above, given a passage, it is not always obvious whether the passage constitutes an *argument* or not. Given that it is an argument, it is not always obvious which sentences are premises, which are conclusions, and which sentences are extraneous (asides which are not a part of the argument).

Some clues are provided by indicator words. For instance, if any of the following words precede a statement which occurs in an argument, then that statement is almost certainly the argument's conclusion:

therefore, ...	hence, ...	so, ...	thus, ...
this entails that...	as a result, ...	for this reason,...	we may conclude ...
consequently,...	accordingly, ...	this implies that...	this entails that...

Similarly, if any of the following words precede a statement in an argument, then that statement is almost certainly one of the argument's premises:

since...	for...	as...	because...
given that...	may be inferred from...	in that...	for the reason that...
seeing that...	seeing as...	as is shown by...	owing to...

However, often, indicator words are missing, and one must infer from the context and other clues both 1) whether the passage is an argument; and 2) which statements are premises and which are conclusions. For (1), it is important to consider the author's *goal* in writing the passage. If their goal is to persuade the reader, then the passage is an argument. If their goal is anything else, then it is not providing an argument. In particular, if the passage is providing an *explanation*, or providing *information*, then it is not an argument. Stories may very well contain indicator words like 'because' and 'consequently', but this does not mean that they are arguments. For instance, if I tell you

Sabeen is visiting New York because her company was hired to do a workshop there.

my goal is not to persuade you that Sabeen is visiting New York. Rather, I'm simply telling you something about *why* she is there. This is not an argument, even though it contains the indicator word 'because'.

For (2), you should work with a principle of *charity*—figure out which potential argument the author might be making is the *best* argument.

PRINCIPLE OF CHARITY: When searching for argumentative structure within a passage, attempt to find the argument which is most persuasive.

For instance, the following passage lacks indicator words:

We must give up some privacy in the name of security. If the homeland is not secure, terrorist attacks orders of magnitudes larger than 9/11 will find their way to our shores. No amount of privacy is worth enduring an attack like this.

So, there are a few arguments we could see the author making. They might be making this argument:

1. We must give up some privacy in the name of security.
2. If the homeland is not secure, terrorist attacks orders of magnitude larger than 9/11 will find their way to our shores.

3. So, no amount of privacy is worth enduring an attack like this.

Alternatively, they might be making *this* argument:

1. We must give up some privacy in the name of security.
2. No amount of privacy is worth enduring an attack orders of magnitude larger than 9/11.

3. So, if the homeland is not secure, terrorist attacks like this will find their way to our shores.

Finally, they might be making *this* argument:

1. If the homeland is not secure, terrorist attacks orders of magnitude larger than 9/11 will find their way to our shores.
2. No amount of privacy is worth enduring an attack like this.

3. So, we must give up some privacy in the name of security.

Which of these is correct? Well, the first two arguments are just *really bad* arguments. With respect to the first one, ask yourself: "suppose that there would be a large attack, and suppose, moreover, that we must give up privacy in the name of security. Does this tell me anything about the relative worth of privacy and avoiding such an attack?" Perhaps the first premise (we must give up some privacy in the name of security) does tell us something about the relative worth of privacy and attacks like this, but then the second premise would be entirely unneeded. So there wouldn't have been any good reason for the arguer to include it. So this argument looks pretty poor.

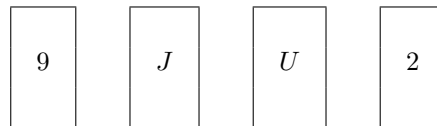
The second argument is even worse. Ask yourself "suppose that we must give up privacy in the name of security, and suppose, moreover, that no amount of privacy is worth enduring an attack worse than 9/11. Does this tell me anything

about whether terrorists will be able to find their way to our shores if we don't secure the homeland?" Again, perhaps the first premise does tell us that there must be *some* reason that we must give up some privacy in the name of security, and perhaps this reason is that if the homeland is not secure, then terrorist attacks will find their way to our shores. However, again, that would make the second premise entirely unnecessary. Moreover, it looks like the only reason one would have for accepting the first premise is that one accepts the conclusion, so the argument is entirely unpersuasive.

The third argument is much stronger. In that argument, both premises are required, and they actually lend support to the conclusion. The principle of charity tells us to attribute this argument to the author.

1.2 Conditionals

Suppose that I have four cards, and I tell you that each of them has a letter printed on one side and a number printed on the other side. I lay them out on the table in front of you, like so:



And I tell you that all four of these cards obey the rule

If there is a vowel printed on one side of the card, then there is an even number printed on the other.

Which of these cards would you have to flip over in order to figure out whether or not I am lying?

Most people get this question wrong. We seem to have a very hard time reasoning about claims like these—claims of the form ‘If P , then Q .’ Claims of this form are known as ‘conditionals.’ That’s because they don’t flat out assert that Q , but rather, they only assert that Q , *conditional on its being the case that P* . Here’s a good way to think about these kinds of claims: ‘if P , then Q ’ says that the truth of P is *sufficient* for the truth of Q .

1.2.1 Necessary and Sufficient Conditions

One condition, X , is **NECESSARY** for another condition, Y , if and only if everything which is Y is also X . That is, X is necessary for Y if and only if there’s no way to be Y without being X .

NECESSARY CONDITION: Being X is necessary for being Y iff there’s no way to be Y without also being X .

NECESSARY CONDITION: The truth of X is necessary for the truth of Y iff there’s no way for X to be true without Y also being true.

For instance, being an American citizen is *necessary* for being the American president. There’s no way to be president without also being an American citizen. For another: being a triangle is *necessary* for being an equilateral triangle. There’s no way to be an equilateral triangle without also being a triangle. The truth of ‘the car is coloured’ is necessary for the truth of ‘the car is red.’ There’s no way for the car to be red without the car being coloured.

One condition, X , is *sufficient* for another condition, Y , if and only if everything which is X is also Y . That is, X is sufficient for Y if and only if there’s no way to be X without also being Y .

SUFFICIENT CONDITION: Being X is sufficient for being Y iff there’s no way to be X without also being Y .

SUFFICIENT CONDITION: The truth of X is sufficient for the truth of Y iff there’s no way for X to be true without Y also being true.

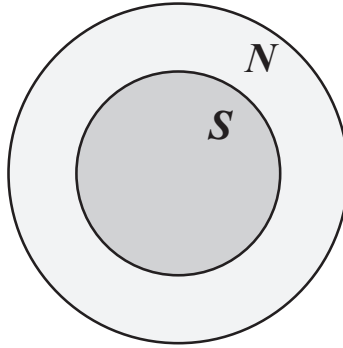


Figure 1.1: : In the diagram, N is necessary for S and S is sufficient for N .

For instance, being French is sufficient for being European. There's no way to be French without also being European. For another: being square is sufficient for being rectangular. There's no way to be square without also being rectangular. And the truth of 'Sabeen is older than 27' is sufficient for the truth of 'Sabeen is older than 20'.

We can visualize this with the Venn Diagram shown in figure 1.1. In that diagram, being inside the circle S is *sufficient* for being inside the circle N —everything inside S is also inside N . And being inside the circle N is *necessary* for being inside the circle S —everything inside S is also inside N . This diagram also makes it clear that S is a sufficient condition for N if and only if N is a necessary condition for S .

1.3 Deductive Validity

Our goal in Logic is to separate out the good arguments from the bad. Here's one very good property that an argument can have: it can be **DEDUCTIVELY VALID**. An argument is deductively valid if and only if the truth of its premises is *sufficient* for the truth of its conclusion.

An argument is **DEDUCTIVELY VALID** if and only if the truth of its premises is sufficient for the truth of its conclusion.

Equivalently, an argument is deductively valid if and only if there is no way for its premises to all be true while its conclusion is simultaneously false.

An argument is **DEDUCTIVELY VALID** if and only if it is impossible for its premises to all be true while its conclusion is simultaneously false.

For instance, each of the following arguments are deductively valid:

1. If Obama is president, then he is the commander in chief.
2. Obama is president.

3. So, Obama is the commander in chief.

1. Gerald is either in Barcelona or in New York.
2. Gerald is not in New York.

3. So, Gerald is in Barcelona.

1. Obama is younger than 30.

2. So, Obama is younger than 40.

(I will often just say that the argument is ‘valid,’ rather than ‘deductively valid.’)

Just because an argument is deductively valid, it doesn’t follow that the conclusion of the argument is true. The third argument above is deductively valid, but its conclusion is false. Obama is not younger than 40. If, however, a deductively valid argument has all true premises, then its conclusion must be true as well. If a deductively valid argument has all true premises, then we say that the argument is DEDUCTIVELY SOUND.

An argument is DEDUCTIVELY SOUND if and only if it is deductively valid and all of its premises are true.

If an argument is deductively sound, then its conclusion will be true. Of all the good making features of arguments that we will discuss today, none is finer than deductive soundness. Of all the honorifics of arguments that we’ll discuss today, there is no finer compliment to an argument than to say that it’s deductively sound.

1.4 Inductive Strength

Not every good argument is deductively valid. For instance, the following argument is not deductively valid:

1. Every human born before 1880 has died.
2. So, I will die.

However, it is still an *excellent* argument. Its premise gives us *spectacular* reason to believe its conclusion. Arguments like these are *inductively strong*, even though they are not deductively valid. An argument is inductively strong if and only if its conclusion is sufficiently probable given its premises.

An argument is INDUCTIVELY STRONG to the extent that its conclusion is probable, given the truth of its premises.

This means that inductive strength, unlike deductive validity, is the kind of thing that comes in degrees. Some arguments can be inductively stronger than others. We could, if we like, set some arbitrary threshold and say that an argument is inductively strong—full stop—if and only if its premises probabilify its conclusion above that threshold. For instance, we could say that

An argument is INDUCTIVELY STRONG if and only if

$$\text{Pr}(\text{conclusion} \mid \text{premises}) > 0.5$$

If an argument is inductively strong with all true premises, then it is *inductively cogent*.

An argument is INDUCTIVELY COGENT if and only if it is inductively strong and all of its premises are true.

- Important Concepts:*
- STATEMENT
 - ARGUMENT
 - PREMISE
 - CONCLUSION
 - NECESSARY CONDITION
 - SUFFICIENT CONDITION
 - CONDITIONAL
 - DEDUCTIVE VALIDITY
 - DEDUCTIVE SOUNDNESS
 - INDUCTIVE STRENGTH
 - INDUCTIVE COGENCY

Chapter 2

Basic Concepts of Logic, Day 2

2.0.1 Review

NECESSARY CONDITION: The truth of X is necessary for the truth of Y iff there's no way for X to be true without Y also being true.

SUFFICIENT CONDITION: The truth of X is sufficient for the truth of Y iff there's no way for X to be true without Y also being true.

An argument is **DEDUCTIVELY VALID** if and only if the truth of its premises is sufficient for the truth of its conclusion.

An argument is **DEDUCTIVELY VALID** if and only if it is impossible for its premises to all be true while its conclusion is simultaneously false.

2.1 Proving Invalidity, take 1

Suppose that you want to show that X is not sufficient for Y . How would you show that? For instance, suppose that you want to show that being human is not sufficient for being a woman. How would you show that? One thing you could do is point to a human man. This is an example of something that is human but not a woman. So, if there is something like that, then it can't be that being human is sufficient for being a woman.

We can do the very same thing with arguments. For instance, suppose that you wanted to show that the truth of the argument's premises is not sufficient for the argument's conclusion. One thing you could do is point to a possibility in which the premises are true, yet the conclusion is false. Call a possibility like that a **COUNTEREXAMPLE** to the validity of an argument.

A **COUNTEREXAMPLE** to the validity of an argument from premises p_1, p_2, \dots, p_N to the conclusion c is a specification of a possibility in which p_1, p_2, \dots, p_N are all true, yet c is false.

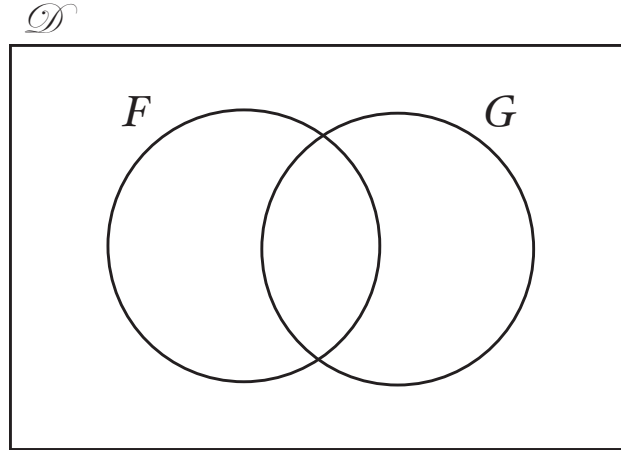


Figure 2.1: : A Venn diagram

2.1.1 Venn Diagrams

Let's talk a bit about Venn diagrams. A Venn diagram has 2 components: a box and some number of labeled circles inside of the box. One example is shown in figure 2.1. In order to interpret this diagram, we must say two things: first, what the *domain*, \mathcal{D} , of the diagram is. That is, we must say what the box contains. Secondly, we must say what each of the circles, F and G , represent.

AN INTERPRETATION of a Venn diagram says
 1) what the domain \mathcal{D} is; and
 2) what each circle represents

In general, a circle will represent a set of things inside the box. An object is represented as belonging to the set if and only if it is inside of the circle. For instance, I could interpret the Venn diagram in figure 2.1 by saying that the domain \mathcal{D} is all animals. That is, every animal is located somewhere inside of the box. I could then say that F is the set of all frogs and that G is the set of all green animals. Alternatively, I could interpret this diagram by saying that the domain is the set of all people, F is the set of all fathers, and G is the set of all grandfathers. Thus, either of the following would be an *interpretation* of the Venn diagram in figure 2.1:

\mathcal{D} = the set of all animals	\mathcal{D} = the set of all people
F = the set of all frogs	F = the set of all fathers
G = the set of all green animals	G = the set of all grandfathers

Let's start with the first interpretation. There are some animals who are neither frogs nor green (zebras). They lie outside of both the circle F and the circle G . There are some animals who are frogs but not green (brown frogs). They lie within the circle F yet outside of the circle G . There are some animals who are both frogs and green (green frogs). They lie inside both the circles F and G . Finally, there are green animals which are not frogs (crocodiles). They lie inside the circle G , but not inside the circle F .

Think now about the second interpretation. There are people who are neither fathers nor grandfathers. There are also people who are fathers but not grandfathers. And there are people who are both fathers and grandfathers. However, there are no people who are grandfathers but not fathers. So there is nobody who is outside of the circle F but still inside of the circle G . Suppose that we want to express the idea that this area is unoccupied. We may do so by crossing out that area of the graph, as shown in figure 2.2. The lines in figure 2.2 make the claim that all G s are F s. Equivalently: they make the claim that there are no G s which are not F . Equivalently: they make the claim that being G is sufficient

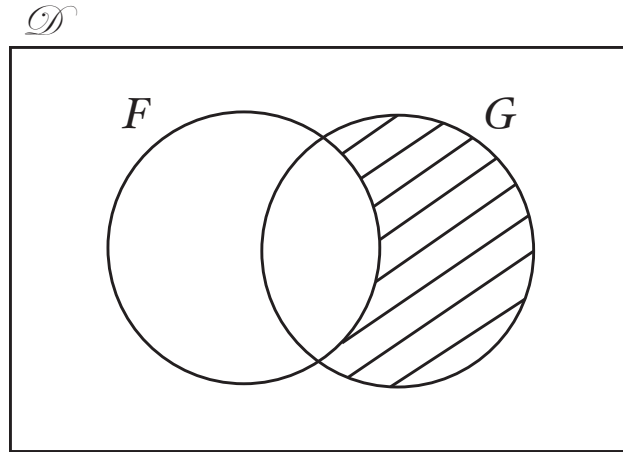


Figure 2.2: : All G s are F s

for being F . Equivalently: they make the claim that being F is necessary for being G . (Make sure that you understand why all of these claims are equivalent.)

Suppose that we wish to say that some area of the Venn diagram *is* occupied. Perhaps, that is, we wish to make the claim that there are some fathers who are not grandfathers. That is, we wish to claim that there are some F s that are not G s. We may indicate this by putting a single 'x' in the diagram which is inside the circle F yet outside of the circle G , as in figure 2.3. In figure 2.3, the 'x' makes the claim that some F s are not G s. Equivalently: it makes the claim that not all

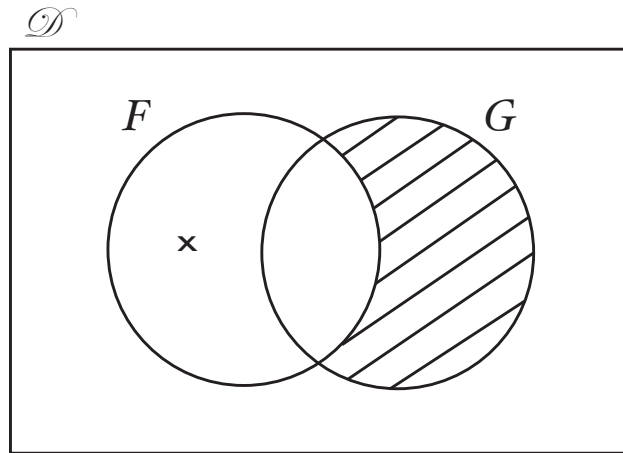


Figure 2.3: : All G s are F s and some F s are not G s

G s are F s. Equivalently: it makes the claim that being F is not sufficient for being G . Equivalently: it makes the claim that being G is not necessary for being F . (Make sure that you understand why all of these claims are equivalent.)

2.1.2 Venn Diagrams, Counterexamples, and Validity

Suppose that we've got an argument from the premises p_1 and p_2 to the conclusion c . This argument is deductively valid if and only if it is impossible for p_1 and p_2 to both be true and yet for c to be simultaneously false. Let's think about this claim using Venn diagrams. Consider the Venn diagram in figure 2.4. Let us give this diagram the following interpretation. The domain \mathcal{D} is the set of all possibilities. If any state of affairs is possible, then that state of affairs is included in \mathcal{D} . P_1 is the set of possibilities in which p_1 is true. P_2 is the set of possibilities in which p_2 is true. And C is the set of possibilities in which c is true.

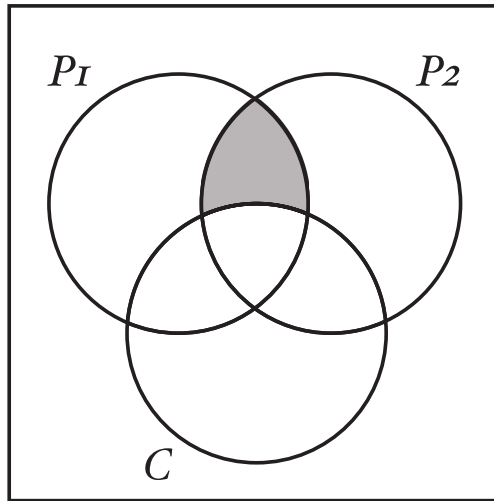


Figure 2.4: : The argument from p_1 and p_2 to c is deductively valid

- \mathcal{D} = the set of all possibilities
- P_1 = the set of possibilities in which p_1 is true
- P_2 = the set of possibilities in which p_2 is true
- C = the set of possibilities in which c is true

The diagram in figure 2.4 makes the claim that there are no possibilities in which both p_1 and p_2 are true, yet c is false. But to say this is just to make the claim that the truth of p_1 and p_2 is sufficient for the truth of c . But to say this is just to make the claim that the argument from p_1 and p_2 to c is deductively valid. (Make sure that you understand why these three claims are equivalent.)

On the other hand, suppose that there is some possibility in which both p_1 and p_2 are true, yet c is false. This claim is illustrated with the Venn diagram in figure 2.5. (There, we are using the same interpretation that we used for the Venn diagram in figure 2.4.) If the claim made in figure 2.5 is correct—if there is some possibility in which p_1 and p_2 are both true, yet c is false—then the claim made in figure 2.4—that there is no possibility in which p_1 and p_2 are both true yet c is false—cannot be true. So, if the claim made in figure 2.5 is correct, then the argument from p_1 to p_2 to c cannot be deductively valid. But the claim made in figure 2.5 is just the claim that there is some *counterexample* to the validity of the argument from p_1 and p_2 to c . So, if there is a counterexample to the validity of an argument, then the argument cannot be deductively valid.

This affords us a new definition of deductive validity which is equivalent to the earlier two.

An argument is DEDUCTIVELY VALID if and only if it has no counterexample.

(Make sure that you understand why this new definition is equivalent to the earlier two.) So, one way to establish that an argument is deductively invalid is to provide a counterexample.

Consider the following arguments:

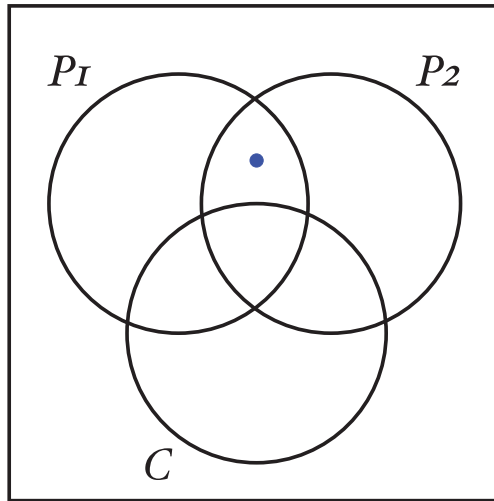


Figure 2.5: : The argument from p_1 and p_2 to c is deductively invalid

1. The earth moves around the sun.
 2. So, the sun does not move.
-
1. Raising the minimum wage reduces employment.
 2. Obama wants to raise the minimum wage.
 3. So, Obama wants to reduce employment.
-
1. We have not discovered life on other planets.
 2. So, there is no life on other planets.

Each of these arguments are deductively invalid. And we may demonstrate that they are deductively invalid by providing the following counterexamples. For the first argument, consider the following state of affairs: the earth moves around the sun, and the sun itself moves. In this state of affairs, the premise of the first argument is true, yet the conclusion is false. So, since this state of affairs is possible (it is actual), the argument is invalid. For the second argument, consider the following state of affairs: raising the minimum wage *does* reduce employment; however, Obama does not know this. Obama wants to raise the minimum wage, but does not want to reduce employment. Since this state of affairs is possible (though perhaps not actual), the argument is invalid. For the third argument, consider the following state of affairs: life on other planets is hidden somewhere we would be unlikely to have yet found it. Though we have not yet found it, it is still out there. In this state of affairs, the premise of the argument is true, yet its conclusion is false. Since this state of affairs is possible (though perhaps not actual), the argument is invalid.

Chapter 3

Basic Concepts of Logic, Day 3

3.1 Review

An argument is DEDUCTIVELY VALID if and only if the truth of its premises is sufficient for the truth of its conclusion.

An argument is DEDUCTIVELY VALID if and only if it is impossible for its premises to all be true while its conclusion is simultaneously false.

A COUNTEREXAMPLE to the validity of an argument is a specification of a possibility in which the premises of the argument are all true, yet the conclusion of the argument is false.

An argument is DEDUCTIVELY VALID if and only if it has no counterexample.

3.2 Formal Deductive Validity

Up until this point, both Hurley and I have been defining DEDUCTIVE VALIDITY as *necessary truth-preservation*—that is, a valid argument is one such that, necessarily, if its premises are all true, then its conclusion will be true as well. In §1.5 of Hurley, however, a new idea shows up: that “the validity of a deductive argument is determined by the argument form.”¹ Understanding this definition requires understanding what an argument form is, as well as what it is for a given argument to *have* a certain form.

Let’s start with the idea of a VARIABLE. A variable is just a kind of place-holder for which you can substitute a certain kind of thing—perhaps a number, perhaps a statement, perhaps a name, perhaps something else entirely. Those entities that can take the place of the variable are the variable’s possible *values*. For instance, we could use ‘*x*’ as a variable whose possible values are names. We could similarly use ‘*p*’ as a variable whose possible values are whole statements. Specifying a variable means specifying what its possible values are—those are known as the values over which the variable *ranges*.

Next, consider the idea of a STATEMENT FORM. A statement form is a string of words containing variables such that, if

¹ Hurley, §1.5.

the variables are substituted for the appropriate values, then you get a statement. For instance, if ‘ p ’ and ‘ q ’ are variables ranging over statements, then

if p , then q

is a statement form. If we plug in statements for p and q , then we get a **SUBSTITUTION INSTANCE** of this statement form. For instance, the following is a substitution instance of ‘if p , then q ’:

If Zoë is hungry, then Barcelona is in France.

Here, we have set $p =$ ‘Zoë is hungry’ and $q =$ ‘Barcelona is in France’. Since both of these are statements, they are both appropriate values for p and q . On the other hand, this is *not* a substitution instance of ‘if p , then q ’:

If Bob, then Mary.

Since ‘Bob’ and ‘Mary’ are not statements, the variables p and q do not range over them, and they may not be substituted in for p and q . Similarly, if ‘ x ’ and ‘ y ’ are variables ranging over names, then

x loves y

is a statement form. A substitution instance of this statement form is

Bob loves Mary.

Since, if we set $x =$ ‘Bob’ and $y =$ ‘Mary’, in the statement form ‘ x loves y ’, we get the statement ‘Bob loves Mary’.

Finally, a **ARGUMENT FORM** is a collection of statements and/or statement forms, one of which is presented as the conclusion, the others of which are presented as the premises. The following are all argument forms (where ‘ p ’ and ‘ q ’ are variables ranging over statements, and ‘ x ’ and ‘ y ’ are variables ranging over names).

$$\frac{p \text{ and } q}{\text{So, } q}$$
$$\frac{\text{If } p, \text{ then } q}{p}$$
$$\frac{p}{\text{So, } q}$$
$$\frac{x \text{ loves } y}{\text{So, } y \text{ loves } x}$$

If we look at the first and second argument form, we might notice that it looks as though we can figure out that, no matter which statements we substitute in for p and q , the resulting argument will be valid. Additionally, we might notice, when we look at the third argument, that it looks as though we can figure out that, no matter which names we substitute in for x and y , the resulting argument will be invalid. This observation suggests the following incredibly bold and daring and provocative thesis about deductive validity: what it is for an argument to be deductively valid is for it to be a substitution instance of a form which necessarily preserves truth.

A bit more carefully: let’s start by defining the notion of a *deductively valid argument form*. An argument form is deductively valid if and only if every substitution instance of the argument form has the following property: if the premises are all true, then the conclusion is true as well.

An argument form is **DEDUCTIVELY VALID** if and only if every substitution instance of the argument form with all true premises has a true conclusion as well.

An argument form is **DEDUCTIVELY INVALID** if and only if there is some substitution instance with true premises and a false conclusion.

Then, we may define a corresponding notion of *formal deductive validity*. An argument is formally deductively valid if and only if it is a substitution instance of a deductively valid argument form.

An argument is **FORMALLY DEDUCTIVELY VALID** if and only if it is a substitution instance of a deductively valid argument form.

Correlatively, we may define the notion of *formal deductive invalidity*. An argument is formally deductively invalid if and only if it is *not* a substitution instance of a deductively valid argument form.

An argument is **FORMALLY DEDUCTIVELY INVALID** if and only if it is *not* a substitution instance of a deductively valid argument form.

Here's the bold and daring and provocative thesis: deductive validity *just is* formal deductive validity.

BOLD AND DARING AND PROVOCATIVE THESIS: An argument is deductively valid if and only if it is formally deductively valid.

To see some *prima facie* motivation for this thesis, consider the examples of deductively valid arguments that we encountered last time.

If Obama is president, then he is the commander in chief.
Obama is president.

So, Obama is the commander in chief.

Either Gerald is in Barcelona or Gerald is in New York.
It is not the case that Gerald is in New York.

So, Gerald is in Barcelona.

Each of these arguments has a deductively valid argument form, namely,

If p , then q
 p

So, q

Either p or q
It is not the case that q

So, p

Despite this strong *prima facie* motivation, the bold and daring and provocative thesis is still controversial; some philosophers dispute it. Nevertheless, I will assume it in what follows. As it turns out, very little of what we will do in this class will depend upon the thesis.

3.3 Proving Invalidity, take 2

Consider the following arguments:

If Russia invades Ukraine, then there will be war.
 It is not the case that Russia will invade Ukraine.

 So, it is not the case that there will be war.

If it's raining, then (it's raining and Romney is president).
 It is not the case that it's raining.

 So, it's not the case that (it's raining and Romney is president).

Both of these arguments are of the same general *form*, namely

If p , then q
 It is not the case that p

 So, it is not the case that q

(In the first argument, p = 'Russia invades the Ukraine' and q = 'there will be war'. In the second argument, p = 'it's raining' and q = 'it's raining and Romney is president'.)

However, this general form is invalid. We can *show* that the general form is invalid by pointing out that it has a substitution instance with true premises and a false conclusion, namely,

If Romney is president, then a man is president.
 It is not the case that Romney is president.

 So, it is not the case that a man is president.

(where p = 'Romney is president' and q = 'a man is president'.) In this substitution instance, the premises are true, yet the conclusion is false. Therefore, the argument form 'if p , then q ; it is not the case that p ; therefore, it is not the case that q ' is invalid.

We will call a substitution instance of an argument form which has true premises and a false conclusion a **FORMAL COUNTEREXAMPLE** to the deductive validity of the argument form.

A **FORMAL COUNTEREXAMPLE** to the deductive validity of an argument form is a substitution instance of the argument form which has all (actually) true premises and an (actually) false conclusion.

This affords us another (equivalent) definition of the deductive validity of argument forms

An argument form is **DEDUCTIVELY VALID** iff it has no formal counterexample.

3.4 Formal Deductive Invalidity

Earlier, I said that

An argument is **FORMALLY DEDUCTIVELY INVALID** if and only if it is *not* a substitution instance of a deductively valid argument form.

What I *didn't* say, because it is *false*, was

THIS IS FALSE!!!
 An argument is **FORMALLY DEDUCTIVELY INVALID** if and only if it is a substitution instance of a deductively invalid argument form.
THIS IS FALSE!!!

To see why this is false, note that the argument considered above, namely,

$$\frac{\begin{array}{l} \text{If it's raining, then it's raining and Romney is president.} \\ \text{It's not raining.} \end{array}}{\text{So, it's not the case that both it is raining and Romney is president.}}$$

is a substitution instance of the deductively invalid form

$$\frac{\begin{array}{l} \text{If } p, \text{ then } q \\ \text{It is not the case that } p \end{array}}{\text{So, it is not the case that } q}$$

However, it is *also* a substitution instance of the deductively *valid* form

$$\frac{\begin{array}{l} \text{If } p, \text{ then both } p \text{ and } q \\ \text{It is not the case that } p \end{array}}{\text{So, it is not the case that both } p \text{ and } q.}$$

This argument form is deductively valid because the conclusion follows straightaway from the second premise. If it's not the case that p , then it can't be the case that p and q . The first premise is unnecessary, but the argument form is still deductively valid.

For another example, consider the deductively valid argument

$$\frac{\begin{array}{l} \text{If Clinton is president, then a woman is president.} \\ \text{Clinton is president.} \end{array}}{\text{So, a woman is president.}}$$

This is a deductively valid argument, since it is of the valid form (known as *modus ponens*)

$$\frac{\begin{array}{l} \text{If } p, \text{ then } q. \\ p. \end{array}}{\text{So, } q.}$$

(with p = 'Clinton is president' and q = 'a woman is president'.) However, it is *also* of the *invalid* form

$$\frac{\begin{array}{l} p. \\ q. \end{array}}{\text{So, } r.}$$

(with p = 'If Clinton is president, then a woman is president', q = 'Clinton is president', and r = 'a woman is president') So, formally deductively valid arguments can have *invalid* forms. In fact, *every* argument whatsoever will have an invalid form. What it takes to show that an argument is deductively invalid is that you've uncovered the *right* form. How much of the form of the argument must we represent in order to be sure that we've uncovered the *right* form? That difficult question will be the one we face when we learn about propositional and predicate logic.

Most Important To Understand:

- COUNTEREXAMPLES and their relationship to DEDUCTIVE VALIDITY
- FORMAL DEDUCTIVE VALIDITY
- FORMAL COUNTEREXAMPLES and their relationship to FORMAL DEDUCTIVE VALIDITY

Very Important to Understand:

- just because an argument is of an invalid form, this does not mean that the argument is formally invalid.

Important to Understand:

- VENN DIAGRAMS

Chapter 4

Dialectics

We are now in a position to recognize two ways to resist the conclusion of a deductive argument (*i.e.*, an argument which is intended by the arguer to be deductively valid):

1. You can reject, or offer an independent argument against, one of the argument's premises.
2. You can deny the validity of the argument, by either providing a counterexample to the argument or by providing a *formal* counterexample to the argument's form.

However, doing this will not in general settle the debate. If you take option (1), the arguer may wish to offer *another* argument in favor of the disputed premise. At that point, you may either reject a premise of their new argument, or question its validity. If you take option (2), the arguer may resist your counterexample, or insist that you have not extracted the proper argument form. The back-and-forth which ensues can very quickly spiral away from the original conclusion being debated. This back-and-forth is known as a DIALECTIC. In any argument, it is important to keep track of the dialectic; failure to do so will lead to confusion about what has been shown and what has not been shown.

Imagine that Rohan presents the following argument for the conclusion that we do not have free will:

1. Actions are free only if we could have not performed them.
 2. The laws of nature determine our actions, so we cannot fail to perform them.
-
3. So, our actions are not free.

Harry then rejects premise (1). Premise (1) claims that the ability to do otherwise is a necessary condition for an action being free. So Harry presents a case in which you are free without the ability to do otherwise. He says:

Suppose that Jones has the kind of freedom which we, according to Rohan, lack. However, a mad scientist named "Dr. Demento" has placed a computer chip in Jones' brain which, when activated, takes away Jones' freedom and makes him do whatever Dr. Demento wants him to do. Initially, the computer chip is inoperative, and does nothing. If Jones does not kill Smith, then Dr. Demento will activate the chip and force Jones to kill Smith. However, if Jones kills Smith on his own, then Dr. Demento will leave the chip deactivated. As it is, Jones kills Smith of his own accord.

Then, Harry argues:

4. Jones killed Smith freely.
 5. Jones could not have failed to kill Smith.
-
6. So, it is not the case that our actions are free only if we could have failed to perform them.

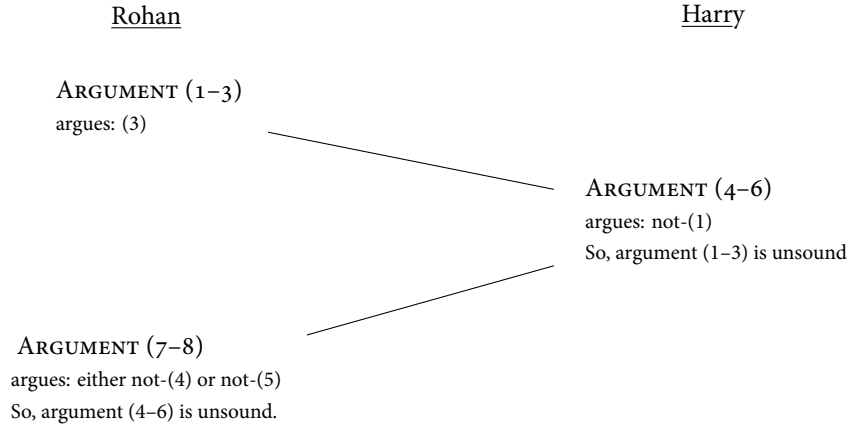


Figure 4.1: : Rohan and Harry's dialectic

Suppose that we accept Harry's argument. Where does this leave us? Has Harry shown that we are free?

No. Harry has merely shown that Rohan's premise (1) is false. This shows that Rohan's argument isn't sound. But showing that Rohan's argument isn't sound *does not* show that his conclusion is false. It could still be that our actions are not performed freely. All we've learned from this exchange—if we agree with Harry—is that Rohan's *argument* for the conclusion that we are not free is not sound.

Suppose that, at that point, Rohan responds to Harry as follows:

7. No physical device, such as a computer chip, can interfere with the exercise of a truly free will.

8. So, either it is not the case that Jones killed Smith freely or it is not the case that Jones could not have failed to kill Smith.

Now, suppose that we accept Rohan's new argument. Where would this leave us? If Rohan's conclusion, (8), is right, then either Harry's premise (4) is false—in which case, his argument (4-6) is unsound—or Harry's premise (5) is false—in which case, his argument (4-6) is unsound. Either way, the argument is unsound. So Harry's objection to Rohan's argument (1-3) has failed. However, just because we accept Rohan's argument (7-8), this *does not* mean that we must accept Rohan's original argument, (1-3). We may think that Harry's argument *against* Rohan's argument (1-3) is not a good one; but that doesn't mean that we must think that Rohan's argument (1-3) has *no* problems with it. We could, if we like, go back to the original argument and raise our own objections to it.

We can picture this back and forth with the diagram in figure 1: first, Rohan presents the argument (1-3), with the aim of establishing (3). Harry responds with an argument of his own that premise (1) of Rohan's argument is false. Rohan responds with an argument *against* Harry's argument against his original argument.

We could, of course, keep going. And philosophers usually do keep going (but I'll spare you). The important thing to keep in mind is that *it is important what each argument is arguing for, and how it fits into the larger dialectic*. You should *not* conclude, just from the fact that Rohan's *final* argument was good, that his *first* argument was. Nor should you conclude from the fact that Harry's objection to Rohan's argument was good that Rohan's conclusion is false.

4.1 Rules for Dialectics

Dialectics in Philosophy can get quite convoluted; and there is not any settled, universally accepted theory about which kinds of moves are dialectically appropriate and which are not. However, we can say some mostly uncontroversial things about how dialectics proceed—confining ourselves, for the moment, to arguments which are intended to be *deductively valid*. Any dialectic will begin with an *opening move*. An opening move may be either an *opening argument* or an *opening statement of a position*.

Opening Move

At the beginning of a dialectic, you may:

1. State your position; or
2. State and provide an argument for your position.

Once one party to the dialectic has stated or argued for their position, the other party to the dialectic may object to the position which has been stated. How it is permissible for them to object depends upon whether their interlocutor (the person with whom they are arguing) has provided an argument for their position, or rather just stated it. Suppose that your interlocutor has simply stated their position. At that point, you may do one of three things: accept your interlocutor's position (thereby bringing the dialectic to an end); reject their position (thereby prompting them to either defend their position further or simply agree to disagree); or provide an independent argument against their position (thereby prompting your interlocutor to respond to your argument).

Responding to a Position

In response to your interlocutor's statement of a position, you may:

1. Accept their position;
2. Reject their position; or
3. Offer an argument against their position.

Suppose, on the other hand, that your interlocutor has not simply stated their position, but rather provided an *argument* for their position. Then, if you disagree with their position, it will not be acceptable for you to simply reject their position or simply provide an argument against their position—you must additionally *respond* to their argument. You must say whether you reject one of its premises or whether you think that the argument is invalid, and why.

Responding to an Argument

In response to your interlocutor providing a deductive argument for their position, you may:

1. Accept their conclusion;
2. Reject one of their argument's premises;
3. Provide an argument against one of their premises; or
4. Claim that their argument is invalid, and provide a (formal) counterexample to the validity of their argument.

Suppose, on the other hand, that *you* state your position, and your interlocutor rejects it. In that case, you may either provide an argument for your position, or simply “agree to disagree” (thus bringing the dialectic to a close).

Responding to a Rejection of a Position

In response to your interlocutor rejecting your position, you may:

1. Provide an argument for your position; or
2. Agree to disagree.

Suppose that you provide an argument for your position, or else that you provide an argument for some claim or other, and your interlocutor provides an argument of their own—either against your position or against one of the premises in your argument. At that point, you must engage with their argument by either accepting their conclusion and therefore revising your own position; rejecting one of the premises of their argument; or else claiming that their argument is invalid.

Responding to an Argument against your Position/Argument

In response to your interlocutor providing an argument against either your position or one of the premises in your argument, you may:

1. Accept their conclusion and revise your position;
2. Reject one of their argument's premises;
3. Provide an argument against one of their premises; or
4. Contend that their argument is invalid, and provide a (formal) counterexample to the validity of their argument.

What if somebody says that your argument is invalid and provides a counterexample? Well, at that point, you face three options: you may either accept the counterexample and abandon the argument; or you may either contend that their counterexample is not a genuine possibility (this will open up a whole can of worms); or that it is not a possibility in which your premises are true and your conclusion is false.

Responding to a Counterexample

In response to your interlocutor providing a counterexample to establish the invalidity of your argument, you may:

1. Accept the counterexample and abandon the argument;
2. Contend that the putative counterexample isn't a possibility; or
3. Contend that the putative counterexample is not a possibility in which the premises are true and the conclusion false.

What if they provide a formal counterexample? At that point, you may accept the formal counterexample and abandon the argument. Alternatively, you may contend that the formal counterexample and your argument do not in fact share an argument form. Alternatively, you may contend that, while they do share an argument form, either the formal counterexample's premises are false or else its conclusion is true. Finally, you may contend that, while the formal counterexample and your argument share a form, and while that form is an invalid form, your argument has *another* form which is a valid argument form.

Responding to a Formal Counterexample

In response to your interlocutor providing a formal counterexample to establish the formal invalidity of your argument, you may:

1. Accept the formal counterexample and abandon the argument;
2. Contend that the formal counterexample and your argument do not in fact share an argument form;
3. Contend that the formal counterexample's premises are not all true;
4. Contend that the formal counterexample's conclusion is not false;
5. Accept that the formal counterexample shows *one* of the argument's forms to be invalid, but insist that the argument has *another* form which *is* valid.

By repeated application of these rules, the dialectic can go on and on and on until one party is persuaded or else agrees to disagree.

4.1.1 Example: A Failed Dialectic

To get a feel for these rules, let's look to some cases in which they are violated. These are cases in which the dialectic *breaks down*. Here's a dialogue between person *A* and person *B* on the issue of abortion:

A: Abortion is the killing of a human life; all killing of human life is immoral; so, abortion is immoral.

B: Well, not *all* killing of human life is immoral. For instance, any time you scratch yourself, you are killing living human cells. But, presumably, you don't think that that's immoral. So you don't think that all killing of human life is immoral.

A: Are you seriously comparing killing a fetus to scratching yourself? Those two things are completely different! Just because I think it's permissible to scratch myself doesn't mean that I have to think that it's permissible to kill a fetus!

This dialectic has broken down. It began with *A* making an argument for their conclusion that abortion is immoral. This argument has two premises, and goes like this:

1. Abortion is the killing of human life.
2. All killing of human life is immoral.

3. So, abortion is immoral.

This is allowed by the rule **Opening Move** (point 2). In response, *B* rejects premise 2. They contend that it is false that all killing of human life is immoral. And they provide an independent argument against this premise. Their independent argument goes like this:

4. Scratching yourself is the killing of human life.
5. Scratching yourself is not immoral.

6. So, not all killing of human life is immoral.

This is allowed by **Responding to an Argument** (point 3). At this point, by **Responding to an Argument against your Argument** *A* may either: a) accept *B*'s point and offer a new argument for their conclusion; b) reject one of the premises in *B*'s argument; or c) claim that *B*'s argument is invalid. However, *A* does none of these things. Rather, they claim that scratching yourself is not analogous to abortion. This may be so, but that scratching yourself is analogous to abortion

was not one of the premises of *B*'s argument, so saying this does not in any way impugn *B*'s argument. Nor is it clear how the disanalogies between scratching yourself and abortion would make *B*'s argument invalid.

A hypothesis: what's happened here is that *A* lost track of the dialectic. *A* began interpreting *B* as though *B* were trying to offer an argument for the conclusion that abortion is permissible; that is what led them to say:

Just because I think it's permissible to scratch myself doesn't mean that I have to think that it's permissible to kill a fetus!

But, while *B*'s argument does not establish that abortion is permissible, *B* was not *trying* to establish that abortion is permissible. *B* is simply trying to establish that one of *A*'s premises is false.

Chapter 5

Informal Fallacies

A **FALLACY** is an error in reasoning. Simply because an argument contains false premises, this is not enough to make the argument fallacious. It must make a mistake in *inferring* the conclusion from the premises. When an argument commits a fallacy, something has gone wrong with the *inference* from the premises to the conclusion.

A **FORMAL FALLACY** is a fallacy that we may diagnose as bad simply by looking at the argument's *form*. For instance, the following is a formal fallacy:

1. If Russia invades Ukraine, then Russia wants war.
2. Russia wants war.

3. So, Russia will invade Ukraine.

We can diagnose this argument as fallacious by noting that it is of a deductively invalid form, namely,

1. If p , then q .
2. q .

3. So, p .

(where p = 'Russia invades Ukraine', and q = 'Russia wants war'.) We may show that this form is invalid by pointing to a substitution instance on which the premises are uncontroversially true, yet the conclusion is uncontroversially false. The following example will do:

1. If Sylvester Stallone was governor of California.
then a former action star was governor of California.
2. A former action star was governor of California.

3. So, Sylvester Stallone was governor of California.

This argument has all true premises, and a false conclusion. And it is a substitution instance of the argument form 'if p , then q ; q ; so, p ' (with p = 'Sylvester Stallone was governor of California' and q = 'a former action star was governor of California'). So the argument form is invalid.

However, there are other common fallacies which we may not detect merely by inspecting the arguments form; we must additionally look to the *content* of the argument.

An **INFORMAL FALLACY** is a fallacy which we cannot diagnose by simply inspecting the argument's form; in order to diagnose the fallacy, we must look additionally to the argument's *content*.

For instance, here is an informal fallacy:

1. Zoë has more energy than Daniel.
2. Energy is proportional to mass.

3. Zoë has more mass than Daniel.

This argument is fallacious; however, if we try to extract its logical form, we might only get the following argument form, which appears to be deductively valid.

1. x has more F than z .
2. F is proportional to G .

3. x has more G than z .

This is an example of the informal fallacy of *equivocation*. The word ‘energy’ has two different meanings in the original argument. In premise 1, it means something like ‘the personality trait of being excitable’ (‘personality energy’, for short); whereas, in premise 2, it means ‘the theoretical physical quantity of energy’ (‘physical energy’, for short). The argument will be valid so long as we mean the same thing by ‘energy’ throughout. However, while both of the following arguments are valid, neither are at all persuasive.

1. Zoë has more physical energy than Daniel.
2. Physical energy is proportional to mass.

3. Zoë has more mass than Daniel.

1. Zoë has more personality energy than Daniel.
2. Personality energy is proportional to mass.

3. Zoë has more mass than Daniel.

Both of these arguments are valid; however, there is no reason whatsoever to accept their premises. In the first argument, premise 1 is obviously false. Just because Zoë is more excitable than Daniel, that doesn’t mean that she has more physical energy than he does. In the second argument, premise 2 is obviously false. Just because $E = mc^2$, this doesn’t mean that the personality trait of being excitable is proportional to mass.

There are three broad classes of informal fallacies that we will study here. They are FALLACIES OF IRRELEVANCE, FALLACIES INVOLVING AMBIGUITY, and FALLACIES INVOLVING UNWARRANTED ASSUMPTIONS. For each informal fallacy we study, we should be on our guard and not be too hasty to call some piece of reasoning fallacious simply because it fits the general mold. For most of these fallacies, though there are a great many arguments that fit the basic mold and which are incredibly poor arguments, there are also some arguments that fit the basic mold but which are perfectly good arguments. For each fallacy, we’ll have to think about *why* an argument of that general character is bad, when it is bad, and why it might be good, when it is good.

5.1 Fallacies of Irrelevance

5.1.1 Argument Against the Person (Ad Hominem)

This is a fallacy in which one fails to properly engage with another person’s reasoning. An *ad hominem* is a way of responding to an argument that attacks the *person* rather than the *argument*. It comes in three flavors: firstly, an ABUSIVE AD HOMINEM attempts to discredit an argument by discrediting the person making that argument.

Example: After Sandra Fluke argued before Congress that healthcare should include birth control, since it is used to combat ovarian cysts, Rush Limbaugh responded with: “What does it say about the college

co-ed Sandra Fluke, who goes before a congressional committee and essentially says that she must be paid to have sex, what does that make her? It makes her a slut, right? It makes her a prostitute.”

Secondly, a *CIRCUMSTANTIAL AD HOMINEM* attempts to discredit an argument by calling attention to some circumstantial features of the person making the argument (even though those features might not in and of themselves be bad-making features).

Example: Robert Kennedy argues that we shouldn't have a wind farm in the Nantucket Sound because the wind turbines would kill thousands of migrating songbirds and sea ducks each year. However, Robert Kennedy is only opposed to the wind farm because he and his family have property in Hyannis Port whose value would be hurt by the building of the wind farms. So songbirds and sea ducks are just a distraction; we should build the wind farm.

Thirdly, a *TU QUOQUE* attempts to discredit an argument by pointing out that the person making the argument themselves hypocritically rejects the conclusion in other contexts. For example,

Example: Newt Gingrich called for Bill Clinton to be impeached for lying about his affair with Monica Lewinsky. However, at the same time, Gingrich was lying about his own affair. So, Clinton ought not to have been impeached.

Why this is fallacious: the argument swings free of the person who happens to be making it. Even if the person who happens to be advancing the argument has some personal flaw, or stands in principle, somebody else without those flaws could make the very same argument.

A closely-related but non-fallacious argument: If the issue under discussion is whether the arguer is a good person, then personal attacks may not be fallacious; they might be entirely relevant to the question at hand. If the arguer is appealing to their own authority, then questioning the arguer's authority could be a perfectly reasonable way of rejecting one of the argument's premises.

5.1.2 Straw Man

A *STRAW MAN FALLACY* occurs when one misrepresents somebody else's position or argument (usually making it more simplistic or naive than their actual position or argument), and then argues against the misrepresented position or argument, rather than the person's actual position or argument.

Example: Mr. Goldberg has argued against prayer in the public schools. Obviously Mr. Goldberg advocates atheism. But atheism is what they used to have in Russia. Atheism leads to the suppression of all religions and the replacement of God by an omnipotent state. Is that what we want for this country? I hardly think so. Clearly Mr. Goldberg's argument is nonsense.

Why this is fallacious: simply because a misrepresentation of somebody's view is false, this doesn't give us any reason to think that their correctly represented view is false.

5.1.3 Appeal to Force (Ad Baculum)

An *AD BACULUM* fallacy occurs when a conclusion is defended, or an argument attacked, by making a threat to the well-being of those who make it (or implying that bad things will happen to those who accept the conclusion or argument).

Example: Anusar argues that workers are entitled to more of the firm's profits than management because they contribute more to the product. But no firm wants to hire an employee with radical views like that.

That's why Anusar's been unemployed for so long. So it doesn't matter how much workers contribute; workers are entitled to what they get. If you think otherwise, you'll end up out of work like Anusar.

Why this is fallacious: simply because you can avoid harm by rejecting a certain statement or argument, that doesn't give you any reason to suppose that the statement is *false* or that the argument is *bad*. So the premises of an *ad baculum* argument don't give you any reason to believe that the conclusion is true; even though they might make it a good idea to *pretend* that the conclusion is true.

A closely-related but non-fallacious argument: if the harm being threatened is *relevant* to the truth of the conclusion, then an *ad baculum* might be perfectly reasonable. E.g.,

Not An Example: You shouldn't smoke, or else you'll likely get lung cancer.

5.1.4 Appeal to the People (Ad Populum)

AD POPULUM is a fallacy which attempts to argue for a conclusion by in some way appealing to people's innate desire to be accepted or desired by others. It can occur when an arguer appeal to nationalism, as in

Example: We Americans have always valued freedom. We understand that this freedom comes with a price, but it is a price we are willing to pay. True Americans resist the more extreme measures of the war on terror, like the Patriot Act. So, we need to repeal the Patriot Act.

Or, it could occur when an arguer appeals to the audience's desire to have mainstream opinions, as in

Example: I can't believe that you think we should curtail the freedom of speech in order to protect minority rights. Only fascists and kooks think that! So you should really reconsider your opinion.

Why this is fallacious: That holding a certain opinion will make you stand out from the group does not, on its own, provide any reason to think that that opinion is false. Even though most people generally *want* to be included in the group and hold the majority opinion, this doesn't give you any reason to think that the majority opinion is *true*.

A closely-related but non-fallacious argument: If the arguer is pointing to the consensus of people who are in a better position to evaluate the evidence, then they could be making an appeal to authority, which needn't be fallacious.

Not An Example: The biological community has reached a near-unanimous consensus that the hypothesis of evolution by natural selection is correct. Since they are experts on the subject, we should trust them that there is excellent reason to believe in the hypothesis of evolution by natural selection.

5.1.5 Appeal to Ignorance (Ad Ignorantiam)

AN APPEAL TO IGNORANCE occurs when somebody argues in favor of a conclusion that we don't antecedently have any reason to accept (or which we antecedently have reason to reject) on the grounds that there's no evidence either way. Alternatively, it occurs when somebody argues against a conclusion that we don't antecedently have any reason to reject (or which we antecedently have reason to accept) on the grounds that there's no evidence either way.

Example: The studies purporting to show that barefoot running is good for you have been discredited. However, there aren't any studies showing that it's *not* good for you—the jury's still out. So, you should keep running barefoot.

Example: There's no evidence showing that there's life on other planets. So we should stop looking—it's not there.

Why it's fallacious: If we don't antecedently have any reason to accept or reject a claim, then, in the absence of evidence, we should suspend judgment. Just because no reason has been offered to think that the conclusion is false, that doesn't mean that we should think that it is true. Similarly, just because no reason has been offered to think that the conclusion is true, that doesn't mean that we should think that it is false.

Two closely-related but non-fallacious arguments:

- a) If you *do* have antecedent reason to accept or reject a conclusion, then the absence of any defeating evidence can provide good reason to continue believing the conclusion.

Not an Example: The studies showing that circumcision reduces HIV transmission were badly methodologically flawed, so circumcision probably doesn't reduce HIV transmission.

- b) If certain evidence was *to be expected* if a certain statement were true (false), and we don't find that evidence, that can count as good reason to think that the statement is false (true).

Not an Example: If he had been poisoned, the toxicology report would have revealed poison in his blood; it didn't; so, he probably wasn't.

5.1.6 Red Herring (Ignoratio Elenchi)

The RED HERRING fallacy occurs when somebody presents premises which might be psychologically compelling, but which are irrelevant to the conclusion. As such, every other fallacy in this section constitutes an instance of the red herring fallacy. It is the most general fallacy of irrelevance. (Nevertheless, we should use 'red herring' to refer only to fallacies of irrelevance which do *not* fall into the other categories of this section. If a fallacy falls into one of the other categories, identifying it as a red herring, on, *e.g.*, a test, will not be correct.)

Example: Jamal says that we shouldn't have a central bank because central banking is responsible for the economic fluctuations of the business cycle. But people have been banking for centuries. Bankers aren't bad people, and they provide the valuable service of providing credit to people who don't have their own capital.

Chapter 6

Informal Fallacies, Day 2

6.1 Fallacies Involving Ambiguity

These are all fallacies that arise because of some ambiguity in the language appearing in the statements in the argument.

6.1.1 Equivocation

The fallacy of EQUIVOCATION occurs when a single word is used in two different ways at two different stages of the argument, where validity would require that the word be used in the same way at both stages.

Example: In order to be a theist, as opposed to an agnostic, you must claim to *know* that God exists. But, even if you *believe* that God exists, you don't *know* it. Thus, you shouldn't be a theist. It follows that you should either be an agnostic or an atheist. However, once you've ruled out theism, what is there to be agnostic about? Once theism has been ruled out, atheism is the only remaining position. Therefore, you shouldn't be agnostic. Hence, you should be an atheist.

'Agnostic' can mean either 1) not claiming knowledge that God exists, or 2) not having belief either way about whether God exists. The first stage of the argument relies upon the first meaning; while the second stage of the argument relies upon the second meaning.

Why it's fallacious: the argument gives the appearance of validity if we don't realize that the word is being used in two different senses throughout the argument. However, once we are clear about what the words mean, the argument either becomes invalid, or else has an obviously false premise.

A closely related but non-fallacious argument: If an argument uses a word that has multiple meanings, but the premises are all true on a single disambiguation, then the argument does *not* equivocate.

Not an Example: [Suppose that I am a fisherman who works at the riverside] I work at the bank, and there are fish at the bank. So there are fish where I work.

6.1.2 Amphiboly

The fallacy of AMPHIBOLY occurs when multiple meanings of a sentence are used in a context where a) validity would require a single meaning, and b) the multiple meanings are due to sentence structure.

Example: You say that you don't keep your promises because it's in your interest to do so. People who don't keep their promises are immoral. So, you are immoral.

'You don't keep your promises because it's in your interest to do so' has two different readings: either that you keep your promises, but not *because* it's in your interest—that is, your *reason* isn't that it's in your interest. Or that you *don't* keep your promises, and that's because it's in your interest—that is, that the fact that it's in your interest is your reason for not keeping your promises. In most contexts, the former would be the reading intended. So the first sentence is only true if the sentence is interpreted in the first way. However, the conclusion only follows if it is interpreted in the second way.

Example: Nothing is better than Game of Thrones, and Duck Dynasty is better than nothing. We can infer that Duck Dynasty is better than Game of Thrones.

'Nothing is better than Game of Thrones' could either mean that there isn't anything which is better than Game of Thrones, or it could mean that not watching anything at all is better than Game of Thrones. The argument is only valid if we interpret the sentence in the second way. However, the sentence is only true if we interpret it in the first way.

Why it's fallacious: the argument gives the appearance of validity if we don't realize that the sentence is being understood in two different ways in the argument. However, once we are clear about what the sentence means, the argument either becomes invalid, or else has an obviously false premise.

A closely related but non-fallacious argument: If an argument uses a sentence that has multiple meanings, but the premises are true and the argument valid on a single disambiguation, then the argument is not amphibolous.

Not an Example: Flying planes can be dangerous. You should avoid dangerous things. So, you should avoid flying planes.

6.1.3 Composition/Division

The FALLACY OF COMPOSITION occurs when 1) a property of the parts of an object is improperly transferred to the object itself, or 2) a property of the individuals belonging to a group is improperly transferred to the group.

Example: Atoms are invisible, and I am made of atoms. So I am invisible.

Example (?): Every part of the world is caused. So, the world is caused.

The FALLACY OF DIVISION occurs when 1) a property of an object is improperly transferred to the parts of the itself, or 2) a property of a group is improperly transferred to the individuals belonging to the group.

Example: About 70 million people watch sitcoms. So *How I Met Your Mother* has about 70 million viewers.

Example: China and India consume more natural resources than America. So, Chinese and India citizens consume more resources than American citizens.

Why it's fallacious: wholes and parts can have different properties from one another, as can individuals and groups. Simply because parts have a property, that doesn't necessarily mean that the whole does; and simply because individuals have a property, that doesn't necessarily mean that the group does. Similarly, simply because the whole has a property, that doesn't necessarily mean that the parts do; and simply because the group has a property, that doesn't necessarily mean that the individuals in the group do.

A closely related but non-fallacious argument: There are some properties which *can* be properly transferred from parts to wholes (or wholes to parts), or from individuals to groups (or groups to individuals). These arguments are valid. We must, therefore, look to the properties in question in order to decide whether the argument is valid or invalid. (That's what makes this an *informal* fallacy.)

Not an Example: Every part of the train is made of metal; so the train is made of metal.

6.2 Fallacies Involving Unwarranted Assumptions

These fallacies all occur when an arguer assumes something in their argument which they are unwarranted in assuming.

6.2.1 Begging the Question (Petitio Principii)

An argument commits the fallacy of BEGGING THE QUESTION when it assumes the very conclusion that it is trying to establish.

Example: Surely Anthony loves me. For he told me he loves me, and he wouldn't lie to someone he loves.

Example: My scale is working perfectly. I weighed this textbook, and it said that it was 12 ounces. And, as I just learned by looking at the scale, it *is* 12 ounces. So the scale got it exactly right!

Note: question-begging arguments *are deductively valid*. They're just not especially persuasive.

A word of caution: it is incredibly difficult in some cases to distinguish good, valid arguments from question-begging arguments. For instance, the argument

Example?: There are numbers greater than 4. Therefore, there are numbers.

might be thought to be question-begging, because we've simply assumed that there are numbers. However, many people end up finding this argument persuasive. While everyone accepts that some arguments are question-begging, and therefore defective, there is no consensus on the question of when arguments are question-begging and when they are not.

6.2.2 False Dilemma

Two statements are *contraries* when they cannot both be true at once, but they *can* both be false at once. Two statements are *contradictories* when they cannot both be true at once, *nor* can they both be false at once (at least, and at most, one of them must be true).

The fallacy of FALSE DILEMMA occurs when an argument makes use of a premise that presents contraries as though they were contradictories.

Example: Either you are with us or you are with the terrorists. If you're leaking classified information about our government, then you're not with us. So, you are with the terrorists.

Example: It would be terrible if the government regulated every aspect of a person's life—their clothes, their love life, their personal beliefs. So we shouldn't have government regulation; let the free market decide.

Example: It would be terrible if there were no government regulation of any behavior. There would be total anarchy, and those with the most money and influence would exert their arbitrary authority over everyone else. So we need the government to regulate the marketplace.

A closely-related but non-fallacious argument: if we have good reason to set certain cases aside, then, so long as the argument is explicit that it is setting those cases aside, the argument will not be posing a false dilemma. What makes the argument a false dilemma is that it pretends as though two contraries are contradictories—not that it asserts, with good reason, that one of two contraries are true.

Not an Example: Given that it's around noon, Dmitri is either in his office or at lunch. But he's not in his office, so he's probably at lunch.

6.2.3 False Cause Fallacy

The FALSE CAUSE fallacy occurs when a merely possible cause is assumed to be a cause without evidence.

Example: Good philosophers write books; so if you want to be a good philosopher, you should write a book.

Example: The weather channel usually knows what the weather will be. The conclusion is inescapable: the weather channel is causing the weather.

This fallacy has some special sub-cases. First, there is the *post-hoc, ergo propter hoc* fallacy. In Latin, this means 'afterwards, therefore because of'. In this fallacy, it is assumed that, because *E* follows *C*, *C* must be a cause of *E*.

The POST HOC, ERGO PROPTER HOC fallacy occurs when one assumes that *C* caused *E* solely on the basis that *E* followed *C*.

Example: Since Obama's policies were enacted, unemployment has stopped growing. We can conclude that Obama's policies worked.

Example: After Obama's speech, the stock market took a nose dive. Good work, Obama.

A closely-related but non-fallacious argument: If we have good antecedent reason to think that two factors may be causally related, then the fact that they regularly appear together could be good reason to think that they might be causally related.

Another flavor is the *slippery slope fallacy*, in which it is assumed, without evidence, that one action will set off a causal chain leading to several other bad actions, and then argued that we shouldn't perform the first action.

The SLIPPERY SLOPE fallacy occurs when one assumes in an argument against some action that performing the action will set off a chain reaction of several bad consequences, when there is insufficient evidence to support the claim that performing the action will have these consequences.

Example: If we legalize gay marriage, then we'll soon be legalizing polygamous marriages, bestiality, dendrophilia, and other perversions. So we shouldn't legalize gay marriage.

A closely related but non-fallacious argument: If there is good reason to suppose that performing the first action *will* set off a chain reaction of consequences which are bad, then one is not reasoning fallaciously to suggest that we shouldn't undertake the first action.

Not an Example: If start going down that slippery slope, you'll just start slipping further and further down, until you fall off the edge of the cliff. So you shouldn't start going down that slippery slope.

Chapter 7

Propositional Logic

The plan: we're going to construct an artificial language, call it '*PL*' (for 'propositional logic') within which we can be rigorous and precise about which arguments are deductively valid and which are deductively invalid. This, together with a method for translating from English into *PL* (and out of *PL* into English) will allow us to theorize about which English-language arguments are deductively valid and which are deductively invalid. One advantage to theorizing about deductive validity in this way is that we won't have to worry about the kinds of ambiguities that we encountered in our discussion of informal fallacies (*e.g.*, equivocation and amphiboly), because the sentences of our artificial language won't admit of any ambiguity. Their meaning will always be perfectly precise.

In general, we can specify a language by doing three things: 1) giving the vocabulary for the language, 2) giving the grammar of the language—that is, specifying which ways of sticking together the expressions from the vocabulary are grammatical, and 3) saying what the *meaning* of every grammatical expression is. For instance, in English, the vocabulary consists of all of the words of English. The grammar for English consists of rules saying when various strings of English words count as grammatical English sentences. 'Bubbie makes pickles' and 'Colorless green ideas sleep furiously' will count as grammatical sentences, whereas 'Up bouncy ball door John variously catapult' does not count as a grammatical sentence. Finally, the meaning of every English sentence is given by providing a dictionary entry for every word of English and providing rules for understanding the meaning of sentences in terms of the meanings of the words appearing in the sentence. The first two tasks are the tasks of specifying the *syntax* of the language. The final task is the task of specifying the *semantics* of the language.

$$\begin{array}{l} \text{SYNTAX} \left\{ \begin{array}{l} 1. \text{ Vocabulary} \\ 2. \text{ Grammar} \end{array} \right. \\ \text{SEMANTICS} \text{ ---} 3. \text{ Meaning} \end{array}$$

That's exactly what we're going to do for our artificial language *PL*. However, our task will be much simpler than the task of specifying English, as we will have a far simpler vocabulary, a far simpler grammar, and a far simpler semantics.

7.1 Syntax for *PL*

7.1.1 Vocabulary

The vocabulary of *PL* includes the following symbols:

1. An infinite number of statement letters:

$$A, B, C, \dots, Y, Z, A_1, B_1, C_1, \dots, Y_1, Z_1, A_2, B_2, C_2, \dots$$

2. logical operators:

$\sim, \bullet, \vee, \supset, \equiv$

3. parentheses

$(,)$

Nothing else is included in the vocabulary of PL .

7.1.2 Grammar

Any sequence of the symbols in the vocabulary of PL is a *formula* of PL . For instance, all of the following are formulae of PL :

$((()A_{23} \bullet \bullet \supset \supset Z$
 $P \supset (Q) \supset \bullet ()$
 $(P \supset (Q \supset (R \supset (S \supset T))))$
 $A \bullet B \bullet (C \sim D))$

However, only one—the third—is a *well-formed formula* (or ‘wff’) of PL . We specify what it is for a string of symbols from the vocabulary of PL to be a wff of PL with the following rules.

SL) Any statement letter, by itself, is a wff.

- \sim) If ‘ p ’ is a wff, then ‘ $\sim p$ ’ is a wff.
- \bullet) If ‘ p ’ and ‘ q ’ are wffs, then ‘ $(p \bullet q)$ ’ is a wff.
- \vee) If ‘ p ’ and ‘ q ’ are wffs, then ‘ $(p \vee q)$ ’ is a wff.
- \supset) If ‘ p ’ and ‘ q ’ are wffs, then ‘ $(p \supset q)$ ’ is a wff.
- \equiv) If ‘ p ’ and ‘ q ’ are wffs, then ‘ $(p \equiv q)$ ’ is a wff.
- $-$) Nothing else is a wff.

Note: ‘ p ’ and ‘ q ’ do not appear in the vocabulary of PL . They are not themselves wffs of PL . Rather, they are being used here as FORMULAE VARIABLES—they are variables whose potential values are formulae of PL .

All and only the strings of symbols that can be constructed by repeated application of the rules above are well-formed formulae. To show that ‘ $(\sim(P \vee Q) \supset R)$ ’ is a wff, we could walk through the following steps to build the formula up:

- a) ‘ P ’ is a wff [from (SL)]
- b) ‘ Q ’ is a wff [from (SL)]
- c) So, ‘ $(P \vee Q)$ ’ is a wff [from (a) and (b) and (\vee)]
- d) So, ‘ $\sim(P \vee Q)$ ’ is a wff [from (c) and (\sim)]
- e) ‘ R ’ is a wff [from (SL)]
- f) So, ‘ $(\sim(P \vee Q) \supset R)$ ’ is a wff [from (d), (e), and (\supset)]

The rule (\supset) requires that we include the outermost parentheses around the expression ‘ $(\sim(P \vee Q) \supset R)$ ’. However, I will adopt the standard convention of omitting the outermost parentheses, writing, e.g., ‘ $\sim(P \vee Q) \supset R$ ’. This convention is harmless, but you should bear in mind that, *strictly speaking*, formula like ‘ $\sim(P \vee Q) \supset R$ ’ are not wffs of PL .

7.1.3 Main Operators and Subformulae

Given the rules for wffs provided above, we can give a simple definition of what a wff's *main operator* is. The wff's *main operator* is just the operator associated with the last rule which would have to be applied if we were building the formula up by applying the rules for wffs above. For instance, if we want to know what the main operator is for the wff ' $\sim P \bullet Q$ ', we would just imagine running through the following proof that ' $\sim P \bullet Q$ ' is a wff of *PL*, by applying to the rules for well formed formulae, *i.e.*,

- a) ' P ' is a wff [from (SL)]
- b) So, ' $\sim P$ ' is a wff [from (a) and (\sim)]
- c) ' Q ' is a wff [from (SL)]
- d) So, ' $(\sim P \bullet Q)$ ' is a wff [from (b), (c), and (\bullet)]

Here, the fact that we had to appeal to the rule (\bullet) in the final step of building up ' $\sim P \bullet Q$ ' tells us that \bullet is the main operator. Imagine that we had tried to build up the formula in some other way. For instance, suppose we had attempted to *first* apply the rule (\bullet) and *then* the rule (\sim). Then, our derivation would have gone like this.

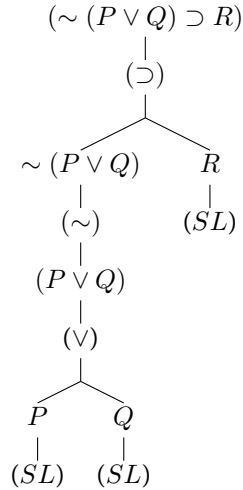
- a) ' P ' is a wff [from (SL)]
- b) ' Q ' is a wff [from (SL)]
- c) So, ' $(P \bullet Q)$ ' is a wff [from (a), (b), and (\bullet)]
- d) So, ' $\sim(P \bullet Q)$ ' is a wff [from (c) and (\sim)]

This is an entirely different wff. ' $\sim(P \bullet Q)$ ' is not the same as ' $(\sim P \bullet Q)$ '. While the main operator of ' $\sim P \bullet Q$ ' is \bullet , the main operator of ' $\sim(P \bullet Q)$ ' is \sim .

We can also use the rules for wffs to give a definition of what a wff's *subformulae* are. p is a subformula of q if and only if, in the course of building up q by applying the rules for wffs, p appears on a line before q . So, for instance ' $\sim P$ ' is a subformula of ' $\sim P \bullet Q$ ' (because it shows up on line (b) of that wff's derivation), whereas ' $\sim P$ ' is *not* a subformula of ' $\sim(P \bullet Q)$ ' (since it does not show up at any point in that wff's derivation).

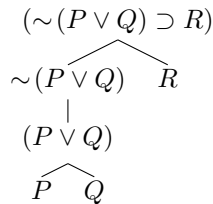
A formula's *immediate subformulae* are those wffs whose lines were appealed to in the final step of building to formula up. For instance, the immediate subformulae of ' $\sim P \bullet Q$ ' are ' $\sim P$ ' and ' Q ', whereas the immediate subformula of ' $\sim(P \bullet Q)$ ' is ' $P \bullet Q$ '. A wff's immediate subformulae are just those formulae on which the wff's main operator operates.

Another way of notating the proofs that certain formulae are wffs of *PL* is with SYNTAX TREES. For instance, we could represent our proof that ' $(\sim(P \vee Q) \supset R)$ ' is a wff of *PL* with the following syntax tree.

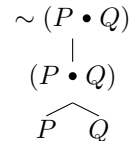
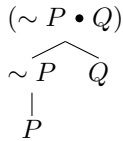


This tree tells us, firstly, that ‘ P ’ and ‘ Q ’ are wffs of PL (by rule (SL)). Then, by rule (\vee) , ‘ $(P \vee Q)$ ’ is a wff. Then, by rule (\sim) , ‘ $\sim (P \vee Q)$ ’ is a wff. And, since ‘ R ’ is a wff, by (SL) , ‘ $\sim (P \vee Q)$ ’ is a wff (by rule (\supset)).

If we want to leave out the rules, we can represent this syntax tree more simply as follows.



We can similarly write out the syntax trees for ‘ $\sim P \bullet Q$ ’ and ‘ $\sim (P \bullet Q)$ ’ like so.



These trees give us the syntactic structure of a wff of PL . They highlight what the parentheses were already telling us about what the main operator of the sentence is, what its subformulae are, and how the various subformulae are interrelated (how the sentence is built up out of its subformulae). For instance, the tree on the left tells us that the immediate subformulae of ‘ $\sim P \bullet Q$ ’ are ‘ $\sim P$ ’ and ‘ Q ’. And the tree on the right tells us that the immediate subformula of ‘ $\sim (P \bullet Q)$ ’ is ‘ $(P \bullet Q)$ ’.

7.2 Semantics for PL

We now need to say something about the *meaning* of the wffs appearing in PL . Throughout, our assumption will be that what it is to understand the meaning of an expression is just to understand the circumstances in which it is true.

There are three components to the vocabulary of PL : the statement letters, the logical operators, and the parentheses. The parentheses do not add anything to the meaning of the sentences of PL . They merely serve as notational tools that help us avoid ambiguity. Put them aside. We must then say what the meanings of the statements letters are and what the meanings of the logical operators are.

7.2.1 The Meaning of the Statement Letters

Each statement letter represents a statement in English. The statement letter is true if and only if the statement in English is true. That is: statement letters inherit their meaning from their English translations.

7.2.2 The Meaning of ‘ \sim ’

The operator ‘ \sim ’ is known as the **TILDE**. A wff whose main operator is the tilde is called a **NEGATION**. Its immediate subformula is called the **NEGAND**. If a wff ‘ p ’, is true, then ‘ $\sim p$ ’ is false. If a wff p is false, then ‘ $\sim p$ ’ is true. To write this a bit more perspicaciously, we can use the letters ‘ T ’ and ‘ F ’ to stand for the truth-values true and false. Then, for any wff ‘ p ’, if ‘ p ’ is T , then ‘ $\sim p$ ’ is F . If ‘ p ’ is F , then ‘ $\sim p$ ’ is T . We can summarize this with the following **TRUTH TABLE**.

p	$\sim p$
T	F
F	T

This table tells us how the truth-value of a wff of the form ‘ $\sim p$ ’ is determined by the truth-value of ‘ p ’. If we understand the circumstances under which ‘ p ’ is true, then the above definition gives us all that we need to understand the circumstances under which ‘ $\sim p$ ’ is true. So we’ve said enough to say what the meaning of ‘ \sim ’ is.

Note that ‘ p ’ is not a wff of *PL*—statement letters must be capitalized. Rather, we are using the lowercase ‘ p ’ and ‘ q ’ as variables ranging over the wffs of *PL*.

7.2.3 The Meaning of ‘ \bullet ’

The operator ‘ \bullet ’ is known as the **DOT**. A wff whose main operator is the dot is known as a **CONJUNCTION**. Its immediate subformulae are called **CONJUNCTS**. A conjunction is true if and only if both of its conjuncts are true. Using a truth-table, this means that:

p	q	$p \bullet q$
T	T	T
T	F	F
F	T	F
F	F	F

This table tells us how the truth-value of a wff of the form ‘ $p \bullet q$ ’ is determined by the truth-values of ‘ p ’ and ‘ q ’. If we understand the circumstances under which ‘ p ’ and ‘ q ’ are true, then this definition gives us enough to understand the circumstances under which ‘ $p \bullet q$ ’ is true. So we’ve said enough to say what the meaning of ‘ \bullet ’ is.

7.2.4 The Meaning of ‘ \vee ’

The operator ‘ \vee ’ is known as the **WEDGE**. A wff whose main operator is the wedge is known as a **DISJUNCTION**. It’s immediate subformulae are called **DISJUNCTS**. A disjunction is true if and only if at least one of its disjuncts is true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

This table tells us how the truth-value of a wff of the form ' $p \vee q$ ' is determined by the truth-value of ' p ' and ' q '. If we understand the circumstances under which ' p ' and ' q ' are true, then this definition gives us enough to understand the circumstances under which ' $p \vee q$ ' is true. So we've said enough to say what the meaning of ' \vee ' is.

7.2.5 The Meaning of ' \supset '

The operator ' \supset ' is known as the HORSESHOE. A wff whose main operator is the horseshoe is known as a MATERIAL CONDITIONAL. The immediate subformulae which precedes the horseshoe is known as the ANTECEDENT. The immediate subformulae which follows the horseshoe is known as the CONSEQUENT. A material conditional is true if and only if either its antecedent is false or its consequent is true.

p	q	$p \supset q$
T	T	T
T	F	F
F	T	T
F	F	T

As before, the above table gives us enough to understand the circumstances under which a wff of the form ' $p \supset q$ ' is true, assuming that we understand the circumstances under which ' p ' and ' q ' are true. So this table defines the meaning of the operator ' \supset '.

Note that this is the only binary operator which is not symmetric. That is, while ' $p \bullet q$ ' has the same meaning as ' $q \bullet p$ ', ' $p \vee q$ ' as the same meaning as ' $q \vee p$ ', and ' $p \equiv q$ ' has the same meaning as ' $q \equiv p$ ', ' $p \supset q$ ' *does not* have the same meaning as ' $q \supset p$ '.

7.2.6 The Meaning of ' \equiv '

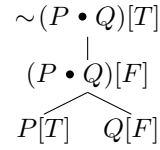
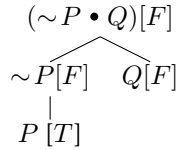
The operator ' \equiv ' is known as the TRIPLE BAR. A wff whose main operator is the triple bar is known as a MATERIAL BICONDITIONAL. The immediate subformula which appears before the triple bar is known as the biconditional's LEFT HAND SIDE, and the immediate subformula which appears after the triple bar is known as the biconditional's RIGHT HAND SIDE. A material biconditional is true if and only if its right hand side and its left hand side have the same truth-value.

p	q	$p \equiv q$
T	T	T
T	F	F
F	T	F
F	F	T

Again, this table gives us enough to understand the circumstances under which a wff of the form ' $p \equiv q$ ' is true, assuming that we understand the circumstances under which ' p ' and ' q ' are true. So this table defines the meaning of the operator ' \equiv '.

7.2.7 Determining the Truth-value of a wff of PL

If we know the truth-value of all the statement letters appearing in a wff of PL , then we can use our knowledge of the syntactic structure of the wff to determine its truth value. For instance, suppose that we know that ' P ' is true and that ' Q ' is false. Then, we know that ' $\sim P \bullet Q$ ' is false, and that ' $\sim(P \bullet Q)$ ' is true.



We can do the very same thing with truth-tables. For instance, to construct the truth-table for the wff ‘ $\sim P \bullet Q$ ’, begin by writing out all the possible truth-values for P and Q .

P	Q	$\sim P$	$\bullet Q$
T	T		
T	F		
F	T		
F	F		

Then, copy the column of truth-values for P , placing it beneath every appearance of the statement letter P , and do the same for Q .

P	Q	$\sim P$	$\bullet Q$
T	T	T	T
T	F	T	F
F	T	F	T
F	F	F	F

Then, begin working your way up the syntactic structure of the sentence by calculating the truth-values of the subformulae appearing in the wff. We know how to calculate the truth-value of ‘ $\sim P$ ’, given the truth-value of ‘ P ’ (from the truth-table for \sim which tells us the meaning of ‘ \sim ’), so do that first, placing the appropriate truth-values beneath the main connective of the subformulae ‘ $\sim P$ ’.

P	Q	$\sim P$	$\bullet Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

Now, we have to calculate the column of truth-values of ‘ $\sim P \bullet Q$ ’, writing them out beneath the main connective of that wff—the ‘ \bullet ’. The truth-value of ‘ $\sim P \bullet Q$ ’ is a function of the truth-values of ‘ $\sim P$ ’ and ‘ Q ’, and *not* the truth values of ‘ P ’ and ‘ Q ’, so we must look at the bolded columns of truth-values below.

P	Q	$\sim P$	$\bullet Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

Now, we can simply look to the truth-table for ‘ \bullet ’ to figure out what column of truth-values ought to go beneath the ‘ \bullet ’ in ‘ $\sim P \bullet Q$ ’. Since ‘ \bullet ’ is the main operator of the wff, this tells us the column of truth-values associated with the wff ‘ $\sim P \bullet Q$ ’. To indicate that this column of truth-values is the column associated with the main operator of the wff ‘ $\sim P \bullet Q$ ’, we put a box around this column.

P	Q	$\sim P$	$\bullet Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	F

This truth-table tells us how the truth-value of $\sim P \bullet Q$ is determined by the truth-values of P and Q . If P is false and Q is true, then $\sim P \bullet Q$ is true. Otherwise, $\sim P \bullet Q$ is false.

If we do the same thing with the wff $\sim(P \bullet Q)$, we will arrive at the following truth-table.

P	Q	\sim	$(P \bullet Q)$
T	T	F	T
T	F	T	F
F	T	T	F
F	F	T	T

This shows us how important it is to pay attention to the syntactic structure of the different wffs of PL —they end up making a difference to the *meaning* of those sentences. If we’re not careful with our parentheses, we’ll lose a big advantage of moving to a formal language—namely, that the sentences in PL are not ambiguous between different meanings.

7.3 Translation from PL to English

The meanings of \sim , \bullet , \vee , \supset , and \equiv are given by the truth-tables in the previous section. However, when we look at those meanings, it is difficult to not see some commonalities between these operators and some common English words. In particular, it appears that there’s a very close connection between the meaning of \sim and the meaning of ‘it is not the case that’; a very close connection between \vee and ‘or’; a very close connection between \bullet and ‘and’.

Submitted for your approval: the following provides a translation guide from PL to English.

$\sim p$	\longrightarrow	It is not the case that p
$p \bullet q$	\longrightarrow	Both p and q
$p \vee q$	\longrightarrow	Either p or q
$p \supset q$	\longrightarrow	If p , then q
$p \equiv q$	\longrightarrow	p if and only if q

This translation guide requires some provisos. In the first place: there appears to be an important difference between the meaning of $p \supset q$ and ‘if p , then q ’. The difference is this: if p is false, then $p \supset q$ is automatically true, *no matter what statement q represents, and no matter what kind of connection there is between p and q* . However, we wouldn’t ordinarily think that the sentence ‘if John Adams was America’s first president, then eating soap cures cancer’ is true, just in virtue of the fact that ‘John Adams was America’s first president’ is false. So it must be that ‘if p , then q ’ differs in meaning from $p \supset q$. I think that this is exactly right. However, there is still some close connection between the meanings of these two claims. To bring that connection out, suppose that I make the following claim:

If it’s a weekday, then I’m on campus.

And suppose that Steve makes the claim,

If I’m on campus, then it’s a weekday.

Think about the circumstances under which you could justly say that Steve or I had lied. If it’s a weekday, but I’m not on campus, then I have lied. If, however, it’s a weekday but Steve is not on campus, then he *hasn’t* lied. After all, he never said that he would be on campus every weekday. He just said that, *if he’s on campus*, then it’s a weekday. But he did not commit himself to ever coming to campus at all. On the other hand, suppose that I’m on campus during the weekend. Then, you wouldn’t be able to say that I had lied. For I never said that I would stay home during the weekend. I just said that, *if it’s a weekday*, then I’m on campus. However, if Steve is on campus during the weekend, then Steve *has* lied.

After all, he said that he'd only be on campus on weekdays. Using ' D ' to represent the statement 'Dmitri is on campus', ' S ' to represent 'Steve is on campus' and ' W ' to represent 'it is a weekday', then it looks like the possibilities in which you can say that I have lied are just the possibilities in which the material conditional ' $W \supset D$ ' is false.

D	W	if W , then D	$W \supset D$
T	T	didn't lie	T
T	F	didn't lie	T
F	T	lied	F
F	F	didn't lie	T

And it looks like the possibilities in which you can say that Steve has lied are just the possibilities in which you can say that the material conditional ' $S \supset W$ ' is false.

S	W	if S , then W	$S \supset W$
T	T	didn't lie	T
T	F	lied	F
F	T	didn't lie	T
F	F	didn't lie	T

So, even though the translation isn't perfect, it's still pretty good. Moreover, even if a *PL* wff of the form ' $p \supset q$ ' might be better translated into English with 'Either it is not the case that p or q ', it appears as though ' $p \supset q$ ' is the best possible *PL*-translation of the English 'if p , then q '. So that's how we'll be translating it here. But if you think the translation is less than perfect, you're absolutely correct. There are more advanced logics which attempt to give a better translation of the English conditional, but they are beyond the purview of this course.

In the second place: 'or' is used in English in two different senses. In one sense, called the 'inclusive or', a statement of the form ' p or q ' is true if and only at least one of ' p ' and ' q ' are true—that is, it is true if and only if either ' p ' is true, or ' q ' is true, or both are true. For instance, if I say to you 'either the elevator or the escalator is working', then I haven't lied to you if they are *both* working. To see this more clearly, think about the sentence 'if either the elevator or the escalator is working, then you will be in compliance with the Americans with Disabilities Act'. If both are working and you are not in compliance with the ADA, then I have lied to you. However, if 'either the elevator or the escalator is working' were false when they are both working, then I couldn't have lied to you.

Inclusive 'or': In the inclusive sense ' p or q ' means 'Either ' p ' or ' q ' or both.'

In another sense, called the 'exclusive or', a statement of the form ' p or q ' is true if and only if at least and at most one of ' p ' and ' q ' are true. That is, in the exclusive sense, ' p or q ' means ' p or q , but not both'. For instance, if your parent tells you, 'Either you clean your room, or you're grounded', you clean your room, and your parent grounds you, then you can fairly complain that they lied.

Exclusive 'or': In the exclusive sense ' p or q ' means 'Either ' p ' or ' q ', but not both.'

When I say that ' $p \vee q$ ' may be translated as ' p or q ', I am using 'or' in its *inclusive* sense—that is, I am using it to mean ' p or q or both'.

' \vee ' translates to the *inclusive* 'or'

Let's call the phrases on the right-hand-side of the translation guide above the *canonical logical expressions* of English. If the logical structure of an English statement is written in this form, then that statement is in *canonical logical form*. For instance, the following claim is in canonical logical form:

If both John loves Andrew and it is not the case that Andrew loves John, then it is not the case that John and Andrew will be friends.

Because the sentence is in canonical logical form, it is simple to translate it into *PL*. We simply introduce the statement letters '*J*', '*A*', and '*F*', where *J* = 'John loves Andrew', *A* = 'Andrew loves John', and *F* = 'John and Andrew will be friends'. Then, the translation into *PL* is

$$(J \bullet \sim A) \supset \sim F$$

On the other hand, *this* English sentence, which has the same meaning as the first, is *not* written in canonical logical form.

John and Andrew won't be friends if John loves Andrew but Andrew doesn't love him back.

So we'll have to say a bit more about how to translate sentences like this into *PL*.

7.4 Translation from English to *PL*

7.4.1 Negation

In English, the word 'not' can show up in many places in a sentence. In order for an English sentence to be translated into a wff of *PL* with a ' \sim ', it need not contain the words 'it is not the case that'. For instance, if we let '*H*' stand in for the English sentence 'Harry likes chestnuts', then we may translate the English sentence

Harry doesn't like chestnuts

as ' $\sim H$ '. The reason is that ' $\sim H$ ' is true if and only if '*H*' is false, and 'Harry doesn't like chestnuts' is true if and only if 'Harry likes chestnuts' is false. So our translation has the same meaning as the sentence we wanted to translate. Here's a more general strategy for translating English sentences into *PL*: re-write the sentences in the *canonical logical form* given by the translation schema from the previous section, and check to see whether the re-written sentence has the same meaning as the sentence that you started out with. If it does, then you may substitute the canonical logical forms for the logical operators of *PL* according to the translation schema of the previous section. If not, then you may not.

For instance, we could re-write 'Harry doesn't like chestnuts' as

It is not the case that Harry likes chestnuts.

Since this contains the canonical logical form 'it is not the case that', we may swap this phrase of English out for *PL*'s ' \sim ' to get

\sim Harry likes chestnuts.

We may then use the statement letter '*H*' to represent 'Harry likes chestnuts', and we will get the *PL* wff

$\sim H$

A word of warning: just because an English statement contains the word 'not', that does not mean that it should be translated into a wff of *PL* with a ' \sim '. In order to see whether it can, we have to see whether re-writing the statement in canonical logical form preserves meaning. For instance, the following sentence contains the word 'not':

I hate not getting what I want and I hate getting what I want.

We might attempt to translate this into canonical logical form like so,

It is not the case that I hate getting what I want, and I hate getting what I want.

substitute ‘ \sim ’ for ‘it is not the case that’ and ‘ \bullet ’ for ‘and’, and get

\sim I hate getting what I want \bullet I hate getting what I want.

If we then used ‘ H ’ to represent the English ‘I hate getting what I want’, we would get the PL wff

$\sim H \bullet H$

However, *this* wff of PL is necessarily false, as the following truth-table shows

H	\sim	H	\bullet	H
T	F	T	F	T
F	T	F	F	F

But the sentence we started with *wasn't* necessarily false. For it is possible that I both hate not getting what I want *and* getting what I want. If this were possible, then I'd hate everything, but surely it's not a logical truth that I don't hate everything. So something went wrong. What went wrong was that ‘I hate not getting what I want’ doesn't have the same meaning as ‘It is not the case that I hate getting what I want’. So we must make sure that translation into canonical logical form preserves meaning in English before we translate that canonical logical form into PL .

7.4.2 Conjunction

Many expressions in English have subtle shades of meaning which must be lost when we translate into PL . In particular, the following two English expressions will both have the same PL translation:

Hannes loves peaches and he loves apples.

Hannes loves peaches but he loves apples.

The second sentence implies some kind of contrast between ‘Hannes loves peaches’ and ‘Hannes loves apples’; whereas the first sentence does not. This subtle difference in meaning will be lost when we translate into PL , since both of these claims are *true* under exactly the same conditions: namely, the condition in which Hannes loves peaches and apples. So, using ‘ P ’ to represent ‘Hannes loves peaches’ and ‘ A ’ to represent ‘Hannes loves apples’, they will both be translated into PL as ‘ $P \bullet A$ ’.

All of the following expressions of English will also be translated into PL with the ‘ \bullet ’:

$$\left. \begin{array}{l} p \text{ and } q \\ p, \text{ but } q \\ p; \text{ however, } q \\ p, \text{ though } q \\ p \text{ as well as } q \end{array} \right\} \longrightarrow p \bullet q$$

7.4.3 Disjunction

Both ‘ p or q ’ and ‘ p unless q ’ are translated into PL as ‘ $p \vee q$ ’. If you're unhappy about this translation, think about the following argument: ‘ p unless q ’ could be translated as ‘if it's not the case that q , then p ’, or: ‘ $\sim q \supset p$ ’. And this

expression has the very same meaning, in PL , as ' $p \vee q$ ' (they have the very same truth-table). Thus, ' $p \vee q$ ' translates ' p unless q '. If you're *still* unhappy about this translation, think about how you would want to change it (think, that is, about what translation into PL you think does a better job than $p \vee q$). My guess is that, if you're unhappy with ' $p \vee q$ ', then you'll probably be more happy with ' $p \equiv \sim q$ '.

$$\left. \begin{array}{l} p \text{ or } q \\ p \text{ unless } q \end{array} \right\} \longrightarrow p \vee q$$

7.4.4 The Material Conditional and Biconditional

Any of the following English expressions are appropriately translated in PL as ' $p \supset q$ '.

$$\left. \begin{array}{l} \text{If } p, \text{ then } q \\ p \text{ only if } q \\ \text{Only if } q, p \\ q \text{ if } p \\ q, \text{ provided that } p \\ q, \text{ given that } p \\ q \text{ is true whenever } p \text{ is} \\ p \text{ is sufficient for } q \\ q \text{ is necessary for } p \end{array} \right\} \longrightarrow p \supset q$$

And any of the following are appropriately translated in PL as ' $p \equiv q$ '.

$$\left. \begin{array}{l} p \text{ if and only if } q \\ p \text{ is necessary and sufficient for } q \\ p \text{ is true when and only when } q \text{ is true} \end{array} \right\} \longrightarrow p \equiv q$$

Chapter 8

Logical Notions of PL: Validity

8.1 How to Construct a Truth-Table

What kind of truth-table we want to create will depend upon which wff/argument/set of wffs of PL we are considering, and which statement letters they contain as subformulae. For instance, we may be interested in the wff of PL

$$(P \cdot Q) \supset \sim R$$

In that case, we will need a truth-table which has columns for each of the statement letters $P, Q,$ and $R.$ Alternatively, we might be interested in the argument of $PL,$

$$\begin{array}{l} (A \cdot B_{23}) \supset D \\ D \equiv Z_{12} \\ \hline ((A \cdot B_{23}) \cdot Z_{12}) \equiv D \end{array}$$

In that case, we will need a truth table which has columns for each of the statement letters $A, B_{23}, D,$ and $Z_{12}.$ Alternatively, we might be interested in the *set* of wffs of PL

$$\left\{ \begin{array}{l} E \vee F \\ \sim F \supset G \\ \sim E \vee \sim G \end{array} \right\}$$

In that case, we will need a truth table which has columns for each of the statement letters $E, F,$ and $G.$

In general, if there are n distinct statement letters appearing in your wff/argument/set of wffs of $PL,$ then create 2^n rows in your truth table. Arrange the statement letters alphabetically (lower subscripts first), and then put $2^n/2$ 'T's, followed by $2^n/2$ 'F's, under the first statement letter. For the next statement letter (if there is one), put $2^n/4$ 'T's, followed by $2^n/4$ 'F's, followed by $2^n/4$ 'T's, followed by $2^n/4$ 'F's. In general, for the i th statement letter, put $2^n/2^i$ 'T's, followed by $2^n/2^i$ 'F's, and so on, until all the rows are filled. Complete this until you've written out a row for every statement letter.

Why we do it this way: because this way, we'll end up representing *every possible* assignment of truth-values to the statement letters appearing in the wff/argument/set of wffs. So we'll be sure to consider every possible case. If we didn't do it in this systematic way, we might end up leaving some possibility out, and incorrectly concluding that something was a tautology when it's not, or that an argument is valid when it's not, or what-have-you.

Example: if $n = 1,$ then we need $2^1 = 2$ rows in our truth-table. Under the first (*i.e., only*) statement letter, we put

$2^1/2 = 1$ 'T' followed by $2^1/2 = 1$ 'F', and we're done.

A	
T	
F	

If $n = 2$, then we need $2^2 = 4$ rows in our truth-table. Under the first statement letter, we put $2^2/2 = 2$ 'T's, followed by $2^2/2 = 2$ 'F's. Under the second statement letter, we put $2^2/2^2 = 1$ 'T', followed by $2^2/2^2 = 1$ 'F', followed by $2^2/2^2 = 1$ 'T', followed by $2^2/2^2 = 1$ 'F', and we're done.

A	B	
T	T	
T	F	
F	T	
F	F	

If $n = 3$, then we need $2^3 = 8$ rows in our truth-table. Under the first statement letter, we put $2^3/2 = 4$ 'T's, followed by $2^3/2 = 4$ 'F's. Under the second statement letter, we put $2^3/2^2 = 2$ 'T's, followed by $2^3/2^2 = 2$ 'F's, followed by $2^3/2^2 = 2$ 'T's, and so on, until we fill the column. Under the third statement letter, we put $2^3/2^3 = 1$ 'T' followed by $2^3/2^3 = 1$ 'F', followed by $2^3/2^3 = 1$ 'T', and so on, until we fill the column.

A	B	C	
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

If $n = 4$, then we need $2^4 = 16$ rows in our truth-table. Under the first statement letter, we write $2^4/2 = 8$ 'T's, followed by $2^4/2 = 8$ 'F's. Under the second statement letter, we write $2^4/2^2 = 4$ 'T's, followed by $2^4/2^2 = 4$ 'F's, followed by $2^4/2^2 = 4$ 'T's, and so on, until we fill the column. Under the third statement letter, we write $2^4/2^3 = 2$ 'T's, followed by $2^4/2^3 = 2$ 'F's, followed by $2^4/2^3 = 2$ 'T's, and so on, until we fill the column. Finally, under the final statement letter, we write $2^4/2^4 = 1$ 'T', followed by $2^4/2^4 = 1$ 'F', followed by $2^4/2^4 = 1$ 'T', and so on, until we fill the column.

<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>
<i>T</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>T</i>	<i>T</i>	<i>F</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>T</i>
<i>T</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>T</i>
<i>T</i>	<i>F</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>T</i>	<i>F</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>T</i>	<i>F</i>
<i>F</i>	<i>F</i>	<i>F</i>	<i>T</i>
<i>F</i>	<i>F</i>	<i>F</i>	<i>F</i>

8.2 What a Truth-Table Represents

Let's introduce the idea of a TRUTH-VALUE ASSIGNMENT. A truth-value assignment is an assignment of truth value—true or false—to every statement letter of *PL*.

A TRUTH-VALUE ASSIGNMENT is an assignment of truth-value—either true or false—to every statement letter of *PL*.

There are, of course, infinitely many statement letters of *PL*, so a truth-value assignment must assign infinitely many truth-values to infinitely-many statement letters.

Note, however, that a truth-value assignment does *not* assign a truth-value to a complex wff of *PL* like

$$(P \bullet Q) \equiv \sim (R \vee S).$$

Of course, given a truth-value assignment, we may *work out*—along with the definitions of the logical operators \bullet , \equiv , \sim , and \vee —the truth-value of this more complicated wff. However, the truth-value assignment, *on its own*, only tells us the truth-value of the wffs of *PL* which consist entirely of individual statement letters—no parentheses, and no logical operators.

Suppose that we don't wish to specify a truth-value assignment completely. That is, we don't wish to specify the truth-values for all of the infinitely many statement letters of *PL*. Then, we may choose to just provide a PARTIAL TRUTH-VALUE ASSIGNMENT. A *partial* truth-value assignment merely assigns truth-values to some set of statement letters.

A PARTIAL TRUTH-VALUE ASSIGNMENT assigns a truth-value—either true or false—to each statement letter in some set of statement letters.

For instance, a partial truth value assignment, for the set of statement letters $\{A, B, C\}$, is given by saying that *A* is true, *B* is false, and *C* is false.

With these notions under our belt, we can see that the rows of a truth-table are providing us with *every possible* partial truth-value assignment to the statement letters appearing in the wff/argument/set of wffs of *PL* that we're interested in. For instance, if our wff/argument/set of wffs of *PL* contains the statement letters *X*, *Y*, and *Z*, then our truth-table will represent every possible partial truth-value assignment to the set of statement letters $\{X, Y, Z\}$

Every possible partial truth-value assignment to $\{X, Y, Z\}$	{	<i>X</i>	<i>Y</i>	<i>Z</i>	
		<i>T</i>	<i>T</i>	<i>T</i>	
		<i>T</i>	<i>T</i>	<i>F</i>	
		<i>T</i>	<i>F</i>	<i>T</i>	
		<i>T</i>	<i>F</i>	<i>F</i>	
		<i>F</i>	<i>T</i>	<i>T</i>	
		<i>F</i>	<i>T</i>	<i>F</i>	
		<i>F</i>	<i>F</i>	<i>T</i>	
		<i>F</i>	<i>F</i>	<i>F</i>	

Each row of the truth-table corresponds to a partial truth-value assignment to the statement letters. For instance, the first row of the truth table above corresponds to the partial truth-value assignment which assigns the truth-value 'true' to each of *X*, *Y*, and *Z*. The penultimate row corresponds to the partial truth-value assignment which assigns the truth-value 'false' to *X*, assigns the truth-value 'false' to *Y*, and assigns the truth-value 'true' to *Z*.

For a wff of *PL* which contain only the statement letters *X*, *Y*, and *Z*—like, for instance,

$$\sim ((X \bullet Y) \bullet Z) \equiv ((\sim X \vee \sim Y) \vee \sim Z)$$

this partial truth-value assignment is all that is needed to work out all of the possible truth-values for the wff. That's because the truth-value of this wff is entirely determined by the truth-values of the statement letters which appear in it. So, if our goal is to determine which possible truth-values the wff ' $\sim ((X \bullet Y) \bullet Z) \equiv ((\sim X \vee \sim Y) \vee \sim Z)$ ' could take on, we need only consider the *partial* truth-value assignments given in the truth-table above.

8.3 *PL*-Validity

We can now define a whole host of interesting logical notions in the language *PL* with the aid of truth-tables. To begin with, bit of new notation: If we have an argument from the premises *p*, *q*, *r* to the conclusion *s*, then, rather than writing this as we have been, like so

$$\frac{p}{q} \quad \frac{r}{s}$$

we will denote the argument by putting single forward slashes between premises, and putting a double forward slash between the premises and the conclusion, like so:

$$p / q / r // s$$

If we have a collection of wffs of *PL*, one of which is designated the conclusion, the others of which are designated premises, then we have what we will call a '*PL*-argument'.

A *PL*-ARGUMENT is a collection of wffs of *PL*, one of which is designated the conclusion, and the others of which are designated the premises.

Recall the definition of DEDUCTIVE VALIDITY. An argument is deductively valid if and only if there is no *possibility* in which all of the premises are true but the conclusion is false. Equivalently: an argument is deductively valid if and only if every possibility in which the premises of the argument are all true is a possibility in which the conclusion is true also. To model deductive validity in the language PL , we will give a definition of validity within the language PL —what we will call ‘ PL -VALIDITY’—which is just the same as the definition of deductive validity, except with a formal substitution for the notion of a *possibility*. The substitution we will make is this: for ‘possibility’, we will substitute ‘truth-value assignment’.

A PL -argument is PL -VALID if and only if there is no truth-value assignment in which all of the premises are true and the conclusion is false.

As we saw in §8.2 above, because the truth-values of the wffs of PL appearing in a PL -argument are determined entirely by the statement letters appearing in those wffs, we need not consider *every* truth-value assignment. Rather, it will be enough to look at all the *partial* truth-value assignments to those statement letters appearing in the PL -argument. Each such partial truth-value assignment corresponds to a row of the truth-table for the PL -argument. So, another, equivalent, definition of PL -validity is this:

A PL -argument is PL -VALID if and only if, in the argument’s truth-table, there is no row in which the premises of the argument are all true and the conclusion is false.

So, for instance, suppose we wish to determine whether the following PL -argument is PL -valid:

$$\sim (P \bullet Q) \parallel \sim P \vee \sim Q$$

To check, we first construct the truth-table for the argument. The argument contains 2 statement letters: P and Q . So we place two columns on the right-most side, and fill in the possible partial truth-value assignments to those sentence letters, according to the directions given in §8.1, like so:

P	Q	
T	T	
T	F	
F	T	
F	F	

Next, we place the argument’s premise in one column, and the argument’s conclusion in another, like so, and, underneath each of the statement letters, we copy over the truth-values from the first two columns, like so:

P	Q	$\sim (P \bullet Q)$	\parallel	$\sim P \vee \sim Q$
T	T	T	\parallel	T
T	F	T	\parallel	T
F	T	F	\parallel	F
F	F	F	\parallel	F

(Here, I’ve placed two vertical lines between the premise and the conclusion just to indicate that the conclusion is the thing to the right of those double vertical lines.)

We then finish up by, in the case of the argument’s premise ‘ $\sim (P \bullet Q)$ ’, determining the appropriate column of truth values beneath $(P \bullet Q)$, and then determining the appropriate column of truth values beneath $\sim (P \bullet Q)$. For the argument’s conclusion, $\sim P \vee \sim Q$, we determine the appropriate column of truth-values beneath $\sim P$, and the appropriate column of truth-values beneath $\sim Q$, and then determine the appropriate column of truth-values beneath $\sim P \vee \sim Q$. When we’re done, we place a box around the columns beneath the premise’s and the conclusion’s main operator.

P	Q	\sim	$(P$	\bullet	$Q)$	\sim	P	\vee	\sim	Q
T	T	F	T	T	T	F	T	F	F	T
T	F	T	T	F	F	F	T	T	T	F
F	T	T	F	F	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

Now, in order to decide whether the argument is *PL*-valid or *PL*-invalid, we need to determine whether every row in which the premise is true is a row in which the conclusion is true also. So first consider all the rows in which the premise is true. That's rows 2–4:

P	Q	\sim	$(P$	\bullet	$Q)$	\sim	P	\vee	\sim	Q
T	T	F	T	T	T	F	T	F	F	T
T	F	T	T	F	F	F	T	T	T	F
F	T	T	F	F	T	T	F	T	F	T
F	F	T	F	F	F	T	F	T	T	F

And, in each of rows 2–4, the conclusion is true. So, there is no row of the truth-table in which the premises are all true yet the conclusion is false. So, the *PL*-argument

$$\sim (P \bullet Q) // \sim P \vee \sim Q$$

is *PL*-valid.

A *PL*-argument is *PL*-invalid if and only if it is not *PL*-valid. Thus:

A *PL*-argument is *PL*-INVALID if and only if there is some truth-value assignment on which all of the argument's premises true yet its conclusion is false.

Or, equivalently:

A *PL*-argument is *PL*-INVALID if and only if there is some row of the truth-table in which all of the premises are true and in which the conclusion is false.

Suppose that we want to show that the following *PL*-argument is *PL*-invalid:

$$A \supset C / \sim A // \sim C$$

Then, we may construct the truth-table for this *PL*-argument. We will arrive at the following:

A	C	A	\supset	C	\sim	A	\sim	C
T	T	T	T	T	F	T	F	T
T	F	T	F	F	F	T	T	F
F	T	F	T	T	T	F	F	T
F	F	F	T	F	T	F	T	F

Both of the premises are true in rows 3 and 4 of the truth-table. So we restrict our attention to those rows. If the conclusion is also true in those rows of the truth table, then the argument is *PL*-valid. If, however, the conclusion is false in one of those rows of the truth-table, then the argument is *PL*-invalid.

A	C	A	\supset	C	\sim	A	\sim	C
T	T	T	T	T	F	T	F	T
T	F	T	F	F	F	T	T	F
F	T	F	T	T	T	F	F	T
F	F	F	T	F	T	F	T	F

Look, however, at the third row of the truth-table. On this row of the truth-table, the premises of the argument are true, yet its conclusion is false. So the argument is *PL*-invalid.

Just as we saw with deductive validity and deductive invalidity, we can give an equivalent definition of *PL*-validity and *PL*-invalidity by introducing the notion of a *PL*-counterexample.

A *PL*-COUNTEREXAMPLE to the *PL*-validity of a *PL*-argument is a truth-value assignment on which the premises of the argument are all true, yet the conclusion is false.

Or, equivalently:

A *PL*-COUNTEREXAMPLE to the *PL*-validity of a *PL*-argument is a row of the truth table in which all of the premises are true and the conclusion is false.

Now, a *PL*-argument is *PL*-valid if and only if it has no *PL*-counterexample.

A *PL*-argument is *PL*-VALID if and only if it has no *PL*-counterexample.

And, thus, a *PL*-argument is *PL*-invalid if and only if it has a *PL*-counterexample.

A *PL*-argument is *PL*-INVALID if and only if it has a *PL*-counterexample.

For instance, the *PL*-argument

$$A \supset C / \sim A // \sim C$$

considered above has the following *PL*-counterexample:

A is false and C is true

(This is the assignment of truth-values to A and C which corresponds to the third row the truth-table above.)

Chapter 9

Propositional Logic Derivations, day 1

The truth-table method of checking for *PL*-validity and *PL*-invalidity can be prohibitively difficult when the number of statement letters appearing in the argument are large. For instance, consider the following argument:

$$(P \equiv Q) \supset R / R \equiv S / S \equiv T / T \equiv U / U \equiv V / \sim V // (P \bullet \sim Q) \vee (\sim P \bullet Q)$$

This argument is *PL*-valid. However, checking the validity of this argument with a truth table would require a table with $2^7 = 128$ rows.

In this section of the course, we're going to learn how to establish the validity arguments involving many statement letters much more simply. We will, at the same time, acquire the ability to *think through* which wffs *PL*-follow from which other wffs.

9.1 The Basics

To begin with: a *PL*-derivation consists of a certain number of lines, each one numbered. On each line of the derivation, we have a wff of *PL* along with a *justification* explaining why we get to write that wff down on that line—*unless* that wff is one of the *premises* of the argument we are attempting to show to be valid. To get a flavor for what these derivations look like, here is a sample derivation:

1	$A \supset B$	
2	$B \supset (C \supset D)$	
3	$A \supset (C \supset D)$	1, 2, <i>HS</i>
4	$(A \bullet C) \supset D$	3, <i>Exp</i>
5	$\sim D \supset \sim(A \bullet C)$	4, <i>Trans</i>
6	$\sim D \supset (\sim A \vee \sim C)$	5, <i>DM</i>

If the derivation is to be *legal*, then the formulae appearing on each line must be wffs of *PL*. Additionally, each line with a justification must *follow* from the lines cited in the justification, along with the rule cited in the justification. Moreover, the lines cited must *precede* the line on which the justification is written. You may *not* justify a line by citing a line beneath it in the derivation. Only lines preceding a given line are *accessible* from that line; and only *accessible* lines may be legally cited in a justification.

There are many possible derivations systems like this. And there's a reason that we're interested in the particular one that we'll be studying. It's because the system we will be studying has the following excellent property: you can derive a

wff of PL , c , from other wffs of PL , p_1, p_2, \dots, p_N , within this system *if and only if* c follows from p_1, p_2, \dots, p_N . That is, c is derivable from p_1, p_2, \dots, p_N if and only if the PL -argument $p_1 / p_2 / \dots / p_N \parallel c$ is a PL -valid argument.

FACT: If there is a *legal* PL -derivation which has the wffs p_1, p_2, \dots, p_N as assumptions and has q appearing on its final line, then $p_1 / p_2 / \dots / p_N \parallel q$ is a PL -valid argument.

FACT: If $p_1 / p_2 / \dots / p_N \parallel q$ is a PL -valid argument, then there is a *legal* PL -derivation which has the wffs p_1, p_2, \dots, p_N as assumptions and has q appearing on its final line

Here, I'll just ask you to take my word that these two facts are true. In more advanced courses on logic, you might be asked to *prove* that these two facts are true.

9.2 Rules of Implication

The first set of rules are rules of *implication*. What makes these rules of *implication* are that they are *one way*. While the lines cited in the justification do entail the wff which is so justified (*i.e.*, the argument from the lines cited in the justification to the justified wff is PL -valid), the justified wff does not entail the lines cited in the justification (*i.e.*, the argument from the justified wff to the lines cited in the justification is not PL -valid). You could check the PL -validity with truth-tables, if you wanted.

9.2.1 Modus Ponens

The first rule is known as *modus ponens*.

Modus Ponens (MP)

$p \supset q$

p

$\triangleright q$

Here's how to read this rule. It says: if you have a wff of the form ' p ' written down on an accessible line, and you have a wff ' $p \supset q$ ' written down on an accessible line, then you can write down ' q '. When you justify your use of this rule, you should cite the line numbers that ' p ' and ' $p \supset q$ ' were written on, and write ' MP '.

9.2.2 Modus Tollens

The next rule is known as *modus tollens*.

Modus Tollens (MT)

$p \supset q$

$\sim q$

$\triangleright \sim p$

This rule says: if you have a wff of the form ' $p \supset q$ ' written down on an accessible line, and you have a wff of the form ' $\sim q$ ' written down on an accessible line, then you may write down ' $\sim p$ '. When you justify your use of this rule, you should cite the line numbers on which ' $p \supset q$ ' and ' $\sim q$ ' appeared and write 'MT'.

A Sample Derivation

1	$\sim C \supset (A \supset C)$	
2	$\sim C$	$/ \sim A$
3	$A \supset C$	1, 2, MP
4	$\sim A$	2, 3, MT

In the derivation, lines 1 and 2 don't have any justifications written next to them. That's because they are the premises of the argument, and don't require justification. The ' $/ \sim A$ ' written on line 2 indicates that ' $\sim A$ ' is the conclusion to be derived from the wffs appearing on lines 1 and 2.

9.2.3 Hypothetical Syllogism

The next rule of implication is known as *hypothetical syllogism*.

Hypothetical Syllogism (HS)

$p \supset q$

$q \supset r$

$\triangleright p \supset r$

This rule says: if you have a wff of the form ' $p \supset q$ ' on an accessible line, and you have a wff of the form ' $q \supset r$ ' on an accessible line, then you may write down ' $p \supset r$ ' on an accessible line. When you justify your use of this rule, you must cite the line numbers on which ' $p \supset q$ ' and ' $q \supset r$ ' appeared and write 'HS'.

9.2.4 Disjunctive Syllogism

The next rule of implication is known as *disjunctive syllogism*.

Disjunctive Syllogism (DS)

$p \vee q$

$\sim p$

$\triangleright q$

This rule says: if you have a wff of the form ' $p \vee q$ ' on an accessible line, and you have a wff of the form ' $\sim p$ ' on an accessible line, then you may write down ' q '. In your justification, you should write the lines on which ' $p \vee q$ ' and ' $\sim p$ ' appear, and 'DS'.

NOTE: In DS, the order of the disjuncts in a disjunction matters. The following is *not* a legal derivation:

1 $A \vee B$
2 $\sim B$
3 A 1, 2, DS ← MISTAKE!!!

For lines 1 and 2 are of the form ' $p \vee q$ ' and ' $\sim q$ '. However, DS only tells us what we can do with lines of the form ' $p \vee q$ ' and ' $\sim p$ '. So DS does not tell us that we may infer ' A ' from ' $A \vee B$ ' and ' $\sim B$ '.

This, however, *is* a legal derivation:

1 $A \vee B$
2 $\sim A$
3 B 1, 2, DS

NOTE: It is not enough to have a line which is *PL-equivalent* to ' $\sim p$ '. The line must actually *be of the form* ' $\sim p$ '. For instance, the following derivation is *not* legal:

1 $\sim A \vee B$
2 A
3 B 1, 2, DS ← MISTAKE!!!

This derivation, on the other hand, *is* legal:

1 $\sim A \vee B$
2 $\sim\sim A$
3 B 1, 2, DS

A Sample Derivation

1	$B \supset S$	
2	$S \supset (T \vee U)$	
3	B	
4	$\sim T$	$/U$
5	$B \supset (T \vee U)$	1, 2, HS
6	$T \vee U$	3, 5, MP
7	U	4, 6, DS

9.2.5 Simplification

The next two rules of implication govern the logical operator ' \bullet '. The first is known as *simplification*, and it allows us to *remove* a conjunct from a conjunction.

Simplification (Simp)

$$\frac{p \bullet q}{\triangleright p}$$

This rule says: if you have a formula of the form ' $p \bullet q$ ' written on an accessible line, then you may write down ' p '. Your justification should cite the line number on which ' $p \bullet q$ ' appears and say '*Simp*'.

NOTE: Here, too, **the order of the conjuncts in ' $p \bullet q$ ' matters**. The following is *not* a legal derivation:

$$\begin{array}{ll} 1 & (A \equiv B) \bullet \sim(C \supset D) \\ 2 & \sim(C \supset D) \qquad 1, \text{Simp} \leftarrow \text{MISTAKE!!!} \end{array}$$

However, the following *is* a legal derivation:

$$\begin{array}{ll} 1 & (A \equiv B) \bullet \sim(C \supset D) \\ 2 & A \equiv B \qquad 1, \text{Simp} \end{array}$$

9.2.6 Conjunction

This rule of implication is known as *conjunction*, and it allows us to *form* a conjunction from two wffs of *PL*.

Conjunction (Conj)

$$\frac{p \quad q}{\triangleright p \bullet q}$$

This rule says that, if you have a wff of the form ' p ' written on an accessible line, and you have a wff of the form ' q ' written on an accessible line, then you may write ' $p \bullet q$ '. Your justification should cite the line number of the line on which ' p ' appears, the line number of the line of which ' q ' appears, and say '*Conj*'.

9.2.7 Addition

This rule of implication is known as *addition*.

<u>Addition (Add)</u>	
	p
\triangleright	$p \vee q$

This rule says that, if you have a wff of the form ' p ' written on an accessible line, then you may write any wff of the form ' $p \vee q$ '. Your justification should cite the line number of the line on which ' p ' appears and say '*Add*'.

NOTE: the order of the disjuncts in ' $p \vee q$ ' matters. For instance, the following is *not* a legal derivation:

- 1 $C \supset (D \supset E)$
- 2 $(Z \equiv W) \vee (C \supset (D \supset E))$ 1, *Add* ← **MISTAKE!!!**

However, this *is* a legal derivation.

- 1 $C \supset (D \supset E)$
- 2 $(C \supset (D \supset E)) \vee (Z \equiv W)$ 1, *Add*

9.2.8 Constructive Dilemma

The final rule of inference is known as *constructive dilemma*.

<u>Constructive Dilemma (CD)</u>	
	$(p \supset q) \bullet (r \supset s)$
	$p \vee r$
\triangleright	$q \vee s$

This rule says the following: if you have a wff of the form ' $(p \supset q) \bullet (r \supset s)$ ' written on an accessible line and a wff of the form ' $p \vee r$ ' written on an accessible line, then you may write down ' $q \vee s$ '. In your justification, you should cite the line numbers of the lines on which ' $(p \supset q) \bullet (r \supset s)$ ' and ' $p \vee r$ ' appear, and write '*CD*'

NOTE: one of the lines appealed to must be of the form ' $(p \supset q) \bullet (r \supset s)$ '. You may *not* appeal to two lines, one of the form ' $p \supset q$ ' and one of the form ' $r \supset s$ '. For instance, the following derivation is *not* legal:

- 1 $A \supset (Q \vee R)$
- 2 $B \supset (T \equiv V)$
- 3 $A \vee B$
- 4 $(Q \vee R) \vee (T \equiv V)$ 1, 2, 3, *CD* \leftarrow MISTAKE!!!

However, the following derivation *is* legal:

- 1 $A \supset (Q \vee R)$
- 2 $B \supset (T \equiv V)$
- 3 $A \vee B$
- 4 $(A \supset (Q \vee R)) \cdot (B \supset (T \equiv V))$ 1, 2, *Conj*
- 5 $(Q \vee R) \vee (T \equiv V)$ 3, 4, *CD*

NOTE: Here, too, the order of both the conjuncts in ' $(p \supset q) \cdot (r \supset s)$ ' and the disjuncts in ' $p \vee r$ ' and ' $q \vee s$ ' matters. For instance, the following derivations are *not* legal:

- 1 $(A \supset B) \cdot (C \supset D)$
- 2 $C \vee A$
- 3 $B \vee D$ 1, 2, *CD* \leftarrow MISTAKE!!!

- 1 $(A \supset B) \cdot (C \supset D)$
- 2 $A \vee C$
- 3 $D \vee B$ 1, 2, *CD* \leftarrow MISTAKE!!!

This, however, *is* a legal derivation:

- 1 $(A \supset B) \cdot (C \supset D)$
- 2 $A \vee C$
- 3 $B \vee D$ 1, 2, *CD*

A Sample Derivation

1	$A \bullet B$	
2	$(A \vee C) \supset ((D \supset E) \bullet F)$	
3	$G \supset H$	
4	$D \vee G$	$/E \vee H$
5	A	1, <i>Simp</i>
6	$A \vee C$	5, <i>Add</i>
7	$(D \supset E) \bullet F$	2, 6, <i>MP</i>
8	$D \supset E$	7, <i>Simp</i>
9	$(D \supset E) \bullet (G \supset H)$	3, 8, <i>Conj</i>
10	$E \vee H$	4, 9, <i>CD</i>

9.3 A Mistake to Avoid

Rules of implication *may not* be applied to subformulae. For instance, the following derivation is *not* legal.

1	$P \supset (Q \supset R)$	
2	Q	
3	$P \supset R$	1, 2, <i>MP</i> ← MISTAKE!!!

Modus Ponens allows you to write down ' R ' if you have ' $Q \supset R$ ' written down on an accessible line and ' Q ' written down on an accessible line. However, it does not allow you to swap out ' R ' for ' $Q \supset R$ ' if you have ' Q ' written on an accessible line, and ' $Q \supset R$ ' is merely a *subformulae* of a wff on an accessible line.

9.4 Rules of Replacement

The rules in this section are known as *rules of replacement*. What makes them rules of *replacement* is 1) that they are *two way*; and 2) that they may be applied to *subformulae* of the wffs of *PL* appearing on the lines of your derivation. They allow you to substitute one wff of *PL* for another, and they allow you to substitute the other for the one. Each rule of replacement encodes a certain *PL*-equivalence. Each of these rules tell you that you may substitute certain subformulae of a wff of *PL* for other certain subformulae of *PL* which are *equivalent* to them. It does this only when the substituted wff of *PL* is *PL*-equivalent to the one it replaces. You could check each of these rules with truth-tables to verify that the two wffs of *PL* that the rules allow you to interchange are *PL*-equivalent, if you wanted.

Let me reiterate this point, because it is important: **Rules of Replacement, unlike Rules of Implication, may be applied to subformulae.**

9.4.1 De Morgan's

Our first rule of replacement is known as *De Morgan's*, after the British mathematician and logician Augustus De Morgan, who first presented the equivalences between $\sim(p \bullet q)$ and $\sim p \vee \sim q$ and between $\sim(p \vee q)$ and $\sim p \bullet \sim q$.

<p><u>De Morgan's (DM)</u></p> $\sim(p \bullet q) \triangleleft \triangleright \sim p \vee \sim q$ $\sim(p \vee q) \triangleleft \triangleright \sim p \bullet \sim q$
--

This rule actually allows four distinct replacements (one corresponding to each ' \triangleright '). It says:

1. if you have a subformula of the form ' $\sim(p \bullet q)$ ' within a wff on an accessible line, you may replace that subformula with ' $\sim p \vee \sim q$ '. When you do so, cite the line on which the wff containing ' $\sim(p \bullet q)$ ' appears and write 'DM'.
2. Similarly, if you have a subformula of the form ' $\sim p \vee \sim q$ ' within a wff on an accessible line, you may replace that subformula with ' $\sim(p \bullet q)$ '. When you do so, cite the line on which the wff containing ' $\sim p \vee \sim q$ ' appears and write 'DM'.
3. Additionally, if you have a wff of the form ' $\sim(p \vee q)$ ' within a wff on an accessible line, you may replace that subformula with ' $\sim p \bullet \sim q$ '. When you do so, cite the line on which the wff containing ' $\sim(p \vee q)$ ' appears and write 'DM'.
4. Similarly, if you have a subformula of the form ' $\sim p \bullet \sim q$ ' within a wff on an accessible line, you may replace that subformula with ' $\sim(p \vee q)$ '. When you do so, cite the line on which the wff containing ' $\sim p \bullet \sim q$ ' appears and write 'DM'.

NOTE: In order for DeMorgan's rule to apply, ' $\sim(p \vee q)$ ' (for example) must *actually be* a subformula of a wff on an accessible line. For instance, the following derivation is *not* legal:

$$\begin{array}{l}
 1 \quad A \supset (\sim B \vee C) \\
 2 \quad A \supset (\sim B \bullet \sim C) \quad 1, DM \leftarrow \text{MISTAKE!!!}
 \end{array}$$

Here, ' $\sim B \vee C$ ' is *not* of the form ' $\sim(p \vee q)$ ' (because it is missing the parentheses). However, the following derivation *is* legal:

$$\begin{array}{l}
 1 \quad A \supset \sim(B \vee C) \\
 2 \quad A \supset (\sim B \bullet \sim C) \quad 1, DM
 \end{array}$$

9.4.2 Commutativity

An operator is said to be *commutative* if and only if it doesn't matter which order the operation is applied. So multiplication is commutative, because $x \times y = y \times x$, for all x and y . Similarly, addition is commutative, because $x + y = y + x$, for all x and y . Our next rule of replacement tells us that conjunction and disjunction is commutative.

<p><u>Commutativity (Com)</u></p> $p \vee q \triangleright q \vee p$ $p \bullet q \triangleright q \bullet p$

This rule says:

1. If you have a subformula of the form ' $p \vee q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $q \vee p$ '. When you do so, cite the line number on which the wff containing ' $p \vee q$ ' appears and write 'Com'.
2. If you have a subformula of the form ' $p \bullet q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $q \bullet p$ '. When you do so, cite the line number on which the wff containing ' $p \bullet q$ ' appears and write 'Com'.

9.4.3 Associativity

In general, an operation \circ is said to be *associative* if and only if the result of applying the operation to two things, x and y , and the applying it to a third, z , $(x \circ y) \circ z$, will always give you the same result as applying it to x and the result of applying it to $y \circ z$, $x \circ (y \circ z)$. This property is not trivial. For instance, it doesn't apply to the material conditional, as $p \supset (q \supset r)$ need not be the same as $(p \supset q) \supset r$.

An operation may be commutative without being associative. For instance, consider the operation ' $[child]$ ', which tells you who the child of two people, a and b , is—and if they have no child together, returns 'none'. Then, the child of a and b is the child of b and a — $a[child]b = b[child]a$ even though the child of the child of a and b and c is not in general the child of a and the child of b and c — $(a[child]b)[child]c \neq a[child](b[child]c)$.

Similarly, an operation may be associative without being commutative. For instance, consider the operation $*$ which simply hands back the thing to the left of the operation—e.g., $x * y = x$, for all x and y . Then, for all x and y , $(x * y) * z = x * (y * z) = x$. However, for any distinct x and y , $x * y \neq y * x$.

The next rule of replacement tells us that conjunction and disjunction are associative.

<p><u>Associativity (Assoc)</u></p> $(p \vee q) \vee r \triangleleft \triangleright p \vee (q \vee r)$ $(p \bullet q) \bullet r \triangleleft \triangleright p \bullet (q \bullet r)$

This rule says:

1. If you have a subformula of the form ' $(p \vee q) \vee r$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \vee (q \vee r)$ '. When you do so, cite the line number on which the wff containing ' $(p \vee q) \vee r$ ' appears and write 'Assoc'.
2. Similarly, if you have a subformula of the form ' $p \vee (q \vee r)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \vee q) \vee r$ '. When you do so, cite the line number on which the wff containing ' $p \vee (q \vee r)$ ' appears and write 'Assoc'.
3. Additionally, if you have a subformula of the form ' $(p \bullet q) \bullet r$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \bullet (q \bullet r)$ '. When you do so, cite the line number on which the wff containing ' $(p \bullet q) \bullet r$ ' appears and write 'Assoc'.
4. Similarly, if you have a subformula of the form ' $p \bullet (q \bullet r)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \bullet q) \bullet r$ '. When you do so, cite the line number on which the wff containing ' $p \bullet (q \bullet r)$ ' appears and write 'Assoc'.

NOTE: It is important that the subformula contains *only* ' \vee 's or *only* ' \bullet 's. For instance, the following derivation is *not* legal:

$$\begin{array}{l}
1 \quad A \equiv (B \vee (C \bullet D)) \\
2 \quad A \equiv ((B \vee C) \bullet D) \quad 1, \text{Assoc} \leftarrow \text{MISTAKE!!!}
\end{array}$$

This derivation, on the other hand, *is* legal:

$$\begin{array}{l}
1 \quad A \equiv (B \vee (C \vee D)) \\
2 \quad A \equiv ((B \vee C) \vee D) \quad 1, \text{Assoc}
\end{array}$$

9.4.4 Distribution

While, with conjunction and disjunction *individually*, it doesn't matter which order we apply the operations in, if you're applying *both* operations, one after the other, then it *does* matter the order you apply them in. This is like addition and multiplication. In general, $x + (y \times z) \neq (x + y) \times z$ and $x \times (y + z) \neq (x \times y) + z$. Similarly, in general, $p \bullet (q \vee r) \neq (p \bullet q) \vee r$, and $p \vee (q \bullet r) \neq (p \vee q) \bullet r$. However, there is a relationship between sequential applications of the operations '+' and '×'. For instance, $x \times (y + z) = (x \times y) + (x \times z)$.

Similarly, there is a relationship between sequential applications of the operations '•' and '∨'. That relationship is encoded into our derivation system with the rule of replacement *distribution*.

<p><i>Distribution (Dist)</i></p> $p \bullet (q \vee r) \triangleleft \triangleright (p \bullet q) \vee (p \bullet r)$ $p \vee (q \bullet r) \triangleleft \triangleright (p \vee q) \bullet (p \vee r)$
--

This rule says:

1. If you have a subformula of the form ' $p \bullet (q \vee r)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \bullet q) \vee (p \bullet r)$ '. When you do so, cite the line number on which the wff containing ' $p \bullet (q \vee r)$ ' appears and write '*Dist*'.
2. Similarly, if you have a subformula of the form ' $(p \bullet q) \vee (p \bullet r)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \bullet (q \vee r)$ '. When you do so, cite the line number on which the wff containing ' $(p \bullet q) \vee (p \bullet r)$ ' appears and write '*Dist*'.
3. Additionally, if you have a subformula of the form ' $(p \bullet q) \bullet r$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \bullet (q \bullet r)$ '. When you do so, cite the line number on which the wff containing ' $(p \bullet q) \bullet r$ ' appears and write '*Dist*'.
4. Similarly, if you have a subformula of the form ' $p \bullet (q \bullet r)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \bullet q) \bullet r$ '. When you do so, cite the line number on which the wff containing ' $p \bullet (q \bullet r)$ ' appears and write '*Dist*'.

NOTE: When you apply Distribution to a subformula, the main operator of that subformula should change either from a '∨' to a '•' or from a '•' to a '∨'. For instance, the following is *not* a legal derivation:

$$\begin{array}{l}
1 \quad P \equiv (A \bullet (B \vee C)) \\
2 \quad P \equiv ((A \vee B) \bullet (A \vee C)) \quad 1, \text{Dist} \leftarrow \text{MISTAKE!!!}
\end{array}$$

This derivation, on the other hand, *is* legal:

- 1 $P \equiv (A \bullet (B \vee C))$
- 2 $P \equiv ((A \bullet B) \vee (A \bullet C))$ 1, *Dist*

NOTE: As with Disjunctive Syllogism and Simplification, the order of the disjuncts and conjuncts matters. The following derivation is *not* legal:

- 1 $P \equiv ((B \vee C) \bullet A)$
- 2 $P \equiv ((B \bullet A) \vee (C \bullet A))$ 1, *Dist* ← MISTAKE!!!

This derivation, however, *is* legal:

- 1 $P \equiv ((B \vee C) \bullet A)$
- 2 $P \equiv (A \bullet (B \vee C))$ 1, *Com*
- 3 $P \equiv ((A \bullet B) \vee (A \bullet C))$ 2, *Dist*

9.4.5 Double Negation

Our next rule of replacement, called *double negation*, tells us that we may always eliminate, or introduce, a pair of adjacent tildes in front of a wff of *PL*.

<p><i>Double Negation (DN)</i></p> $p \triangleleft \triangleright \sim\sim p$
--

This rule says:

1. If you have a subformula of the form ' p ' within a wff appearing on an accessible line, then you may replace that subformula with ' $\sim\sim p$ '. When you do so, cite the line number on which the wff containing ' p ' appears and write '*DN*'.
2. If you have a subformula of the form ' $\sim\sim p$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' p '. When you do so, cite the line number on which the wff containing ' p ' appears and write '*DN*'.

A Sample Derivation

1	$A \vee (B \cdot C)$	
2	$\sim B$	
3	$\sim\sim A \supset \sim(D \vee E)$	
4	$\sim D \supset F$	$/F$
5	$(A \vee B) \cdot (A \vee C)$	1, <i>Dist</i>
6	$A \vee B$	5, <i>Simp</i>
7	$B \vee A$	6, <i>Com</i>
8	A	2, 7, <i>DS</i>
9	$\sim\sim A$	8, <i>DN</i>
10	$\sim\sim A \supset (\sim D \cdot \sim E)$	3, <i>DM</i>
11	$\sim D \cdot \sim E$	9, 10, <i>MP</i>
12	$\sim D$	11, <i>Simp</i>
13	F	4, 12, <i>MP</i>

9.4.6 Transposition

The next rule of replacement is known as *transposition* (or, perhaps more commonly, as *contraposition*—though we'll stick to Hurley's terminology here).

<u>Transposition (Trans)</u>
$p \supset q \triangleleft \sim q \supset \sim p$

This rule says:

1. If you have a subformula of the form ' $p \supset q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $\sim q \supset \sim p$ '. When you do so, cite the line number on which the wff containing ' $p \supset q$ ' appears and write '*Trans*'.
2. If you have a subformula of the form ' $\sim q \supset \sim p$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \supset q$ '. When you do so, cite the line number on which the wff containing ' $\sim q \supset \sim p$ ' appears and write '*Trans*'.

NOTE: The subformula which you replace and the one with which you replace it **must actually be of the forms ' $p \supset q$ ' and ' $\sim q \supset \sim p$ '**. It is not enough that they are *PL*-equivalent to wffs of those forms. For instance, the following derivation is *not* legal:

1	$A \equiv (B \supset \sim C)$	
2	$A \equiv (C \supset \sim B)$	1, <i>Trans</i> ← MISTAKE!!!

This derivation, however, *is* legal:

- 1 $A \equiv (B \supset \sim C)$
- 2 $A \equiv (\sim \sim C \supset \sim B)$ 1, *Trans*
- 3 $A \equiv (C \supset \sim B)$ 2, *DN*

9.4.7 Material Implication

The next rule of inference is known as *material implication*. It tells us, essentially, that a material conditional $p \supset q$ is true if and only if either its antecedent is false or its consequent is true, $\sim p \vee q$.

<p><i>Material Implication (Impl)</i></p> $p \supset q \triangleq \sim p \vee q$
--

This rule says:

1. If you have a subformula of the form ' $p \supset q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $\sim p \vee q$ '. When you do so, cite the line number on which the wff containing ' $p \supset q$ ' appears and write '*Impl*'.
2. If you have a subformula of the form ' $\sim p \vee q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \supset q$ '. When you do so, cite the line number on which the wff containing ' $\sim p \vee q$ ' appears and write '*Impl*'.

9.4.8 Material Equivalence

This rule of replacement is the only one governing the material biconditional, \equiv . It encodes the fact that a material biconditional $p \equiv q$ is equivalent both to the conjunction of two material conditional, $(p \supset q) \bullet (q \supset p)$ and to the disjunction of the two conjunctions, $(p \bullet q) \vee (\sim p \bullet \sim q)$. The first equivalence tells us why the material biconditional is appropriately translated as 'if and only if', and the second equivalence holds because a material biconditional is true if and only if its left and right hand sides have the same truth-value.

<p><i>Material Equivalence (Equiv)</i></p> $p \equiv q \triangleq (p \supset q) \bullet (q \supset p)$ $p \equiv q \triangleq (p \bullet q) \vee (\sim p \bullet \sim q)$

This rule says:

1. If you have a subformula of the form ' $p \equiv q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \supset q) \bullet (q \supset p)$ '. When you do so, cite the line number on which the wff containing ' $p \equiv q$ ' appears and write '*Equiv*'.
2. Similarly, if you have a subformula of the form ' $(p \supset q) \bullet (q \supset p)$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \equiv q$ '. When you do so, cite the line number on which the wff containing ' $(p \supset q) \bullet (q \supset p)$ ' appears and write '*Equiv*'.
3. Additionally, if you have a subformula of the form ' $p \equiv q$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' $(p \bullet q) \vee (\sim p \bullet \sim q)$ '. When you do so, cite the line number on which the wff containing ' $p \equiv q$ ' appears and write '*Equiv*'.

4. Similarly, if you have a subformula of the form $(p \bullet q) \vee (\sim p \bullet \sim q)$ within a wff appearing on an accessible line, then you may replace that subformula with $p \equiv q$. When you do so, cite the line number on which the wff containing $(p \bullet q) \vee (\sim p \bullet \sim q)$ appears and write *Equiv*.

NOTE: Here, as with Disjunctive Syllogism and Simplification, the order of the disjuncts and the conjuncts matter. The following derivations are *not* legal:

$$\begin{array}{l} 1 \quad (A \vee B) \equiv Q \\ 2 \quad (\sim(A \vee B) \bullet \sim Q) \vee ((A \vee B) \bullet Q) \quad 1, \text{Equiv} \leftarrow \text{MISTAKE!!!} \end{array}$$

$$\begin{array}{l} 1 \quad (A \vee B) \equiv Q \\ 2 \quad (Q \bullet (A \vee B)) \vee (\sim Q \bullet \sim(A \vee B)) \quad 1, \text{Equiv} \leftarrow \text{MISTAKE!!!} \end{array}$$

This derivation, however, *is* legal:

$$\begin{array}{l} 1 \quad (A \vee B) \equiv Q \\ 2 \quad ((A \vee B) \bullet Q) \vee (\sim(A \vee B) \bullet \sim Q) \quad 1, \text{Equiv} \end{array}$$

9.4.9 Exportation

The next rule of replacement is known as *exportation*. It encodes the fact that $p \supset (q \supset r)$ is *PL*-equivalent to $(p \bullet q) \supset r$.

<u>Exportation (Exp)</u> $(p \bullet q) \supset r \triangleq p \supset (q \supset r)$
--

This rule says:

1. If you have a subformula of the form $(p \bullet q) \supset r$ within a wff appearing on an accessible line, then you may replace that subformula with $p \supset (q \supset r)$. When you do so, cite the line number on which the wff containing $(p \bullet q) \supset r$ appears and write *Exp*.
2. If you have a subformula of the form $p \supset (q \supset r)$ within a wff appearing on an accessible line, then you may replace that subformula with $(p \bullet q) \supset r$. When you do so, cite the line number on which the wff containing $p \supset (q \supset r)$ appears and write *Exp*.

<u>Tautology (Taut)</u> $p \triangleq p \vee p$ $p \triangleq p \bullet p$
--

This rule says:

1. If you have a subformula of the form ' p ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \vee p$ '. When you do so, cite the line number on which the wff containing ' p ' appears and write '*Taut*'.
2. Similarly, if you have a subformula of the form ' $p \vee p$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' p '. When you do so, cite the line number on which the wff containing ' $p \vee p$ ' appears and write '*Taut*'.
3. Additionally, if you have a subformula of the form ' p ' within a wff appearing on an accessible line, then you may replace that subformula with ' $p \bullet p$ '. When you do so, cite the line number on which the wff containing ' p ' appears and write '*Taut*'.
4. Similarly, if you have a subformula of the form ' $p \bullet p$ ' within a wff appearing on an accessible line, then you may replace that subformula with ' p '. When you do so, cite the line number on which the wff containing ' $p \bullet p$ ' appears and write '*Taut*'.

A Sample Derivation

1	$A \equiv \sim A$	$/A \bullet \sim A$
2	$(A \bullet \sim A) \vee (\sim A \bullet \sim \sim A)$	1, <i>Equiv</i>
3	$(\sim A \bullet A) \vee (\sim A \bullet \sim \sim A)$	2, <i>Com</i>
4	$\sim A \bullet (A \vee \sim \sim A)$	3, <i>Dist</i>
5	$\sim A \bullet (A \vee A)$	4, <i>DN</i>
6	$\sim A \bullet A$	5, <i>Taut</i>
7	$A \bullet \sim A$	6, <i>Com</i>

Chapter 10

Propositional Logic Derivations, Day 2

10.1 Four Final Rules of Inference

Last time, we covered 38 rules. We've just got four more to cover. However, these rules are very special—in part, because they are so powerful; and in part, because they are altogether different from the rules which preceded them.

10.1.1 Subderivations

First, we need to introduce the idea of a SUBDERIVATION. A subderivation is a kind of *suppositional* derivation which takes place within another derivation. To indicate that the subderivation is suppositional, we indent those lines of the derivation which are taking place in the subderivation and place a *scope line* to the left of all those wffs which are within the scope of the supposition. For instance, the following is a derivation utilizing a subderivation.

1	$(A \supset B) \supset C$	
2	$B \bullet D$	$/C$
3	A	ACP
4	B	$2, Simp$
5	$A \supset B$	$3-4, CP$
6	C	$1, 5, MP$

The subderivation takes place from lines 3–4, as indicated by the indentation and the vertical scope line which runs from line 3 to line 4.

The intuitive idea behind a subderivation is this: even if our premises don't tell us that p , we might just want to *suppose* that p is true, and see what follows from this supposition. Our first two new rules tell us that we may suppose anything that we wish—bar none.

Assumption for Conditional Proof (ACP)

You may, at any point in a derivation, begin a new SUBDERIVATION, and write *any wff of PL whatsoever* on the first line of that subderivation. In the justification line, you should write 'ACP'.

Assumption for Indirect Proof (AIP)

You may, at any point in a derivation, begin a new SUBDERIVATION, and write *any wff of PL whatsoever* on the first line of that subderivation. In the justification line, you should write 'AIP'.

The only difference between these two rules is the justification that you provide. Those justifications will become relevant later on, as they will end up making a difference for what you get to use your subderivations to show *outside* of the subderivation.

You may also decide to *end* a subderivation *whenever you wish*. Now, given the way that we defined *accessibility* last time, these new rules threaten to make it far too easy to prove anything whatsoever. For instance, given *ACP*, there is as yet nothing to rule out the following derivation:

1	$A \supset B$	$/ \sim A$	
2	$\sim B$	ACP	
3	$\sim A$	$1, 2, MT$	\leftarrow MISTAKE!!!

If our derivation system could be used to derive ' $\sim A$ ' from ' $A \supset B$ ', that would be disaster, since the argument $A \supset B // \sim A$ is not *PL*-valid. Fortunately, we don't allow this, since we place the following new restriction on which lines are *accessible*, and thus available to be legally cited, at a given line in the derivation:

At a given line in a derivation, n , another line of the derivation, m , is ACCESSIBLE if and only if 1) line m precedes line n ($m < n$), and 2) either i) line m lies outside the scope of any subderivation, or ii) line m lies within a subderivation whose vertical scope line extends to line n .

Moreover, since the two new rules below will allow us to cite, not just individual *lines* within a derivation, but rather *entire subderivations*, we will have to define which subderivations are accessible at a given line:

An *entire subderivation* is ACCESSIBLE at line n so long as 1) the subderivation precedes line n , and 2) either i) that subderivation is outside the scope of any other subderivation, or else ii) the subderivation lines within another subderivation whose vertical scope line extends to line n .

Another, simpler way of putting the same point is this: while you may end a subderivation whenever you wish, once you do so, none of the lines or subderivations appearing within the scope of that subderivation are accessible any longer.

For illustration, consider the following (legal) *PL*-derivation.

1	$C \bullet Z$	
2	A	ACP
3	B	ACP
4	C	$1, Simp$
5	$B \supset C$	$3-4, CP$
6	$A \supset (B \supset C)$	$2-5, CP$
7	$B \supset A$	ACP
8	C	$1, Simp$
9	$(B \supset A) \supset C$	$7-8, CP$
10	$(A \supset (B \supset C)) \bullet ((B \supset A) \supset C)$	$6, 9, Conj$

The following table shows all of the accessibility relations amongst the lines and subderivations in this derivation (subderivations are identified by the range of line numbers they span; e.g., the subderivation running from line 3 through line 4 is identified with ‘3–4’; this is very different than the citation ‘3, 4’, which cites the lines 3 and 4. ‘3–4’ does not cite either line 3 or line 4, both of which are inaccessible at line 5; rather, it cites the entire subderivation running from line 3 to line 4, which *is* accessible at line 5):

Line	Accessible Lines/ Subderivations	Inaccessible Lines/Subderivations (of those which precede the line)
2	1	
3	1, 2	
4	1, 2, 3	
5	1, 2, 3–4	3, 4
6	1, 2–5	2, 3, 4, 3–4, 5
7	1, 2–5, 6	2, 3, 4, 3–4, 5
8	1, 2–5, 6, 7	2, 3, 4, 3–4, 5
9	1, 2–5, 6, 7–8	2, 3, 4, 3–4, 5, 7, 8
10	1, 2–5, 6, 7–8, 9	2, 3, 4, 3–4, 5, 7, 8

On line 4, line 1 is accessible because it lies outside the scope of any subderivation and it precedes line 4. Line 2 is accessible because, even though it lies within a subderivation, that subderivation continues through to line 4 (its vertical scope line continues through to line 5). Similarly, line 3 is accessible, since, even though it lies within a subderivation, that subderivation continues through to line 4.

Once we leave the subderivation running from lines 3–4, on line 5, line 1 is still accessible, as it lies outside the scope of any subderivation. Additionally, line 2 is still accessible, since the vertical scope line of the subderivation to which it belongs continues through to line 5. However, lines 3 and 4 are *no longer* accessible. They occur within the scope of a subderivation which does not continue through to line 5. *Nevertheless*, the *entire subderivation* running from lines 3–4 *is* still accessible. It may be legitimately cited in applying a rule at line 5 (as it is here in this derivation).

Note that this changes once we end the subderivation running from lines 2–5. On line 6, the subderivation running from lines 3–4 is *no longer accessible*. Neither are any of the individual lines 2, 3, 4, or 5. *Nevertheless*, the *entire subderivation* running from lines 2–5 *is* accessible at line 6.

Similarly, down on line 9, neither line 2, 3, 4, 5, 7, nor 8 is accessible. However, lines 1 and 6 are accessible, as are the subderivations running from lines 2–6 and from lines 7–8.

10.1.2 Conditional Proof

With that background on subderivations out of the way, here is our third new rule of inference:

<i>Conditional Proof (CP)</i>		
n	p	ACP
\vdots	\vdots	
m	q	
\triangleright	$p \supset q$	$n-m, CP$

This rule says: if you have an accessible subderivation whose first line, n , is a wff of the form ' p '—and that wff is justified by the rule *ACP*—and whose last line, m , is a wff of the form ' q ', then you may write down ' $p \supset q$ '. When you do so, you should cite the entire subderivation running from line n to line m (' $n-m$ ') and write '*CP*'.

The intuitive thought here is this: we make a *supposition* that p is true. From this supposition, we are able to *derive* that q is true. So, it should be that case, *without* any supposition, that *if* p is true, *then* q is true.

Sample Derivations

1	$A \supset B$	
2	$B \supset C$	$/A \supset C$
3	A	ACP
4	B	$1, 3, MP$
5	C	$2, 4, MP$
6	$A \supset C$	$3-5, CP$

1	$A \vee (B \supset Q)$	
2	B	$/\sim A \supset Q$
3	$\sim A$	ACP
4	$B \supset Q$	$1, 3, DS$
5	Q	$2, 4, MP$
6	$\sim A \supset Q$	$3-5, CP$

1	A	$\neg B \supset (A \supset B)$
2	B	ACP
3	A	ACP
4	$B \vee B$	$2, Taut$
5	B	$4, Taut$
6	$A \supset B$	$3-5, CP$
7	$B \supset (A \supset B)$	$2-6, CP$

10.1.3 Indirect Proof

Here is the final—and most powerful—rule of inference.

<u>Indirect Proof (IP)</u>		
n	p	AIP
\vdots	\vdots	
m	$q \bullet \sim q$	
	$\triangleright \sim p$	$n-m, IP$

This rule says: If you have a subderivation whose first line, n , is a wff of the form ' p '—and that line is justified by *AIP*—and whose last line is an *explicit contradiction* of the form $q \bullet \sim q$, then you may write down ' $\sim p$ '. When you do so, you should cite the entire subderivation running from line n to line m (' $n-m$ ') and write '*IP*'.

NOTE: the explicit contradiction *must be of the form* ' $q \bullet \sim q$ '. The following derivation is *not* legal.

1	$A \bullet B$	
2	$\sim A$	AIP
3	A	$1, Simp$
4	$\sim A \bullet A$	$2, 3, Conj$
5	$\sim \sim A$	$2-4, IP \leftarrow$ MISTAKE!!!

This derivation, however, *is* legal:

1	$A \bullet B$	
2	$\sim A$	AIP
3	A	$1, Simp$
4	$A \bullet \sim A$	$2, 3, Conj$
5	$\sim \sim A$	$2-4, IP$

NOTE: what you conclude outside of the subderivation *must be the negation of the thing you assumed*. The following derivation is *not* legal.

1		$\sim(A \vee \sim A)$	<i>AIP</i>
2		$\sim A \cdot \sim\sim A$	1, <i>DM</i>
3		$A \vee \sim A$	1-2, <i>IP</i> ← MISTAKE!!!

This derivation, however, is legal.

1		$\sim(A \vee \sim A)$	<i>AIP</i>
2		$\sim A \cdot \sim\sim A$	1, <i>DM</i>
3		$\sim\sim(A \vee \sim A)$	1-2, <i>IP</i>
4		$A \vee \sim A$	3, <i>DN</i>

Sample Derivations

1		$P \equiv Q$	
2		$P \vee Q$	<i>IP</i>
3		$\sim P$	<i>AIP</i>
4		Q	2, 3, <i>DS</i>
5		$(P \supset Q) \cdot (Q \supset P)$	1, <i>Equiv</i>
6		$(Q \supset P) \cdot (P \supset Q)$	5, <i>Com</i>
7		$Q \supset P$	6, <i>Simp</i>
8		P	4, 7, <i>MP</i>
9		$P \cdot \sim P$	3, 8, <i>Conj</i>
10		$\sim\sim P$	3-9, <i>IP</i>
11		P	10, <i>DN</i>

1		$\sim(A \cdot B)$	<i>IA $\supset \sim B$</i>
2		A	<i>ACP</i>
3		B	<i>AIP</i>
4		$A \cdot B$	2, 3, <i>Conj</i>
5		$(A \cdot B) \cdot \sim(A \cdot B)$	1, 4, <i>Conj</i>
6		$\sim B$	3-5, <i>IP</i>
7		$A \supset \sim B$	2-6, <i>CP</i>

1	A	$\neg A \supset (Z \equiv W)$
2	$\neg A$	<i>ACP</i>
3	$\neg(Z \equiv W)$	<i>AIP</i>
4	$A \cdot \neg A$	1, 2, <i>Conj</i>
5	$\neg\neg(Z \equiv W)$	3-4, <i>IP</i>
6	$Z \equiv W$	5, <i>DN</i>
7	$\neg A \supset (Z \equiv W)$	2-6, <i>CP</i>

10.2 *PL*-Derivability and the Logical Notions of *PL*

10.2.1 Some New Notation

Thus far, we've been showing how to use derivations to show that *arguments* are *PL*-valid. However, we can also use derivations to establish other interesting facts about the logical notions of *PL* that we previously defined in terms of truth-tables.

First, let's introduce a more compact way of representing the claim that the argument $p_1 / p_2 / \dots / p_N // c$ is *PL*-valid. If and only if this argument is valid, I will write:

$$p_1, p_2, \dots, p_N \models_{PL} c$$

This expression just means 'the *PL*-argument whose premises are p_1, p_2, \dots , and p_N and whose conclusion is c is *PL*-valid'.

And similarly, if and only if it is possible to construct a legal *PL*-derivation whose assumptions are p_1, p_2, \dots, p_N and whose final line is c , I will write

$$p_1, p_2, \dots, p_N \vdash_{PL} c$$

This expression just means 'there is a possible legal *PL*-derivation whose assumptions are p_1, p_2, \dots, p_N , and whose final line is c '. Or, for short ' c is *PL*-derivable from p_1, p_2, \dots, p_N '.

We can use this notion of *PL*-derivability to characterize the logical notions of *PL* that we previously defined in terms of truth-tables. (Those notions, by the way, are *still* defined in terms of truth tables. The relationships I'm going to tell you about below are not mere stipulations. In more advanced logic courses, I would ask you to *prove* that these relationships hold.)

10.2.2 *PL*-Validity

Fact 1: $p_1 / p_2 / \dots / p_N // c$ is *PL*-valid if and only if c is *PL*-derivable from p_1, p_2, \dots , and p_N .

$$p_1, p_2, \dots, p_N \models_{PL} c \quad \text{if and only if} \quad p_1, p_2, \dots, p_N \vdash_{PL} c$$

We've seen this fact before. It just tells us that our derivation system is one that can be used to show that an argument of *PL* is *PL*-valid.

10.2.3 *PL*-Tautologies and *PL*-Self-Contradictions

We can, additionally, use our derivation system to show that a wff of *PL* is a *PL*-tautology, if and only if it is a *PL*-tautology; and we can use it to show that a wff of *PL* is a *PL*-self-contradiction, if and only if it is a *PL*-self-contradiction.

We defined a *PL*-tautology to be a wff of *PL* that was true in every row of the truth table. However, it turns out (in a more advanced logic course, I would ask you to *prove* this) that a wff of *PL*, p , is a *PL*-tautology if and only if there is a legal *PL*-derivation without any assumptions whose final line is p . In that case, let's say that p is '*PL*-derivable' from *no* assumptions.

Fact 2: A wff of *PL*, p , is a *PL*-tautology if and only if

$$\vdash_{PL} p$$

This is really a fantastic fact, and we should pause momentarily to marvel at it. This tells us that if there's some way of constructing a *PL*-derivation according to the 40 rules that we've encountered here which has no assumptions and whose final line is p , then p will be true in every row of the truth-table. Isn't it *fantastic*—isn't it nothing short of *amazing*—that these two procedures for discovering whether something is a *PL*-tautology should line up so nicely?

It's no accident, since the derivation system was specifically designed for this purpose; but it is a grand accomplishment that we got a derivation system which lines up *so perfectly* with the truth-table method for determining both *PL*-validity and *PL*-tautology, and—as we'll see below—all of the other logical notions of *PL* as well.

Now that we've marveled appropriately: What is it for a *PL*-derivation to have no assumptions? We have already seen a *PL*-derivation without any assumptions. Look back at the *PL*-derivation whose final line is ' $A \vee \sim A$ '. Every line of that *PL*-derivation has a justification written next to it, and all of the justifications are legal. So, it is a legal *PL*-derivation without any assumptions. Given the astonishing fact above, that *PL*-derivation tells us that ' $A \vee \sim A$ ' is a *PL*-tautology.

Here is an example of a *PL*-derivation with no assumptions establishing that $(\sim P \supset Q) \vee (P \supset R)$ is a *PL*-tautology.

1	$\sim((\sim P \supset Q) \vee (P \supset R))$	<i>AIP</i>
2	$\sim(\sim P \supset Q) \bullet \sim(P \supset R)$	1, <i>DM</i>
3	$\sim(\sim P \supset Q)$	2, <i>Simp</i>
4	$\sim(\sim\sim P \vee Q)$	3, <i>Impl</i>
5	$\sim\sim\sim P \bullet \sim Q$	4, <i>DM</i>
6	$\sim\sim\sim P$	5, <i>Simp</i>
7	$\sim(P \supset R) \bullet \sim(\sim P \supset Q)$	2, <i>Com</i>
8	$\sim(P \supset R)$	7, <i>Simp</i>
9	$\sim(\sim P \vee R)$	8, <i>Impl</i>
10	$\sim\sim P \bullet \sim R$	9, <i>DM</i>
11	$\sim\sim P$	10, <i>Simp</i>
12	$\sim\sim P \bullet \sim\sim\sim P$	6, 11, <i>Conj</i>
13	$\sim\sim((\sim P \supset Q) \vee (P \supset R))$	1–12, <i>IP</i>
14	$(\sim P \supset Q) \vee (P \supset R)$	13, <i>DN</i>

This derivation establishes that $(\sim P \supset Q) \vee (P \supset R)$ is *PL*-derivable from no assumptions,

$$\vdash_{PL} (\sim P \supset Q) \vee (P \supset R),$$

and, therefore, given **Fact 2** above, that $(\sim P \supset Q) \vee (P \supset R)$ is a *PL*-tautology.

Alternatively, we could provide the following derivation to show that $(\sim P \supset Q) \vee (P \supset R)$ is a *PL*-tautology:

1	$\sim(\sim P \supset Q)$	<i>ACP</i>
2	$\sim(\sim\sim P \vee Q)$	1, <i>Impl</i>
3	$\sim\sim\sim P \bullet \sim Q$	2, <i>DM</i>
4	$\sim\sim\sim P$	3, <i>Simp</i>

5	$\sim P$	4, <i>DN</i>
6	$\sim P \vee R$	5, <i>Add</i>
7	$P \supset R$	6, <i>Impl</i>
8	$\sim(\sim P \supset Q) \supset (P \supset R)$	1-7, <i>CP</i>
9	$\sim\sim(\sim P \supset Q) \vee (P \supset R)$	8, <i>Impl</i>
10	$(\sim P \supset Q) \vee (P \supset R)$	9, <i>DN</i>

Similarly, it turns out that a wff of *PL* is a *PL*-self-contradiction if and only if there is a legal *PL*-derivation whose only assumption is p and whose final line is $A \bullet \sim A$.

Fact 3: A wff of *PL*, p , is a *PL*-self-contradiction if and only if

$$p \vdash_{PL} A \bullet \sim A$$

Thus, **Fact 3** tells us that the following derivation establishes that $\sim(P \supset Q) \bullet \sim(Q \supset P)$ is a *PL*-self-contradiction:

1	$\sim (P \supset Q) \bullet \sim (Q \supset P)$	
2	$\sim (Q \supset P) \bullet \sim (P \supset Q)$	1, <i>Com</i>
3	$\sim (A \bullet \sim A)$	<i>AIP</i>
4	$\sim (P \supset Q)$	1, <i>Simp</i>
5	$\sim (Q \supset P)$	2, <i>Simp</i>
6	$\sim (\sim P \vee Q)$	4, <i>Impl</i>
7	$\sim \sim P \bullet \sim Q$	6, <i>DM</i>
8	$\sim (\sim Q \vee P)$	5, <i>Impl</i>
9	$\sim \sim Q \bullet \sim P$	8, <i>DM</i>
10	$\sim P \bullet \sim \sim Q$	9, <i>Com</i>
11	$\sim P$	10, <i>Simp</i>
12	$\sim \sim P$	7, <i>Simp</i>
13	$\sim P \bullet \sim \sim P$	11, 12, <i>Conj</i>
14	$\sim \sim (A \bullet \sim A)$	3–13, <i>IP</i>
15	$A \bullet \sim A$	14, <i>DN</i>

10.2.4 *PL*-Equivalence and *PL*-Contradiction

Similarly, we may use *PL*-derivations to establish that two wffs of *PL* are *PL*-equivalent by appealing to the following fact (which is, again, the kind of thing that we could prove to be true—*PL*-equivalence is still defined in terms of truth-value assignments; it is a fantastic achievement that we were able to get a derivation system for which this fact is true).

Fact 4: Two wffs of *PL*, p and q are *PL*-equivalent if and only if

$$\vdash_{PL} p \equiv q$$

That is: if you have a *PL*-derivation with no assumptions and whose final line is of the form $p \equiv q$, then you have shown that p and q are *PL*-equivalent.

For instance, given **Fact 4**, the following derivation shows that $\sim (P \equiv Q)$ and $P \equiv \sim Q$ are *PL*-equivalent.

1	$\sim (P \equiv Q)$	<i>ACP</i>
2	$\sim ((P \bullet Q) \vee (\sim P \bullet \sim Q))$	1, <i>Equiv</i>
3	$\sim (P \bullet Q) \bullet \sim (\sim P \bullet \sim Q)$	2, <i>DM</i>
4	$(\sim P \vee \sim Q) \bullet \sim (\sim P \bullet \sim Q)$	3, <i>DM</i>
5	$(\sim P \vee \sim Q) \bullet (\sim \sim P \vee \sim \sim Q)$	4, <i>DM</i>
6	$(P \supset \sim Q) \bullet (\sim \sim P \vee \sim \sim Q)$	5, <i>Impl</i>
7	$(P \supset \sim Q) \bullet (\sim \sim Q \vee \sim \sim P)$	6, <i>Com</i>
8	$(P \supset \sim Q) \bullet (\sim \sim Q \vee P)$	7, <i>DN</i>
9	$(P \supset \sim Q) \bullet (\sim Q \supset P)$	8, <i>Impl</i>
10	$P \equiv \sim Q$	9, <i>Equiv</i>
11	$\sim (P \equiv Q) \supset (P \equiv \sim Q)$	1–10, <i>CP</i>
12	$P \equiv \sim Q$	<i>ACP</i>
13	$(P \supset \sim Q) \bullet (\sim Q \supset P)$	12, <i>Equiv</i>
14	$(P \supset \sim Q) \bullet (\sim \sim Q \vee P)$	13, <i>Impl</i>
15	$(P \supset \sim Q) \bullet (\sim \sim Q \vee \sim \sim P)$	14, <i>DN</i>
16	$(P \supset \sim Q) \bullet (\sim \sim P \vee \sim \sim Q)$	15, <i>Com</i>
17	$(\sim P \vee \sim Q) \bullet (\sim \sim P \vee \sim \sim Q)$	16, <i>Impl</i>
18	$\sim (P \bullet Q) \bullet (\sim \sim P \vee \sim \sim Q)$	17, <i>DM</i>
19	$\sim (P \bullet Q) \bullet \sim (\sim P \bullet \sim Q)$	18, <i>DM</i>
20	$\sim ((P \bullet Q) \vee (\sim P \bullet \sim Q))$	19, <i>DM</i>
21	$\sim (P \equiv Q)$	20, <i>Equiv</i>
22	$(P \equiv \sim Q) \supset \sim (P \equiv Q)$	12–21, <i>CP</i>
23	$(\sim (P \equiv Q) \supset (P \equiv \sim Q)) \bullet ((P \equiv \sim Q) \supset \sim (P \equiv Q))$	11, 22, <i>Conj</i>
24	$\sim (P \equiv Q) \equiv (P \equiv \sim Q)$	23, <i>Equiv</i>

In a similar fashion, we can appeal to the following fact to use our derivation system to show that two wffs of *PL* are *PL*-contradictories.

Fact 5: Two wffs of *PL*, p and q are *PL*-contradictories if and only if

$$\vdash_{PL} p \equiv \sim q$$

Fact 5 tells us that, if $p \equiv \sim q$ is *PL*-derivable from no assumptions, then p and $\sim q$ are *PL*-contradictories: that is, whenever p is true, q is false; and whenever p is false, q is true.

Therefore, the following *PL*-derivation shows that the two wffs of *PL*, $J \supset K$ and $J \bullet \sim K$, are *PL*-contradictories.

1	$J \supset K$	<i>ACP</i>
2	$\sim J \vee K$	<i>1, Impl</i>
3	$\sim J \vee \sim \sim K$	<i>2, DN</i>
4	$\sim (J \bullet \sim K)$	<i>3, DM</i>
5	$(J \supset K) \supset \sim (J \bullet \sim K)$	<i>1-4, CP</i>
6	$\sim (J \bullet \sim K)$	<i>ACP</i>
7	$\sim J \vee \sim \sim K$	<i>6, DM</i>
8	$J \supset \sim \sim K$	<i>7, Impl</i>
9	$J \supset K$	<i>8, DN</i>
10	$\sim (J \bullet \sim K) \supset (J \supset K)$	<i>6-9, CP</i>
11	$((J \supset K) \supset \sim (J \bullet \sim K)) \bullet (\sim (J \bullet \sim K) \supset (J \supset K))$	<i>5, 10, Conj</i>
12	$(J \supset K) \equiv \sim (J \bullet \sim K)$	<i>11, Equiv</i>

10.2.5 *PL*-Inconsistency

We can similarly use *PL*-derivations to show that a set of wffs of *PL* is *PL*-inconsistent, by appealing to the following fact.

Fact 6: A set of wffs of *PL*, $\{q_1, q_2, \dots, q_N\}$ is *PL*-inconsistent if and only if

$$q_2, \dots, q_N \vdash_{PL} \sim q_1$$

That is: if, by beginning a *PL*-derivation with all but one of the members of a set of wffs of *PL*, we can construct a legal derivation whose final line is the negation of the remaining member, then the original set of wffs of *PL* is *PL*-inconsistent.

Fact 6 tells us that the following *PL*-derivation establishes that the set $\{A \supset (B \supset C), A \bullet B, A \supset \sim C\}$ is *PL*-inconsistent.

1	$A \supset (B \supset C)$	
2	$A \bullet B$	
3	$A \supset \sim C$	<i>AIP</i>
4	$(A \bullet B) \supset C$	<i>1, Exp</i>
5	C	<i>2, 4, MP</i>

6	A	<i>2, Simp</i>
7	$\sim C$	<i>3, 6, MP</i>
8	$C \bullet \sim C$	<i>5, 7, Conj</i>
9	$\sim (A \supset \sim C)$	<i>3-8, IP</i>

Alternatively, we could have begun with $A \supset (B \supset C)$ and $A \supset \sim C$ as assumptions and derived $\sim (A \bullet B)$, as so:

1	$A \supset (B \supset C)$	
2	$A \supset \sim C$	
3	$A \bullet B$	<i>AIP</i>
4	$B \bullet A$	3, <i>Com</i>
5	A	3, <i>Simp</i>
6	$B \supset C$	1, 5, <i>MP</i>
7	B	4, <i>Simp</i>
8	C	6, 7, <i>MP</i>
9	$\sim C$	2, 5, <i>MP</i>
10	$C \bullet \sim C$	8, 9, <i>Conj</i>
11	$\sim (A \bullet B)$	3–10, <i>IP</i>

Finally, we could have begun with the assumptions $A \bullet B$ and $A \supset \sim C$ and derived $\sim (A \supset (B \supset C))$, as so:

1	$A \bullet B$	
2	$A \supset \sim C$	
3	$A \supset (B \supset C)$	<i>AIP</i>
4	A	1, <i>Simp</i>
5	$B \supset C$	3, 4, <i>MP</i>
6	$B \bullet A$	1, <i>Com</i>
7	B	6, <i>Simp</i>
8	C	5, 7, <i>MP</i>
9	$\sim C$	2, 4, <i>MP</i>
10	$C \bullet \sim C$	8, 9, <i>Conj</i>
11	$\sim (A \supset (B \supset C))$	3–10, <i>IP</i>

Fact 6 assures us that any one of these derivations, on its own, is sufficient to show that the set

$$\{A \supset (B \supset C), A \bullet B, A \supset \sim C\}$$

is *PL*-inconsistent.

Chapter 11

Categorical Propositions

Before looking at the modern theory of Quantificational Logic—*QL*—we’re going to spend a bit of time looking at the ancient and medieval system of logic which theorized about *categorical syllogisms*. Hopefully, this will serve two goals. Firstly, you will come first to an informal understanding of what’s going on in Quantificational Logic (and how its assumptions about, for instance, the meaning of ‘some’ and ‘all’ might differ from your intuitive conception of those words). Secondly, by comparison with the relatively impoverished system of logic that we get from the theory of categorical syllogisms, you will come to appreciate just how much of an improvement modern Quantificational Logic represents.

11.1 Categorical Propositions

All of the following are examples of *categorical propositions*:

1. Everyone loves a Georgia peach.
2. No animal eats its young.
3. Somebody loves Kesha.
4. Some hats are not fashionable.

What makes them *categorical propositions* is that they say something about how the members of one class are related to the members of another class. Translated into talk of classes, the above statements say the following things:

1. Everything in the class of people is contained in the class of things that love a Georgia peach.
2. Nothing in the class of animals is contained in the class of things that eat their young.
3. Something in the class of people is contained in the class of things that love Kesha.
4. Something in the class of hats is not contained in the class of things that are fashionable clothing.

For our purposes, we will look at four *standard form categorical propositions*:

- A) All *S* are *P*
- E) No *S* are *P*

- I) Some S are P
- O) Some S are not P

Claim (1) above could easily be translated into the form of A above, by setting S to “people” and setting P to “Georgia peach lovers”. Claim (2) could easily be translated into the form of E above by setting S to “animals” and setting P to “young-eaters”. Claim (3) could easily be translated into the form of I above by setting S to “people” and setting P to “Kesha-lovers”. Finally, claim (4) could easily be translated into the form of O above by setting S to “hats” and setting P to “fashionable clothing”.

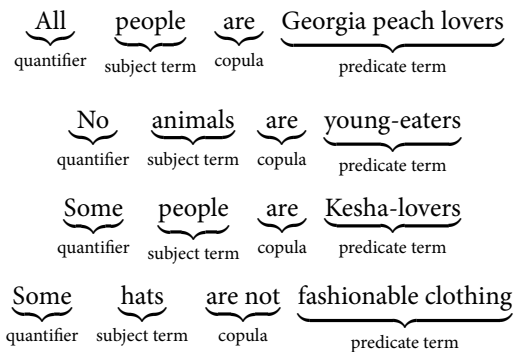
Because our interest in the logical system of categorical syllogisms will be fleeting, we won’t worry too much about how to translate statements into standard form categorical propositions; hopefully it is clear enough how to proceed in the clear cases we’ll be considering here.

11.1.1 The Components of a Categorical Proposition

A categorical proposition has four components:

1. A *quantifier*—either ‘All’, ‘No’, or ‘Some’
2. A *subject term*—the thing denoted in A, E, I, and O above with an ‘ S ’
3. A *copula*—either ‘are’ or ‘are not’
4. A *predicate term*—the thing denoted in A, E, I, and O above with a ‘ P ’

Applied to the categorical propositions with which we began—translated into standard form categorical propositions—these four components are as listed below.



One important point to note about the way that the words ‘subject term’ and ‘predicate term’ are being used here is that they *need not line up with the grammatical subject and the grammatical predicate*. For instance, “are” is included in the grammatical predicate in the sentence “Some people are Kesha-lovers”, but “are” is not included in the *predicate term*, as we are using that notion here.

An important thing to note about the word ‘some’ is that ‘Some S are P ’ does *not* imply that some S are not P . It could of course *be* that some S are P while some other S are not P . However, it could also be that some S are P while *no* S are not P . That is, if there are some S , and all of those S are P , then it is also true that some of those S are P . That is, the following two claims are consistent:

1. Everyone loves Kesha
2. Somebody loves Kesha

11.1.2 Quality, Quantity, and Distribution

There are three properties of categorical propositions which will be relevant later on: their *quality* (which may be either *affirmative* or *negative*), their *quantity* (which may be either *universal* or *particular*), and whether they *distribute* their subject or predicate terms or not.

Quality

Categorical propositions may either *affirm* membership in the predicate class, for either some or all of the things in the subject class, or they may *deny* membership in the predicate class, for either some or all of the things in the subject class. For instance, in both of the claims

- 1) All people are Georgia peach lovers.
- 3) Some people are Kesha-lovers.

membership in the predicate class is *affirmed*, for either some or all of the things in the subject class. In the case of (1), it is affirmed that *everything* in the class of people is also in the class of Georgia peach lovers. In the case of (2), it is affirmed that *something* in the class of people is also in the class of Kesha-lovers.

On the other hand, in both of the claims

- 2) No animals are young-eaters.
- 4) Some hats are not fashionable clothing.

membership in the predicate class is *denied*, for either some or all of the things in the subject class. In the case of (2), it is denied that there are *any* things in the class of animals which are in the class of young-eaters, and in the case of (4), it is denied that all hats are in the class of fashionable clothing.

This property of a categorical proposition—whether it is *affirming* or *denying* membership in the predicate class—is known as its **QUALITY**. The quality of a categorical proposition may be either affirmative (if it affirms membership in the predicate class) or negative (if it denies membership in the predicate class). Type A and type I categorical propositions are *affirmative*, while type E and type O categorical propositions are *negative*.

Quantity

Some categorical propositions say something about *all* members of the subject class; whereas some categorical propositions say something only about *some* members of the subject class. For instance, in both of the claims

- 1) All people are Georgia peach lovers.
- 2) No animals are young-eaters.

we are told something about *all* of the things in the subject class. In (1), we are told that *everything* in the class of people is also in the class of Georgia peach lovers. In (2), we are told that *everything* in the class of animals is *not* in the class of young-eaters.

On the other hand, in both of the claims

- 3) Some people are Kesha-lovers.

4) Some hats are not fashionable clothing.

We are told something only about *some* of the things in the subject class. In (3), we are told that *something* in the class of people is also in the class of Kesha-lovers. In (4), we are told that *something* in the class of hats is not in the class of fashionable clothing.

This property of a categorical proposition—whether it is talking about *some* or *all* members of the subject class—is known as its QUANTITY. Its quantity may be either universal (if it says something about *all* members of the subject class) or particular (if it only says something about *some* members of the subject class).

The names given to the four standard form categorical propositions,

- A) All *S* are *P*
- E) No *S* are *P*
- I) Some *S* are *P*
- O) Some *S* are not *P*

is meant to reflect these first two properties. For the latin for “affirm” is *affirmo*, and the latin for “deny” is *nego*. The *first* vowel of *affirmo*, **A**, is used to denote the universal affirmative categorical proposition. The second vowel of *affirmo*, **I**, is used to denote the particular affirmative categorical proposition. The first vowel of *nego*, **E**, is used to denote the universal negative categorical proposition, and the second vowel of *nego*, **O**, is used to denote the particular negative categorical proposition.

	<i>Affirmative</i>	<i>Negative</i>
		n
<i>Universal</i>	A	E
	f	
	f	g
<i>Particular</i>	I	O
	r	
	m	
	o	

Distribution

Quality and Quantity are both properties of entire categorical propositions. DISTRIBUTION, on the other hand, is a property of either the subject term or the predicate term *within* a categorical proposition. A class term is distributed in a categorical proposition if and only if the proposition tells you something about *every* member of that class.

For instance, in the standard form categorical proposition **A**,

- A) All *S* are *P*.

S is distributed, because **A** tells us that everything in the class *S* is also contained within the class *P*. So it tells us something about *everything* in the class *S*. On the other hand, it does *not* tell us something about everything in the class *P*—it could be that everything in the class *P* is *S* also, or it could be that some things in the class *P* are not *S*. So the predicate term *P* is *not* distributed in **A**.

In the standard form categorical proposition **E**,

- E) No *S* are *P*.

	Distributed?	
	<i>S</i>	<i>P</i>
A: All <i>S</i> are <i>P</i>	✓	×
E: No <i>S</i> are <i>P</i>	✓	✓
I: Some <i>S</i> are <i>P</i>	×	×
O: Some <i>S</i> are not <i>P</i>	×	✓

Figure 11.1: Distribution of terms in standard form categorical propositions

both *S* and *P* are distributed. For E tells us that everything in the *S* class is *not* in the *P* class, and it tells us that everything in the *P* class is *not* in the *S* class.

In the standard form categorical proposition I,

I) Some *S* are *P*

neither *S* nor *P* are distributed, for I does not tell us something about *all* of the members of *S*, nor something about *all* of the members of *P*. It does, of course, tell us that *at least one* thing in the *S* class is also in the *P* class, and that at least one thing in the *P* class is also in the *S* class, but it does not tell us something about *all* of the members of the *S* class nor something about *all* of the members of the *P* class.

The standard form categorical proposition O,

O) Some *S* are not *P*

is a bit tricky. You might think that in O, just like in I, neither term is distributed, since, after all, O just tells us about a particular thing in the *S* class—it tells us that that thing is not in the *P* class. What could this allow us to conclude about *all* of the things in the *S* class or *all* of the things in the *P* class? The trick is this: if you know that there's at least one particular thing which is in the *S* class, but not in the *P* class, then you know that *everything* in the *P* class is distinct from that one particular thing. So O does tell you something about everything in the *P* class—namely, that they are all distinct from something which is *S*. So, in O, *P* is distributed, though *S* is not.

11.2 The Square of Opposition

One way that logicians used to theorize about the logical relationships between the standard form categorical propositions A, E, I, and O was through the SQUARE OF OPPOSITION. Here are three claims about the logical relations between A, E, I, and O:

1. A and E cannot both be true.
 - (a) That is, A and E are *contraries*.
2. A and O necessarily have different truth-values (if A is true, then O is false, and if A is false, then O is true).
 - (a) That is, A and O are *contradictories*.
3. I and E necessarily have different truth-values (if I is true, then E is false, and if I is false, then E is true).
 - (a) That is, I and E are *contradictories*.

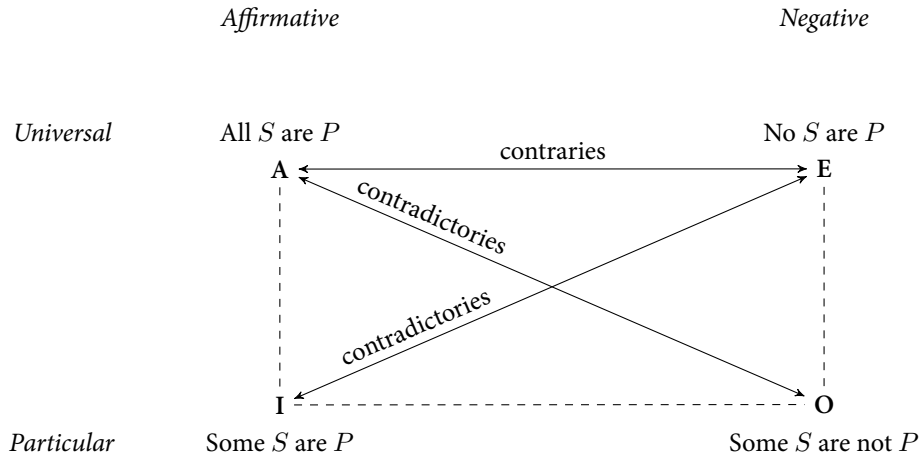


Figure 11.2

These three claims are displayed visually in figure 11.2. From these three logical relations, we can establish others. For instance, suppose that I is false. Then, E must be true (since I and E are contradictories). But, if E is true, then A must be false (since E and A are contraries, and they can't both be true). But, if A is false, then O must be true (since A and O are contradictories). Putting it all together: if I is false, then O must be true.

Similarly, suppose that O is false. Then, A must be true (since they are contradictories). But, at most one of A and E can be true (since they are contraries), so if A is true, then E is false. But, if E is false, then I must be true (since they are contradictories). Putting it all together: if O is false, then I must be true.

The previous two paragraphs have established that, if I is false, then O is true; and, if O is false, then I is true. This tells us that I and O cannot both be false. If two claims cannot both be false, then they are called *subcontraries*. So, from our first three claims, we have derived another:

4. I and O cannot both be false.
 - (a) that is, I and O are *subcontraries*

We can also show that A entails I, and that E entails O. That is to say: we may show that, if A is true, then I must be true also. That is: we may show that the argument whose premise is A and whose conclusion is I is deductively valid.

Suppose that A is true. Then, E must be false (since they are contraries). Then, I must be true (since E and I are contradictories). So, if A is true, then I must be true. So A entails I.

5. If A is true, then I must be true.
 - (a) That is, A *entails* I.

Similarly, suppose that E is true. Then, A must be false (since A and E are contraries). Then, O must be true (since A and O are contradictories). So, if E is true, then O must be true. So, E entails O.

6. If E is true, then O must be true.
 - (a) That is, E *entails* O.

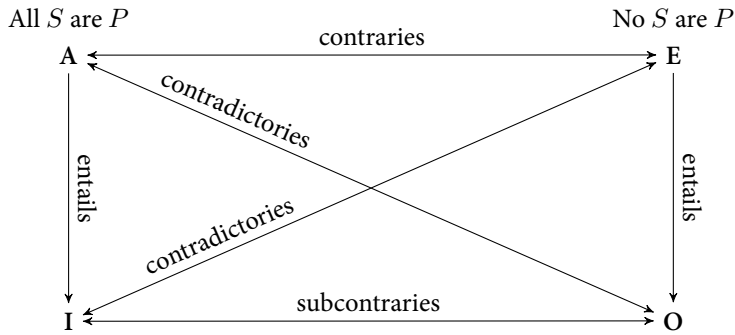
Putting together (1–6), we get the TRADITIONAL SQUARE OF OPPOSITION shown in figure 11.3.

According to the traditional square of opposition, the following arguments are formally deductively valid:

Affirmative

Negative

Universal



Particular

Some S are P Some S are not P

Figure 11.3: The Traditional Square of Opposition

1. All libertarian presidents are still alive.

 2. So, some libertarian presidents are still alive.

1. No libertarian presidents are still alive.

 2. So, some libertarian presidents are not still alive.

1. It is false that some libertarian presidents are still alive.

 2. So, some libertarian presidents are not still alive.

1. It is false that some libertarian presidents are not still alive.

 2. So, some libertarian presidents are still alive.

For they are of the following forms, respectively:

1. All S are P . (A)

 2. So, some S are P . (I)

1. No S are P . (E)

 2. So, some S are not P . (O)

1. It is false that some S are P . (\sim I)

 2. So, some S are not P . (O)

1. It is false that some S are not P . (\sim O)

 2. So, some S are P . (I)

The square tells us that A entails I and E entails O, so the first and second arguments are valid. It also tells us that, if I is false, then O must be true, so the third argument is valid. And it tells us that, if O is false, then I must be true, so the fourth argument is valid as well.

However, the class of libertarian presidents is empty. There are no libertarian presidents. The traditional square tell us that, even if there are no libertarian presidents, it cannot be that both “All libertarian presidents are still alive” and “No libertarian presidents are still alive” are true. They are, on this theory, *contraries*—at most one of them can be true. Similarly, even if there are no libertarian presidents, it cannot be that both “Some libertarian presidents are still alive”

	Traditional Understanding	Modern Understanding
A: All libertarian presidents are still alive	<i>False</i>	<i>True</i>
E: No libertarian presidents are still alive	<i>True</i>	<i>True</i>
I: Some libertarian presidents are still alive	<i>False</i>	<i>False</i>
O: Some libertarian presidents are not still alive	<i>True</i>	<i>False</i>

Table 11.1: The Traditional and the Modern verdicts about the A, E, I, and O standard form categorical propositions when the subject class is empty.

and “Some libertarian presidents are not still alive” are false. They are, on this theory, *subcontraries*—at most one of them can be false.

Which one should we say is false? The traditional view was that the I sentence is always false if there are no *Ss*, and (therefore) the E sentence is always (vacuously) true when there are no *Ss*. Thus, “Some libertarian presidents are still alive” is false and (therefore) “Some libertarian presidents are not still alive” is (vacuously) true. Interestingly, this stance implies that “Some libertarian presidents are dead” is false, while “Some libertarian presidents are not still alive” is true. That’s very strange.

What can we do to avoid this strange consequence? Well, the entire traditional square follows from the following three assumptions:

1. A and E are *contraries*.
2. A and O are *contradictories*.
3. I and E are *contradictories*.

The *modern* approach to these issues has been to deny the first assumption. That is, to deny that “All *S* are *P*” and “No *S* are *P*” cannot both be true. The modern approach is to say that, when there are no *Ss*, both “All *S* are *P*” and “No *S* are *P*” are (vacuously) true. The difference between the traditional and the modern verdicts about what to say about A, E, I, and O when the subject class is empty are shown in table 11.1.

Essentially, the modern approach is built on the assumption that it’s far better to say that “All libertarian presidents are still alive” is vacuously true than it is to say that “Some libertarian presidents are not still alive” is vacuously true. To get yourself in the headspace of the modern understanding, think about the truth of A sentence in the following way: there are no libertarian presidents. So, all of the none of them are still alive. Similarly, all of the none of them are dead. This feels slightly unnatural, in part because we almost never would have occasion to utter sentences like these, since we know that there aren’t any libertarian presidents, but the thought is that it feels far more natural than trying to say that some of the none of the libertarian presidents are not still alive, yet it is false that some of the none of the libertarian presidents are still alive. There’s much more to say here, and there are other approaches than the two that we’ve considered, but hopefully we’ve seen enough for you to understand some of the motivation for the modern perspective. The language *QL*, which we’ll learn about next week, will presuppose the modern understanding of quantifiers like “all” and “some”.

Without the assumption (1), none of the other relationships in the traditional square of opposition follow. In particular, A does not entail I, E does not entail O, and E and O are not subcontraries (they can both be false when there are no *Ss*). The modern square is shown in figure 11.4.

Note that, from the modern standpoint, the argument form

$$\frac{\text{All } S \text{ are } P}{\text{Some } S \text{ are } P} \quad (\text{invalid on the modern theory})$$

is invalid—as is the argument form:

$$\frac{\text{No } S \text{ are } P}{\text{Some } S \text{ are not } P} \quad (\text{invalid on the modern theory})$$

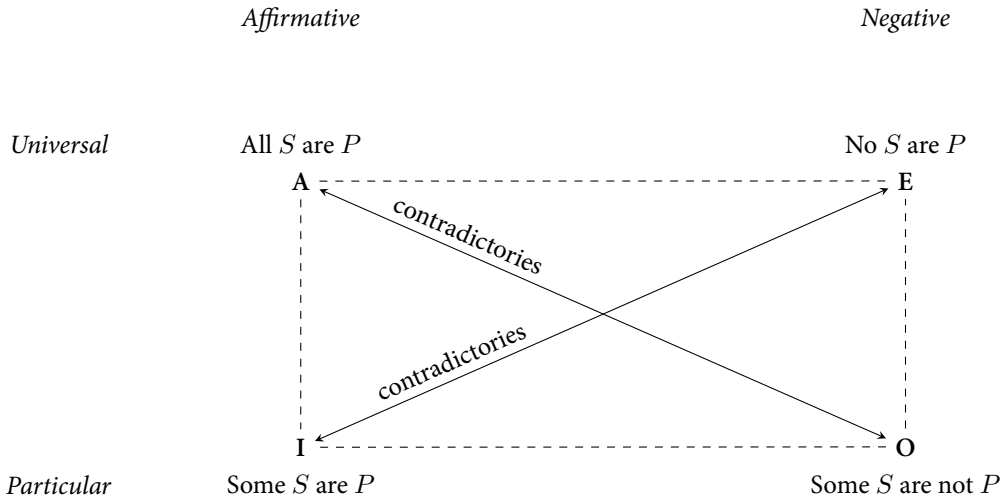


Figure 11.4: The Modern Square of Opposition: A and O are contradictory; and I and E are contradictory; However, there are no other logical relationships between A, E, I, and O on the modern understanding. In particular, A and E could both be true on the modern understanding (if there are no *S*s); I and O could both be false on the modern understanding (if there are no *S*s); A could be true while I is false (if there are no *S*s); and E could be true while O is false (if there are no *S*s).

and the argument forms

$$\frac{\text{All } S \text{ are } P}{\text{It is false that no } S \text{ are } P} \quad (\text{invalid on the modern theory})$$

$$\frac{\text{No } S \text{ are } P}{\text{It is false that all } S \text{ are } P} \quad (\text{invalid on the modern theory})$$

and the argument forms

$$\frac{\text{It is false that some } S \text{ are } P}{\text{Some } S \text{ are not } P} \quad (\text{invalid on the modern theory})$$

$$\frac{\text{It is false that some } S \text{ are not } P}{\text{Some } S \text{ are } P} \quad (\text{invalid on the modern theory})$$

All of these argument forms are valid on the traditional theory, and invalid on the modern one. For each argument form, the modern theory says that there are substitution instances with true premises and false conclusions when the subject class *S* is empty.

11.3 Categorical Syllogisms

A *syllogism* is an argument containing two premises and a conclusion. A *categorical syllogism* is an argument containing two premises and a conclusion, where both premises and the conclusion are *categorical propositions*. Thus, the following are categorical syllogisms:

1. All people are Kesha-lovers.
2. Some Kesha-lovers are bacon-eaters.

3. So, some people are bacon-eaters.

1. All hats are fashionable clothing.
 2. All fashionable clothing is purple.
-
3. So, some hats are purple.

1. All hats are fashionable clothing.
 2. All fashionable clothing is purple.
-
3. So, all hats are purple.

The first categorical syllogism is invalid (it could be that the bacon-eating Kesha-lovers are not people). The second is valid on the traditional theory, but invalid on the modern theory (think about the case in which there are no hats). The final argument is valid on either theory.

11.3.1 Major Terms, Minor Terms, Middle Terms

To understand the theory of categorical syllogisms, we will need to introduce some terminology. Consider the following categorical syllogism form:

1. All M are P .
 2. Some S are M .
-
3. So, some S are P

In a categorical syllogism, the *major term* is the predicate term of the conclusion. So, in the categorical syllogism form above, P is the major term. In a categorical syllogism, the *minor term* is the subject term of the conclusion. So, in the categorical syllogism form above, S is the minor term. In a STANDARD FORM CATEGORICAL SYLLOGISM, the *middle term* is a term which doesn't show up in the conclusion, but does show up in both of the premises. So, in the categorical syllogism form above, M is the middle term.

In a standard form categorical syllogism, the *major premise* is the premise which contains the major term. The *minor premise* is the premise which contains the minor term. So, in the categorical syllogism form above, (1) is the major premise and (2) is the minor premise.

Every standard form categorical syllogism must contain a middle term. That is, there must be some term that occurs in both premises. And every standard form categorical syllogism must contain a major premise and a minor premise, and they must be distinct—that is to say, every categorical syllogism must have the subject term of the conclusion appear in one premise, and the predicate term of the conclusion appear in another, distinct, premise. Additionally, we will require that a standard form categorical syllogism must have its major premise first and its minor premise second.

A STANDARD FORM CATEGORICAL SYLLOGISM contains:

1. Two premises and a conclusion, each a standard form categorical proposition.
2. A major term which appears only in the first premise and the predicate of the conclusion.
3. A minor term which appears only in the second premise and the subject of the conclusion
4. A middle term which appears in both premises but not in the conclusion.

11.3.2 Mood and Figure

Given that we require arguments to be written in this order, we may categorize categorical syllogisms by the kind of standard form categorical proposition which appears as its first (*i.e.*, major) premise, the kind of standard form categorical

proposition which appears as its second (*i.e.*, minor) premise, and the kind of standard form categorical propositions which appears as its conclusion. We will call such a specification the **MOOD** of the sentence. Consider, for instance, the argument form below.

$$\frac{\begin{array}{l} \text{All } P \text{ are } M \\ \text{All } S \text{ are } M \end{array}}{\text{Some } S \text{ are } P}$$

The first premise here is of the form **A**; the second is of the form **A**; and the conclusion is of the form **I**. Therefore, the mood of this argument form is **AAI**.

Now, consider the argument form

$$\frac{\begin{array}{l} \text{All } M \text{ are } P \\ \text{All } S \text{ are } M \end{array}}{\text{Some } S \text{ are } P}$$

The mood of this argument is *also* **AAI**. However, the first argument form is invalid, while, on the traditional theory (though not the modern one) the second argument form is valid. To deal with this, the theory of categorical syllogisms need to distinguish categorical syllogisms which are of the same mood, but have the middle terms distributed differently in the major and the minor premises. The way that the middle term is distributed in the major and minor premises is known as the **FIGURE** of the categorical syllogism. There are four possible figures, which are enumerated below.

1	2	3	4
$\begin{array}{l} \text{M } P \\ \text{S } M \\ \hline \text{S } P \end{array}$	$\begin{array}{l} P \text{ M} \\ \text{S } M \\ \hline \text{S } P \end{array}$	$\begin{array}{l} \text{M } P \\ \text{M } S \\ \hline \text{S } P \end{array}$	$\begin{array}{l} P \text{ M} \\ \text{M } S \\ \hline \text{S } P \end{array}$

That is, if the middle term is the subject term in the major premise and the predicate term in the minor premise, then the figure of the syllogism is 1; if the middle term is the predicate term in both the major and the minor premise, then the figure of the syllogism is 2; if the middle term is the subject term in both the major and the minor premise, then the figure of the syllogism is 3; and if the middle term is the predicate term of the major premise and the subject term of the minor premise, then the figure of the syllogism is 4.

The first argument form, then, has mood **AAI** and figure 2 (**AAI-2**); whereas the second has mood **AAI** and figure 1 (**AAI-1**).

There are then $4^3 = 64$ different moods that a standard form categorical syllogism could have; and, for each mood, 4 different figures it could have. There are therefore $64 \times 4 = 256$ different kinds of standard form categorical syllogisms. The traditional theory then just—incredibly enough—*listed out* the standard form categorical syllogisms which were valid.

On the traditional theory, *all* of the forms below are valid. On the modern theory, only the forms listed *above* the line are valid.

AAA-1	EAE-2	IAI-3	AEE-4
EAE-1	AEE-2	AII-3	IAI-4
AII-1	EIO-2	OAo-3	EIO-4
EIO-1	AOO-2	EIO-3	
AAI-1	AEO-2	AAI-3	AEO-4
EAO-1	EAO-2	EAO-3	EAO-4
			AAI-4

And that, it turns out, is the traditional theory of categorical syllogisms. Everything it has to say about deductive validity is contained in the list above. If memorizing the list above is too onerous, then there is another way that was provided

to figure out which categorical syllogisms were deductively valid. There is a list of *rules* which all the standard form categorical syllogisms listed above obey, and none of the other standard form categorical syllogisms obey. If you're interested, I've provided these rules below.

A standard form categorical syllogism is valid on the modern theory if and only if each of the following five propositions are all true of it. A standard form categorical syllogism is valid on the traditional theory if and only if each of the first four propositions are true of it.

1. The middle term is distributed at least once.
2. If a term is distributed in the conclusion, then that term is distributed in one of the premises.
3. There is at least one affirmative premise.
4. There is a negative premise if and only if there is a negative conclusion.
5. If both premises are universal, then the conclusion is universal.

However, I won't ask you to memorize these rules (or the list of valid standard form categorical syllogisms above). Rather, we'll introduce a way of establishing the validity of standard form categorical syllogisms (on both the traditional and the modern theory) by utilizing *Venn diagrams*.

11.4 Venn Diagrams

We've already encountered VENN DIAGRAMS in the course. To refresh your memory: A Venn diagram consists of a box and some number of labeled circles inside of the box. One such example is shown in figure 11.5. In order to interpret the diagram, we must say two things: 1) what the box represents (what, that is, the *domain*, \mathcal{D} , of the Venn diagram is), and 2) what each circle represents (that is, which *class* of things each circle corresponds to).

AN INTERPRETATION of a Venn diagram says
1) what the box, or the domain, \mathcal{D} is; and
2) which class of things each circle represents.

If there is an 'x' in a region of the Venn diagram, then the diagram makes the claim that that area of the Venn diagram is occupied. For instance, the Venn diagram shown in figure 11.5 makes the claim that there is something which is *both* not in the class of *F*s and not in the class of *G*s—that is, that there is something which is neither *F* nor *G*. If an area of the Venn diagram is shaded in, then the diagram makes the claim that that area of the diagram is *unoccupied*. For instance, the Venn diagram shown in figure 11.5 makes the claim that there is nothing which is in the class of *G* things but not in the class of *F* things. If there is neither an 'x' nor shading in a region of the Venn diagram, then the Venn diagram says nothing at all about that region of the diagram—neither that it is occupied nor that it is unoccupied. It is consistent with everything that the Venn diagram says that the region is occupied and it is consistent with everything that the Venn diagram says that the region is unoccupied. For instance, in the Venn diagram shown in figure 11.5, no claim whatsoever is made about whether there is anything which is both *F* and *G*.

Let's introduce an additional wrinkle into this picture. Suppose that we've got a Venn diagram containing three circles, *F*, *G*, and *H*, as shown in figure 11.6, and we wish to say that there is something which is inside of the *F* circle and outside of the *G* circle, but we don't wish to make any claim whatsoever about whether or not that thing is inside of the *H* circle. Then, we may place an 'x' on top of the line separating the *F*, not-*G*, and not-*H* area and the *F*, not-*G*, and

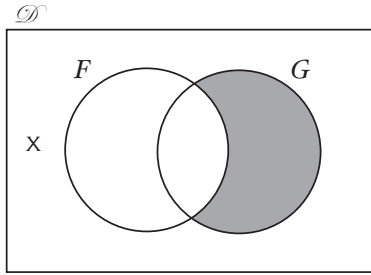


Figure 11.5: A Venn diagram which says that some things are neither F nor G , and that all G are F .

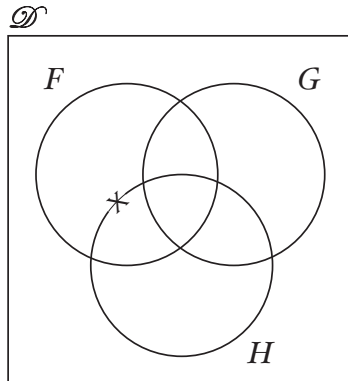


Figure 11.6: A Venn diagram which says that some F are not G .

H area, as in figure 11.6. This tells us that there's something which is F and not- G , but it doesn't tell us anything at all about whether that thing is H or not.

11.4.1 Representing Standard Form Categorical Propositions with Venn Diagrams (Modern Understanding)

Here, we will see how to represent the standard form categorical propositions A, E, I, and O using Venn diagrams. In each of these cases, we will simply take \mathcal{D} to be the class of *all things*. Thus, the S circles below represent *all things which are S*; and the P circles represents *all things which are P*.

On the modern understanding, “All S are P ” is true if and only if there is nothing in the class of S things which is not in the class of P things. In terms of the Venn diagram, “All S are P ” is true if and only if the region inside of the S circle but outside of the P circle is unoccupied, as shown in figure 11.7a.

Similarly, on the modern understanding, “No S are P ” is true if and only if there is nothing in the class of S things which is also in the class of P things. In terms of the Venn diagram, “No S are P ” is true if and only if the region inside of both the S circle and the P circle is unoccupied, as shown in figure 11.7b.

On the modern understanding, “Some S are P ” is true if and only if there is something in the class of S things which is also in the class of P things. In terms of the Venn diagram, “Some S are P ” is true if and only if the region inside of both the S circle and the P circle is occupied, as shown in figure 11.8a.

And similarly, on the modern understanding, “Some S are not P ” is true if and only if there is some in the class of S things which is not in the class of P things. In terms of the Venn diagram, “Some S are not P ” is true if and only if the region inside of the S circle but outside of the P circle is occupied, as shown in figure 11.8b.

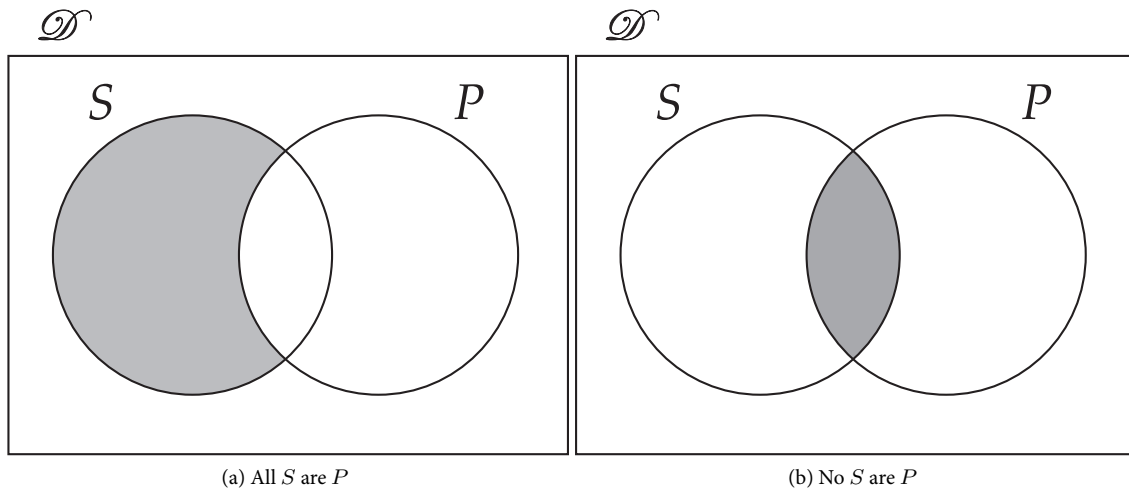


Figure 11.7: 'All S are P ' and 'No S are P ', on the modern understanding.

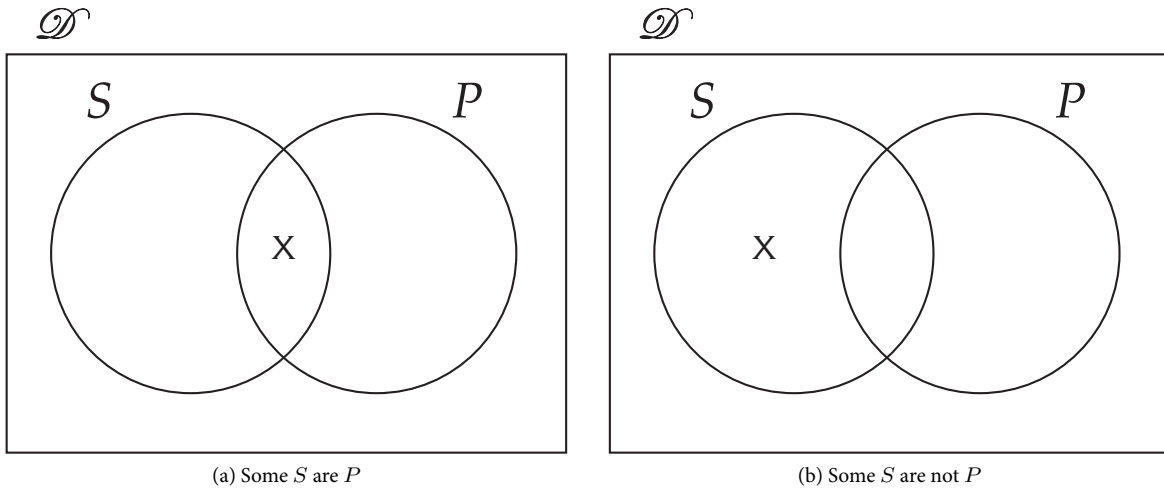


Figure 11.8: 'Some S are P ' and 'Some S are not P ', on the modern understanding.

11.4.2 Testing the Validity of Categorical Syllogisms with Venn Diagrams (Modern Understanding)

Here is a method for testing the validity of standard form categorical syllogisms (on the modern understanding): first, use a Venn diagram to represent the claims of the premises, and then check to see whether the Venn diagram contains within it the claim of the conclusion. If it does, then the categorical syllogism is valid. If it does not, then the categorical syllogism is invalid.

When we do so, it will help us if we *first* represent the claims of *universal* premises, and *then* represent the claims of particular premises (assuming that there are both universal and particular premises in the syllogism—if not, then it won't matter which order we represent the claims in).

Here's an example. Suppose that we wish to evaluate the standard form categorical syllogism AAA-1,

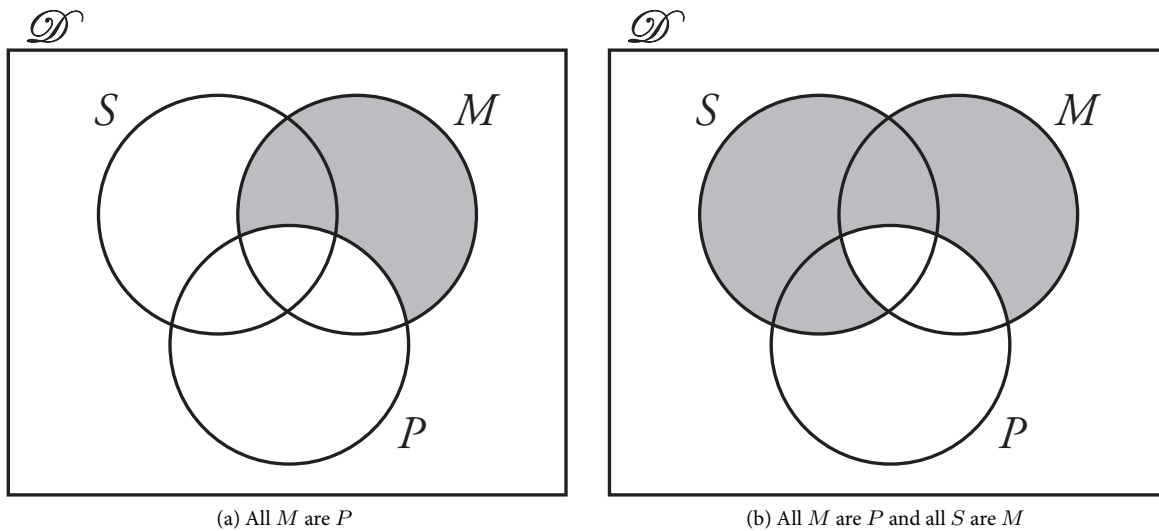


Figure 11.9: Establishing that AAA-1 is valid on the modern understanding.

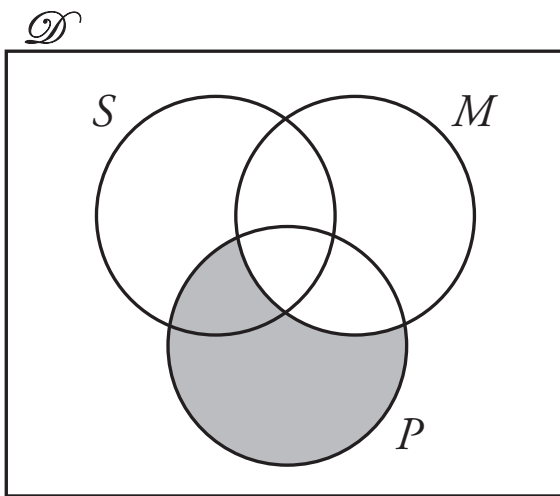
$$\frac{\begin{array}{l} \text{All } M \text{ are } P \\ \text{All } S \text{ are } M \end{array}}{\text{All } S \text{ are } P}$$

First, we represent the claim of the major premise, as shown in figure 11.9a. Then, add to that Venn diagram the claim of the minor premise, as shown in figure 11.9b. And finally, we check to see whether this Venn diagram tells us that all *S* are *P*. And it does! So, on the modern understanding, the standard form categorical syllogism AAA-1 is valid.

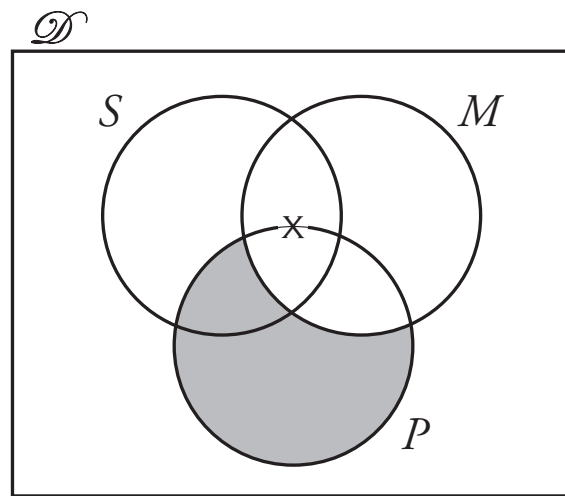
Now, suppose that we wish to determine whether AIA-4, that is,

$$\frac{\begin{array}{l} \text{All } P \text{ are } M \\ \text{Some } M \text{ are } S \end{array}}{\text{All } S \text{ are } P}$$

is valid on the modern understanding. Here, we have a universal premise (the major premise) and a particular premise (the minor one). As I said, it makes sense to start with the universal premise, so we will first represent the claim that all *P* are *M*, as shown in figure 11.10a. Then, we represent the claim that some *M* are *S*, as shown in figure 11.10b. (Here, we don't know whether the thing(s) that is (are) *M* and *S* are also *P* or not, so we place our '×' on the edge of the *P*-circle.) Now, we ask ourselves, does this Venn diagram tell us that all *S* are *P*? In this case, the answer is 'no'. It is consistent with everything that the Venn diagram representing the premises says that some *S* are not *P*, and so, that it is false that all *S* are *P*. So AIA-4 is invalid on the modern understanding.



(a) All P are M



(b) All P are M and some M are S

Figure 11.10: Establishing that AIA-4 is invalid on the modern understanding.

Chapter 12

Quantificational Logic: Syntax and Semantics

12.1 The Language QL

Before getting into the nitty-gritting, some preliminary orientation: we're going to use capital letters to denote *properties* that a thing might or might not have and *relations* things might or might not bear to one another, and we're going to use lowercase letters to denote the things that may or may not have those properties or may and may not bear those relations to one another. So, for instance, we could use the capital letters T , L , and K to represent the following properties and relations:

$$\begin{aligned}Tx &= \text{x was tall} \\Lxy &= \text{x loved y} \\Kxy &= \text{x killed y}\end{aligned}$$

and we could use l , b , c , and p to represent the following individuals:

$$\begin{aligned}l &= \text{Abraham Lincoln} \\b &= \text{John Wilkes Booth} \\c &= \text{Caesar} \\p &= \text{Pompey}\end{aligned}$$

If we put the lowercase letters representing individuals in the place of 'x' and 'y' above, we get statements like the following:

$$\begin{aligned}Tl &= \text{Abraham Lincoln was tall} \\Kbl &= \text{John Wilkes Booth killed Abraham Lincoln} \\Lcp &= \text{Caesar loved Pompey}\end{aligned}$$

We can treat these statements the same way that we treated the statement letters of PL —they can be the negands of negations, the antecedents of conditionals, the disjuncts of disjunctions, and so on and so forth.

$$\begin{aligned}\sim Lbl &= \text{John Wilkes Booth didn't love Abraham Lincoln} \\Kcp \supset \sim Lcp &= \text{If Caesar killed Pompey, then he didn't love him} \\Tc \vee Tb &= \text{Either Caesar or John Wilkes Booth is tall}\end{aligned}$$

We're also going to be able to translate claims like 'everyone loves someone' and 'no one loves anyone who killed them'. They will be translated like so:

$$\begin{aligned}(x)(\exists y)Lxy &= \text{Everyone loves someone} \\ \sim(\exists x)(\exists y)(Kyx \bullet Lxy) &= \text{No one loves anyone who killed them}\end{aligned}$$

But in order to understand that, we'll have to get into the nitty-gritty.

12.1.1 The Syntax of QL

In this section, I'm going to tell you what the vocabulary of QL is and I'm going to tell you which expressions of QL are grammatical—which are *well-formed*—just as we did for PL .

Vocabulary

The vocabulary of QL includes the following symbols:

1. for each $n \geq 0$, an infinite number of n -place predicates (any capital letter, along with a superscript n —perhaps with subscripts)

$$\begin{array}{cccccccc} A^1 & B^1 & \dots & Z^1 & A_1^1 & \dots & Z_1^1 & A_2^1 & \dots \\ A^2 & B^2 & \dots & Z^2 & A_1^2 & \dots & Z_1^2 & A_2^2 & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \\ A^n & B^n & \dots & Z^n & A_1^n & \dots & Z_1^n & A_2^n & \dots \\ \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \end{array}$$

2. An infinite number of *constants* (any lowercase letter between a and w —perhaps with subscripts)

$$a, b, c, \dots, u, v, w, a_1, b_1, \dots, v_1, w_1, a_2, b_2, \dots$$

3. An infinite number of *variables* (lowercase x, y , or z —perhaps with subscripts)

$$x, y, z, x_1, y_2, z_2, x_3 \dots$$

4. Logical operators

$$\sim, \vee, \bullet, \supset, \equiv, \exists$$

5. parentheses

$$(,)$$

Nothing else is included in the vocabulary of QL .

Terminology: Let's call both constants and variables *terms*. That is, both ' a ' and ' x ' are *terms* of QL .

Grammar

Any sequence of the symbols in the vocabulary of QL is a *formula* of QL . For instance, all of the following are formulae of QL :

$$\begin{array}{l} V^{2800} x \sim ((\supset \supset) anv \\ P^1 Q^2 R^3 S^4 T^5 \sim \sim \\ ((x)F^3 xab \supset \sim (\exists y)P^4 ynst) \\ N^{54} xy \vee \sim \sim (\exists x)B^2 x \end{array}$$

However, only one—the third—is a *well-formed formula* (or 'wff') of QL . We specify what it is for a string of symbols from the vocabulary of QL to be a wff of QL with the following rules.

\mathcal{F}) If ' \mathcal{F}^n ' is an n -place predicate and ' t_1 ', ' t_2 ', \dots , ' t_n ' are n terms, then ' $\mathcal{F}^n t_1 t_2 \dots t_n$ ' is a wff.

\sim) If ' P ' is a wff, then ' $\sim P$ ' is a wff.

-) If 'P' and 'Q' are wffs, then '(P • Q)' is a wff.
- ∨) If 'P' and 'Q' are wffs, then '(P ∨ Q)' is a wff.
- ⊃) If 'P' and 'Q' are wffs, then '(P ⊃ Q)' is a wff.
- ≡) If 'P' and 'Q' are wffs, then '(P ≡ Q)' is a wff.
- x) If 'P' is a wff and x is a variable, then '(x)P' is a wff.
- ∃) If 'P' is a wff and x is a variable, then '(∃x)P' is a wff.
- Nothing else is a wff.

Note: none of ' \mathcal{F} ', 'a', 'P', and 'Q' appear in the vocabulary of QL . They are not *themselves* wffs of QL . Rather, we are using them here as VARIABLES ranging over the formulae of QL . In PL , we used lowercase letters for this purpose. However, in QL , lowercase letters are *terms* of the language, so we must use other symbols for the variables ranging over the formulae of QL . We have chosen to use boldface and script letters for this purpose. Capital script letters are variables ranging over the *predicates* of QL ; boldface capital letters are variables ranging over *wffs* of QL ; and boldface lowercase letters are variables ranging over the *terms* of QL .

All and only the strings of symbols that can be constructed by repeated application of the rules above are well-formed formulae. For instance, if we wanted to show that ' $((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx)$ ' is a wff of QL , we could walk through the following steps to build it up:

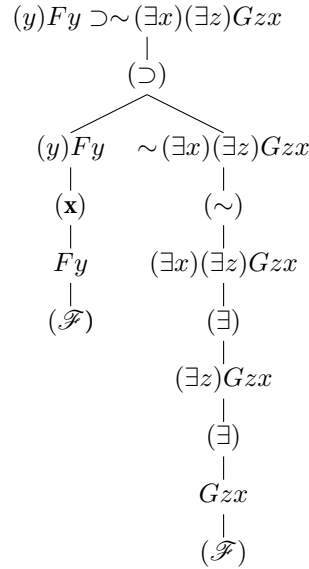
- | | |
|--|------------------------------------|
| a) ' F^1y ' is a wff | [from (\mathcal{F})] |
| b) So, ' $(y)F^1y$ ' is a wff | [from (a) and (x)] |
| c) ' G^2zx ' is a wff | [from (\mathcal{F})] |
| d) So, ' $(\exists z)G^2zx$ ' is a wff | [from (c) and (\exists)] |
| e) So, ' $(\exists x)(\exists z)G^2zx$ ' is a wff | [from (d) and (\exists)] |
| f) So, ' $\sim (\exists x)(\exists z)G^2zx$ ' is a wff | [from (e) and (\sim)] |
| g) So, ' $((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx)$ ' is a wff | [from (b), (f), and (\supset)] |

As before, we will adopt the convention of dropping the outermost parentheses in a wff of QL . We will *additionally* adopt the convention of dropping the superscripts on the predicates of QL . So, abiding by our informal conventions, we would write the wff of QL ' $((y)F^1y \supset \sim (\exists x)(\exists z)G^2zx)$ ' as:

$$(y)Fy \supset \sim (\exists x)(\exists z)Gzx$$

I'll adopt these conventions from here on out.

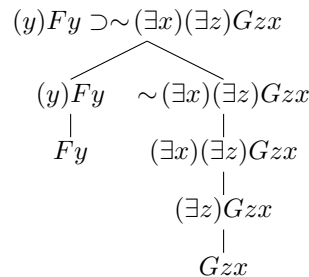
We could, just as before, use SYNTAX TREES to represent the way that a wff of QL is built up according to the rules for wffs given above. For instance, we could notate the proof given above as follows:



The left-hand branch of this syntax tree tells us that ‘ Fy ’ is a wff, by rule (\mathcal{F}) ; and that, therefore, ‘ $(y)Fy$ ’ is a wff, by rule (\mathbf{x}) . The right-hand branch tells us that ‘ Gzx ’ is a wff, by rule (\mathcal{F}) ; and that, therefore, ‘ $(\exists z)Gzx$ ’ is a wff, by rule (\exists) . Thus, ‘ $(\exists x)(\exists z)Gzx$ ’ is a wff, by rule (\exists) again; and, finally, that, therefore, ‘ $\sim (\exists x)(\exists z)Gzx$ ’ is a wff, by rule (\sim) . Putting together what we have from the left-hand branch and the right-hand branch, we can conclude that ‘ $(y)Fy \supset \sim (\exists x)(\exists z)Gzx$ ’ is a wff, by rule (\supset) .

That is to say: the syntax tree tells us exactly what the proof above tells us. It tells us how we may show that ‘ $(y)Fy \supset \sim (\exists x)(\exists z)Gzx$ ’ is a wff of QL by building it up out of its components, according to the rules for wffs for QL .

If we want a simpler way of notating a syntax tree like this, then we may simply remove the justifications (recognizing that they are clear from the context of what lies above each wff on the syntax tree), and write it out as follows:



Free and Bound Variables

Our rules for wffs count ‘ Fx ’ and ‘ Ayc ’ as well-formed formulae. However, the variables that appear in these wffs are FREE. On the other hand, the variables appearing in ‘ $(x)(y)Fxy$ ’ are BOUND. In ‘ $(x)Px \supset Qx$ ’, the first occurrence of the variable ‘ x ’ is bound, whereas the second occurrence is free.

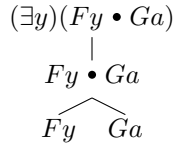
To make these ideas precise, let’s introduce the idea of a QUANTIFIER. For any variable x , both ‘ (\mathbf{x}) ’ and ‘ $(\exists x)$ ’ are *quantifiers*. We call ‘ (\mathbf{x}) ’ the UNIVERSAL QUANTIFIER, and we call ‘ $(\exists x)$ ’ the EXISTENTIAL QUANTIFIER. These quantifiers are logical operators. They can be the MAIN OPERATOR of a wff of QL or they can be the main operator of a wff’s subformulae. Each quantifier has one and only one associated *variable*. For instance, the variable associated with the quantifier ‘ (z) ’ is ‘ z ’. The variable associated with the quantifier ‘ $(\exists y)$ ’ is ‘ y ’.

As before, we can define the MAIN OPERATOR of a wff of QL to be the logical operator whose associated rule is *last* appealed to when building the wff up according to the rules given above. So, for instance, the main operator of

' $(y)Fy \supset \sim (\exists x)(\exists z)Gzx$ ' is the horseshoe ' \supset '. The main operator of ' $(x)Fx$ ', on the other hand, whose syntax tree is shown below, is the universal quantifier ' (x) '.



Similarly, the main operator of ' $(\exists y)(Fy \bullet Ga)$ ' is ' $(\exists y)$ '.



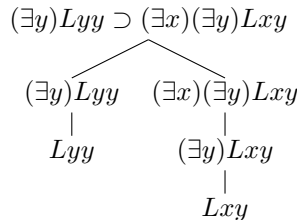
We can define **SUBFORMULA** in the same way that we defined it before: **P** is a subformula of **Q** if and only if **P** must show up on a line during the proof that **Q** is a wff of QL . In terms of the syntax trees: **P** is a subformula of **Q** if and only if **P** lies somewhere on **Q**'s syntax tree.

Similarly, we can define **IMMEDIATE SUBFORMULA** in precisely the same way as before: **P** is an immediate subformula of **Q** iff a line asserting that **P** is a wff must be appealed to in the final line of a proof showing that **Q** is a wff, according to the rules for wffs given above. In terms of the syntax tree: **P** is an immediate subformula of **Q** iff **P** lies immediately below **Q** on the syntax tree. Then, the immediate subformula of ' $(x)Fx$ ' is ' Fx ', and the immediate subformula of ' $(\exists y)(Fy \bullet Ga)$ ' is ' $Fy \bullet Ga$ '.

The **SCOPE** of a quantifier is the immediate subformula of the wff of QL of which it is the main operator.

The scope of a quantifier— (x) or $(\exists x)$ —is the subformula of which that quantifier is the main operator.

So, for instance, in the wff $(\exists y)Lyy \supset (\exists x)(\exists y)Lxy$, whose syntax tree is shown below,



The scope of the very first existential quantifier ' $(\exists y)$ ' is the formula ' Lyy '. The scope of the second existential quantifier ' $(\exists x)$ ' is ' $(\exists y)Lxy$ '. And the scope of the final existential quantifier ' $(\exists y)$ ' is ' Lxy '.

Now, we can define the notions of a **FREE** and a **BOUND** variable.

A variable **x** in a wff of PL is **BOUND** if and only if it occurs within the scope of a quantifier, (x) or $(\exists x)$, whose associated variable is **x**.

A variable **x** in a wff of PL is **FREE** if and only if it does not occur within the scope of a quantifier, (x) or $(\exists x)$, whose associated variable is **x**.

For instance, in the wff

$$(x)(y)Fy \supset (\exists z)Gzx$$

The final occurrence of ‘ x ’ is *free*. Even though there is a universal quantifier ‘ (x) ’ in the wff, the final ‘ x ’ does not occur within the scope of this universal quantifier, so it is not bound by it.

We can similarly define the notion of what it is for a quantifier to **BIND** a variable.

In a wff of QL , a quantifier (x) or $(\exists x)$ **BINDS** a variable x if and only if x occurs *free* within that quantifier’s scope.

This means that a variable can only be bound by a single quantifier. So, for instance, in the following wff of QL ,

$$(\exists x)(x)Fx$$

The variable ‘ x ’ is bound by the *universal* quantifier ‘ (x) ’. It is *not* bound by the existential quantifier ‘ $(\exists x)$ ’.

Note: variable symbols (x, y, z) are only free or bound when they occur after a predicate letter. The variable symbols that appear *within the quantifiers themselves* are neither free nor bound. So, for instance, in the wff ‘ $(x)(x)Lxx$ ’, the symbol ‘ x ’ appearing in the second (innermost) universal quantifier is *not* bound by the first (outermost) universal quantifier. The only occurrences of the symbol ‘ x ’ which are either free or bound are the final two, after ‘ L ’, and both of them are bound by the second (innermost) universal quantifier.

Important Syntactic Features in QL

Parentheses will serve an important rule in distinguishing wffs of QL , just as they played an important role in distinguishing the wffs of PL . Thus, ‘ $(x)((\exists y)Lxy \supset Gx)$ ’ is a different wff than ‘ $(x)(\exists y)Lxy \supset Gx$ ’; for they have different syntax trees, as shown below:



And, in fact, one of these wffs can be true while the other is false (that is to say, they *mean* different things). So it will be very important in QL to keep track of your parentheses.

Similarly, **the order of the terms following the predicates in QL matter**. ‘ Lab ’ is a different wff than ‘ Lba ’; similarly, ‘ $(x)(\exists y)Lxy$ ’ is a different wff than ‘ $(x)(\exists y)Lyx$ ’. For they have different syntax trees,



(since they have different wffs on each line). Moreover, this difference is also one that will end up making a difference to the *meaning* of the wffs of QL . Again, if $Lxy = x$ loves y , and we’re considering only people, then we’ll end up seeing that ‘ $(x)(\exists y)Lxy$ ’ says that everybody loves somebody else; whereas ‘ $(x)(\exists y)Lyx$ ’ says that everybody is loved *by* somebody else. And these two mean very different things—we’ll end up seeing that one could be true while the other is false.

Moreover, **the order of quantifiers plays an important role in distinguishing the wffs of QL** . For instance, ‘ $(\exists x)(y)Lxy$ ’ is a different wff of QL than ‘ $(y)(\exists x)Lxy$ ’. For these wffs of QL have different syntax trees, as shown below:

$$\begin{array}{c}
 (\exists x)(y)Lxy \\
 | \\
 (y)Lxy \\
 | \\
 Lxy
 \end{array}$$

$$\begin{array}{c}
 (y)(\exists x)Lxy \\
 | \\
 (\exists x)Lxy \\
 | \\
 Lxy
 \end{array}$$

And, again, this is a difference that will end up making a difference. If $Lxy = x$ loves y , and we're considering only people, then we'll end up seeing—once we've given the semantics for QL , below—that ' $(\exists x)(y)Lxy$ ' says that somebody loves everybody; whereas ' $(y)(\exists x)Lxy$ ' says that everybody is loved by somebody. And these two mean very different things. So we must be careful to pay attention to the order of the quantifiers.

12.1.2 Semantics for QL

In PL , we defined the semantics for the language in terms of TRUTH-VALUE ASSIGNMENTS. A truth-value assignment, recall, was just an assignment of truth-value (either true or false), to all of the *statement letters* of PL .

A TRUTH-VALUE ASSIGNMENT is an assignment of truth-value—either true or false—to every statement letter of PL .

We then gave definitions for $\sim, \vee, \cdot, \supset,$ and \equiv that allowed us to say, for any given wff of PL , whether it was true or false on that truth-value assignment. Since this allowed us to understand the circumstances under which the wffs of PL were true or false, this provided us with the *meaning* of the wffs of PL .

However, we saw that, since there were an infinite number of statement letters of PL , specifying a truth-value assignment was prohibitively difficult. So, instead, we realized that we could look at a PARTIAL TRUTH-VALUE ASSIGNMENT. Where, recall,

A PARTIAL TRUTH-VALUE ASSIGNMENT assigns a truth-value—either true or false—to each statement letter in some set of statement letters.

Each row of a truth-table represented a partial truth-value assignment. The definitions we gave of $\sim, \cdot, \vee, \supset,$ and \equiv then allowed us to work out the truth-value of a given wff of PL in every partial truth-value assignment (that is, in every row of the truth-table).

We're going to do exactly the same thing with QL . However, rather than dealing with TRUTH-VALUE ASSIGNMENTS, we're going to deal with QL -INTERPRETATIONS.

A QL -INTERPRETATION, \mathcal{I} , provides

1. A specification of which things fall in the *domain*, \mathcal{D} , of the interpretation.^a
2. A unique constant of QL to name every thing in the domain.
3. For every term (constant or variable) of QL , a specification of which thing in the domain \mathcal{D} it represents.
4. For every predicate of QL , a specification of the property or relation it represents.

^a Note: the domain must be non-empty, and it must be countable.

Because a QL interpretation requires us to say of every term in the language which thing in the domain that term denotes—and because it requires us to say of every predicate in the language which property or relation it denotes—and because there are an infinite number of terms and predicates in our language, specifying a full QL -interpretation is just as difficult as specifying a full truth-value assignment. Therefore, just as we introduced the idea of a PARTIAL TRUTH-VALUE ASSIGNMENT (which were just the rows of the truth-tables in PL), we will also introduce the idea of a PARTIAL QL -INTERPRETATION.

Given a wff, set of wffs, or argument of QL , a PARTIAL QL -INTERPRETATION, \mathcal{I}_p provides:

1. A specification of which things fall in the domain, \mathcal{D} , of the partial interpretation.^a
2. For the constants and free variables appearing in the wff, set of wffs, or argument of QL , a specification of which thing in the domain \mathcal{D} they represent.
3. For the predicates appearing in the wff, set of wffs, or argument of QL , a specification of the property or relation they represent.

^a Note: we require that the domain be non-empty and countable.

For instance, suppose that we have the following wff of QL ,

$$(y)Lya \vee (\exists y) \sim Lya$$

Here is a *partial interpretation* of this wff:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \{ \text{Adam, Betsy, Carol} \} \\ a & = \text{Adam} \\ Lxy & = \text{x loves y} \end{cases}$$

We specified the domain, \mathcal{D} . Since all of the variables are bound, we do not need to say which thing they refer to. There is only one constant in the wff, ‘ a ’, and we said what that constant referred to—Adam. And there is just a single 2-place predicate in the wff, ‘ L ’. And we said that Lxy referred to the relation ‘ x loves y ’. So we’ve provided a partial QL interpretation for this wff.

Truth on an Interpretation

Suppose that we’ve got an interpretation \mathcal{I} . Then, we can lay down the following rules which tell us what the wffs of QL mean on that interpretation—that is, under which conditions they are *true* on that interpretation. (Rules 2–6 should be familiar from PL .)

1. A wff of the form ‘ $\mathcal{F}^n t_1 \dots t_n$ ’ is true on the interpretation \mathcal{I} if the things in the domain represented by $t_1 \dots t_n$ have the property/bear to each other the relation represented by \mathcal{F}^n . Otherwise, ‘ $\mathcal{F}^n t_1 \dots t_n$ ’ is false on the interpretation \mathcal{I} .
2. A wff of the form ‘ $\sim P$ ’ is true on the interpretation \mathcal{I} if P is false on the interpretation \mathcal{I} . Otherwise, ‘ $\sim P$ ’ is false on the interpretation \mathcal{I} .
3. A wff of the form ‘ $P \vee Q$ ’ is true on the interpretation \mathcal{I} if either P is true on the interpretation \mathcal{I} or Q is true on the interpretation \mathcal{I} . Otherwise, ‘ $P \vee Q$ ’ is false on the interpretation \mathcal{I} .
4. A wff of the form ‘ $P \bullet Q$ ’ is true on the interpretation \mathcal{I} if both P is true on the interpretation \mathcal{I} and Q is true on the interpretation \mathcal{I} . Otherwise, ‘ $P \bullet Q$ ’ is false on the interpretation \mathcal{I} .

5. A wff of the form ' $P \supset Q$ ' is true on the interpretation \mathcal{I} if either P is false on the interpretation \mathcal{I} or Q is true on the interpretation \mathcal{I} . Otherwise, ' $P \supset Q$ ' is false on the interpretation \mathcal{I} .
6. A wff of the form ' $P \equiv Q$ ' is true on the interpretation \mathcal{I} if both P and Q have the same truth value on the interpretation \mathcal{I} . Otherwise, ' $P \equiv Q$ ' is false on the interpretation \mathcal{I} .

Before getting to the rules for the quantifiers, $(\forall x)$ and $(\exists x)$, we have to introduce one more idea—but it's one we've seen several times already in the course: that of a SUBSTITUTION INSTANCE. A substitution instance of a quantified wff of the form ' $(\forall x)P$ ' or ' $(\exists x)P$ ' is the wff that you get by removing the quantifier, leaving behind just its immediate subformula, and *uniformly replacing every instance of the variable x which is bound by the quantifier with some constant a or some free variable y* —note: it must be the same constant or free variable throughout.

For instance, all of the following are substitution instances of the quantified wff ' $(\exists y)((Ay \bullet \sim Lay) \supset (x) \sim Lax)$ ':

$$(Ab \bullet \sim Lab) \supset (x) \sim Lax$$

$$(Ac \bullet \sim Lac) \supset (x) \sim Lax$$

$$(Az \bullet \sim Laz) \supset (x) \sim Lax$$

$$(Aa \bullet \sim Laa) \supset (x) \sim Lax$$

$$(Ay \bullet \sim Lay) \supset (x) \sim Lax$$

since, we may get each of the above wffs by taking ' $(\exists y)((Ay \bullet \sim Lay) \supset (x) \sim Lax)$ ', removing its outermost quantifier, ' $(\exists y)$ ', and uniformly replacing the variables ' y ' which $(\exists y)$ bound with a single term. In the first wff, we uniformly replaced each ' y ' with the constant ' b '. In the second wff, we uniformly replaced ' y ' with the constant ' c '. In the third wff, we uniformly replaced ' y ' with the variable ' z '; in the fourth, we uniformly replaced ' y ' with ' a '; and in the fifth, we uniformly replaced ' y ' with ' y '. It does not matter that ' a ' already appeared in ' $(\exists y)((Ay \bullet \sim Lay) \supset (x) \sim Lax)$ '. Nor does it matter that ' y ' was the original bound variable in ' $(\exists y)((Ay \bullet \sim Lay) \supset (x) \sim Lax)$ '. We may replace ' y ' throughout with *any term of QL whatsoever*, and what we'll get is a substitution instance of ' $(\exists y)((Ay \bullet \sim Lay) \supset (x) \sim Lax)$ '.

However, the following are *not* substitution instances of ' $(\exists y)(\sim Lay \supset (x) \sim Lax)$ ':

$$(Ax \bullet \sim Laz) \supset (x) \sim Lax$$

$$(Aa \bullet \sim Lab) \supset (x) \sim Lax$$

$$(Ab \bullet \sim Lbb) \supset (x) \sim Lcx$$

$$(Ar \bullet \sim Lra) \supset (x) \sim Lax$$

The first wff is not a substitution instance because the first instance of ' y ' which was bound by $(\exists y)$ ' in the original wff was replaced with ' x ', whereas the second instance of the bound ' y ' was replaced with ' z '. So the replacement was not *uniform*; so it is not a substitution instance. The second wff is not a substitution instance since the first ' y ' was replaced with ' a ', whereas the second ' y ' was replaced with ' b '. The third wff is not a substitution instance because, even though ' y ' was uniformly replaced with ' b ', the consequent was changed from ' $(x) \sim Lax$ ' to ' $(x) \sim Lcx$ '. The fourth wff is not a substitution instance because, whereas the first term after the first ' L ' in the original wff was ' a ', the first term after the first ' L ' in the fourth wff above is ' r '.

We're now in a position to give the rules for quantified statements being *true on an interpretation, \mathcal{I}* :

7. A wff of the form ' $(\forall x)P$ ' is true on the interpretation \mathcal{I} if every substitution instance of ' $(\forall x)P$ ' is true on the interpretation \mathcal{I} . Otherwise, ' $(\forall x)P$ ' is false on the interpretation \mathcal{I} .
8. A wff of the form ' $(\exists x)P$ ' is true on the interpretation \mathcal{I} if there is *some* substitution instance of ' $(\exists x)P$ ' which is true on the interpretation \mathcal{I} . Otherwise, ' $(\exists x)P$ ' is false on the interpretation \mathcal{I} .

12.2 Translations from QL into English

In order to translate from QL into English, we will first need a (partial) QL -interpretation. This interpretation will tell us the meanings of the predicates of QL as well as the constants and (free) variables of QL .

12.2.1 Translating Simple Quantified wffs of QL

We already know how to translate expressions involving the operators \sim , \bullet , \vee , \supset , and \equiv into English. What's needed is a method for translating the quantifiers (\mathbf{x}) and $(\exists \mathbf{x})$ into English. The following will do as a good translation guide in the simple case where the quantifier scopes over a wff consisting of just an 1-place predicate followed by 1 bound variable.

$$\begin{aligned} (\mathbf{x})\mathcal{F}\mathbf{x} &\longrightarrow \text{Everything is } \mathcal{F}. \\ (\exists \mathbf{x})\mathcal{F}\mathbf{x} &\longrightarrow \text{Something is } \mathcal{F}. \end{aligned}$$

For instance, given the partial interpretation \mathcal{I}_p (the interpretation is only partial because I haven't given a name to every thing in the domain, nor a specification of which things in the domain the terms of QL refer to, nor a specification of which properties/relations the predicates of QL refer to),

$$\mathcal{I}_p = \begin{cases} \mathcal{D} &= \text{the set of all actually existing things} \\ B\mathbf{x} &= \mathbf{x} \text{ is beautiful} \end{cases}$$

We can give the following translations from QL into English:

$$\begin{aligned} (y)By &\longrightarrow \text{Everything is beautiful.} \\ (\exists z)Bz &\longrightarrow \text{Something is beautiful.} \end{aligned}$$

12.2.2 Translating More Complicated Quantified wffs of QL

Often, a quantified wff of QL will have a more complicated wff in its scope. There are four kinds of quantified wffs of QL that you should be familiar with, and which you should be able to translate from QL to English (and *vice versa*). They are just the A, E, I, and O sentences from categorical logic.

$$\begin{aligned} (\mathbf{x})(\mathcal{S}\mathbf{x} \supset \mathcal{P}\mathbf{x}) &\longrightarrow \text{All } \mathcal{S} \text{ are } \mathcal{P} && \text{(A)} \\ (\mathbf{x})(\mathcal{S}\mathbf{x} \supset \sim \mathcal{P}\mathbf{x}) &\longrightarrow \text{No } \mathcal{S} \text{ are } \mathcal{P}. && \text{(E)} \\ (\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x}) &\longrightarrow \text{Some } \mathcal{S} \text{ are } \mathcal{P}. && \text{(I)} \\ (\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim \mathcal{P}\mathbf{x}) &\longrightarrow \text{Some } \mathcal{S} \text{ are not } \mathcal{P}. && \text{(O)} \end{aligned}$$

Some \mathcal{S} are \mathcal{P}

To see why these wffs of QL translate into these English sentences, we should think about the Venn diagrams that we learned about earlier in the course. There, we saw that the way to represent a sentence of the form 'Some \mathcal{S} are \mathcal{P} ' with a Venn Diagram is as shown in figure 12.1. That is, 'Some \mathcal{S} are \mathcal{P} ' is true if and only if there is something which is both \mathcal{S} and \mathcal{P} —*i.e.*, if and only if there is something which is inside both of the circles \mathcal{S} and \mathcal{P} . There will be something like that—call it 'a'—if and only if there is some true substitution instance of $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x})$, namely, $\mathcal{S}\mathbf{a} \bullet \mathcal{P}\mathbf{a}$. But there will be a true substitution instance of $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x})$ if and only if $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x})$ is true, since (from above):

8. A wff of the form $(\exists \mathbf{x})\mathbf{P}$ is true on the interpretation \mathcal{I} if there is *some* substitution instance of $(\mathbf{x})\mathbf{P}$ which is true on the interpretation \mathcal{I} . Otherwise, it is false on the interpretation \mathcal{I} .

So 'Some \mathcal{S} are \mathcal{P} ' is a good translation of $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x})$ (and *vice versa*).

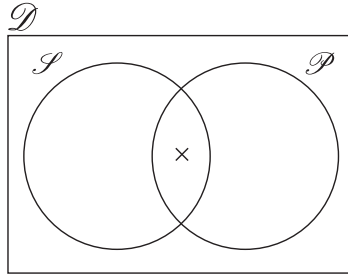


Figure 12.1:

Some \mathcal{S} are not \mathcal{P}

Next, consider ‘Some \mathcal{S} are not \mathcal{P} ’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 12.2. That is, ‘Some \mathcal{S} are not \mathcal{P} ’ is true if and only if there is something which is \mathcal{S} but not \mathcal{P} —*i.e.*,

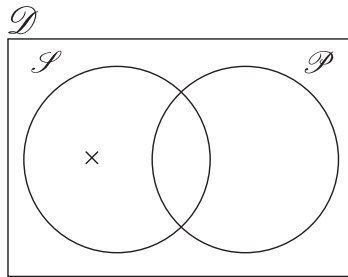


Figure 12.2:

if and only if there is something which is inside the circle \mathcal{S} yet outside of the circle \mathcal{P} . There will be something like that—call it ‘ \mathbf{a} ’—if and only if there is some true substitution instance of $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim \mathcal{P}\mathbf{x})$, namely, $\mathcal{S}\mathbf{a} \bullet \sim \mathcal{P}\mathbf{a}$. But there will be a true substitution instance of $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim \mathcal{P}\mathbf{x})$ if and only if $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim \mathcal{P}\mathbf{x})$ is true, since (again):

8. A wff of the form ‘ $(\exists \mathbf{x})\mathbf{P}$ ’ is true on the interpretation \mathcal{I} if there is *some* substitution instance of ‘ $(\mathbf{x})\mathbf{P}$ ’ which is true on the interpretation \mathcal{I} . Otherwise, it is false on the interpretation \mathcal{I} .

So ‘Some \mathcal{S} are not \mathcal{P} ’ is a good translation of ‘ $(\exists \mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim \mathcal{P}\mathbf{x})$ ’ (and *vice versa*).

All \mathcal{S} are \mathcal{P}

Next, consider ‘All \mathcal{S} are \mathcal{P} ’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 12.3. That is, ‘All \mathcal{S} are \mathcal{P} ’ is true if and only if there is nothing which is \mathcal{S} yet outside of the circle \mathcal{P} . Think about what it would take for this claim to be *false*. This claim would be false if and only if there were something which were \mathcal{S} but not \mathcal{P} . Otherwise, it would be true. Suppose that there were something—call it ‘ \mathbf{a} ’—which were \mathcal{S} but not \mathcal{P} . Then, the wff of *QL* $\mathcal{S}\mathbf{a} \supset \mathcal{P}\mathbf{a}$ would be false—since its antecedent is true, yet its consequent is false. On the other hand, if anything \mathbf{b} in the domain is either both \mathcal{S} and \mathcal{P} or not \mathcal{S} , then $\mathcal{S}\mathbf{b} \supset \mathcal{P}\mathbf{b}$ would still be true (by the definition of ‘ \supset ’).

So, there is something which is \mathcal{S} and not \mathcal{P} if and only if there is some \mathbf{a} such that $\mathcal{S}\mathbf{a} \supset \mathcal{P}\mathbf{a}$ is false.

So, there is something which is \mathcal{S} and not \mathcal{P} if and only if $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \mathcal{P}\mathbf{x})$ is false, since (from above):

7. A wff of the form ‘ $(\mathbf{x})\mathbf{P}$ ’ is true on the interpretation \mathcal{I} if every substitution instance of ‘ $(\mathbf{x})\mathbf{P}$ ’ is true on the interpretation \mathcal{I} . Otherwise, it is false on the interpretation \mathcal{I} .

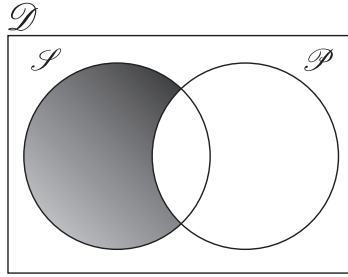


Figure 12.3:

By the same token, if there is nothing which is \mathcal{S} and not \mathcal{P} , then $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \mathcal{P}\mathbf{x})$ will be true, since all of its substitution instances will be true.

So ‘All \mathcal{S} are \mathcal{P} ’ is true in exactly the same circumstances as ‘ $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \mathcal{P}\mathbf{x})$ ’. So the former provides a good translation of the latter (and *vice versa*).

No \mathcal{S} are \mathcal{P}

Finally, consider ‘No \mathcal{S} are \mathcal{P} ’. We saw that the way to represent a sentence like this with a Venn diagram is as shown in figure 12.4. That is: the claim ‘No \mathcal{S} are \mathcal{P} ’ is true if and only if there is nothing which is both \mathcal{S} and \mathcal{P} . Think about

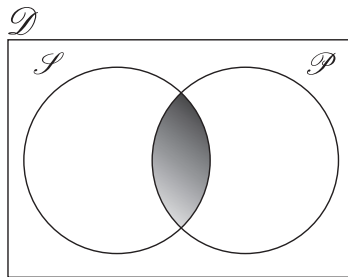


Figure 12.4:

the circumstances under which this claim would be *false*. It would be false if and only if there were something—call it ‘ \mathbf{a} ’—which were both \mathcal{S} and \mathcal{P} . Then, $\mathcal{S}\mathbf{a}$ would be true and $\sim \mathcal{P}\mathbf{a}$ would be false. So $\mathcal{S}\mathbf{a} \supset \sim \mathcal{P}\mathbf{a}$ would be false (by the definition of ‘ \supset ’). So $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \sim \mathcal{P}\mathbf{x})$ would be false (since it has a false substitution instance).

If there were nothing which were both \mathcal{S} and \mathcal{P} , then ‘No \mathcal{S} are \mathcal{P} ’ would be true. And, similarly, $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \sim \mathcal{P}\mathbf{x})$ would be true, since the only way that could be false would be if it had a false substitution instance,

$$\mathcal{S}\mathbf{a} \supset \sim \mathcal{P}\mathbf{a}$$

but the above wff would be false only if \mathbf{a} were both \mathcal{S} and \mathcal{P} —since that is the only thing that would make its antecedent true and its consequent false.

So ‘No \mathcal{S} are \mathcal{P} ’ is true in exactly the same circumstances as ‘ $(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \sim \mathcal{P}\mathbf{x})$ ’. So the former provides a good translation of the latter (and *vice versa*).

12.3 Translations from English into QL

The English expressions appearing in the translation guides from the previous section constitute the *canonical logical form* of English. In general, if we have an English expression in canonical logical form, we may translate it into QL

directly according to that translation schema:

Everything is \mathcal{F}	\longrightarrow	$(\mathbf{x})\mathcal{F}\mathbf{x}$
Something is \mathcal{F}	\longrightarrow	$(\exists\mathbf{x})\mathcal{F}\mathbf{x}$
Some \mathcal{S} are \mathcal{P}	\longrightarrow	$(\exists\mathbf{x})(\mathcal{S}\mathbf{x} \bullet \mathcal{P}\mathbf{x})$
Some \mathcal{S} are not \mathcal{P}	\longrightarrow	$(\exists\mathbf{x})(\mathcal{S}\mathbf{x} \bullet \sim\mathcal{P}\mathbf{x})$
All \mathcal{S} are \mathcal{P}	\longrightarrow	$(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \mathcal{P}\mathbf{x})$
No \mathcal{S} are \mathcal{P}	\longrightarrow	$(\mathbf{x})(\mathcal{S}\mathbf{x} \supset \sim\mathcal{P}\mathbf{x})$

There are, however, other ways of translating these English sentences into QL . For instance, given the following (partial) interpretation,

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \text{the set of all people} \\ R\mathbf{x} & = \mathbf{x} \text{ is a Republican} \\ S\mathbf{x} & = \mathbf{x} \text{ is socially liberal} \end{cases}$$

each of the following wffs of QL correctly translate the English sentence ‘Some Republicans are socially liberal’.

$$\text{Some Republicans are socially liberal} \longrightarrow \begin{cases} (\exists x)(Rx \bullet Sx) \\ \sim(x)(Rx \supset \sim Sx) \end{cases}$$

These wffs are QL -equivalent—they are true in all the same QL -interpretations and false in all the same QL -interpretations. This follows from a fact that we have already seen in the course: **I** is the contradictory of **E**; which means that **I** is equivalent to $\sim\mathbf{E}$. And ‘ $\sim(x)(Rx \supset \sim Sx)$ ’ is just the negation of the **E**-form categorical proposition ‘No Republicans are socially liberal’.

Likewise, each of the following wffs correctly translate the English ‘Some Republicans are not socially liberal’:

$$\text{Some Republicans are not socially liberal} \longrightarrow \begin{cases} (\exists x)(Rx \bullet \sim Sx) \\ \sim(x)(Rx \supset Sx) \end{cases}$$

These wffs are QL -equivalent—they are true in all the same QL -interpretations and false in all the same QL -interpretations. Again, this follows from a fact that we have already seen in the course: **O** is the contradictory of **A**; which means that **O** is equivalent to $\sim\mathbf{A}$. And ‘ $\sim(x)(Rx \supset Sx)$ ’ is just the negation of the **A**-form categorical proposition ‘All Republicans are socially liberal’.

Also, each of the following wffs of QL correctly translate the English sentence ‘No Republicans are socially liberal’.

$$\text{No Republicans are socially liberal} \longrightarrow \begin{cases} (x)(Rx \supset \sim Sx) \\ \sim(\exists x)(Rx \bullet Sx) \end{cases}$$

These wffs are QL -equivalent—they are true in all the same QL -interpretations and false in all the same QL -interpretations. Again, this follows from a fact that we have already seen in the course: **E** is the contradictory of **I**; which means that **E** is equivalent to $\sim\mathbf{I}$. And ‘ $\sim(\exists x)(Rx \bullet Sx)$ ’ is just the negation of the **I**-form categorical proposition ‘Some Republicans are socially liberal’.

Similarly, each of the following wffs correctly translate the English ‘All Republicans are socially liberal’:

$$\text{All Republicans are socially liberal} \longrightarrow \begin{cases} (x)(Rx \supset Sx) \\ \sim(\exists x)(Rx \bullet \sim Sx) \end{cases}$$

These wffs are QL -equivalent—they are true in all the same QL -interpretations and false in all the same QL -interpretations. Again, this follows from a fact that we have already seen in the course: **A** is the contradictory of **O**; which means that **A** is equivalent to $\sim\mathbf{O}$. And ‘ $\sim(\exists x)(Rx \bullet \sim Sx)$ ’ is just the negation of the **O**-form categorical proposition ‘Some Republicans are not socially liberal’.

Chapter 13

Quantificational Logic: Validity, Equivalence, and the Rest

13.1 Notation

A bit of notation: let's use expressions like

$$P[x], Q[x]$$

as variables ranging over the wffs of QL in which the variable x occurs freely (x is itself a variable ranging over the variables of QL ; it is not itself a part of the language QL). And we'll use expressions like

$$P[x \rightarrow t], Q[x \rightarrow t]$$

to refer to the wffs of QL that you get when you replace every free occurrence of x in $P[x]$ and $Q[x]$ with the term t . That is: given a wff $P[x]$, you get the wff $P[x \rightarrow t]$ by going through $P[x]$, and every time x appears free, you swap it out for the term t .

Using this notation,

$$P[x \rightarrow t]$$

refers to a SUBSTITUTION INSTANCE of the quantified formulae

$$(\forall x)P[x]$$

and

$$(\exists x)P[x]$$

(We will return to the notion of a substitution instance, and refine our definition, when we talk about QL -Derivations.)

13.2 QL -Validity

In PL , we defined validity in terms of truth-preservation on truth-value assignments—that is, an argument was PL -valid if and only if every truth-value assignment which made all of the premises true was a truth-value assignment which made the conclusion true also.

An argument $p_1 / p_2 / \dots / p_N \parallel c$ is PL -valid if and only if every truth-value assignment on which p_1, p_2, \dots, p_N are all true is a truth-value assignment on which c is true also.

In QL , we're going to define validity in precisely the same way, except that we're going to exchange 'truth-value assignment' for ' QL -interpretation'.

A QL -argument $P_1 / P_2 / \dots / P_N // C$ is QL -VALID if and only if every QL -interpretation on which P_1, P_2, \dots, P_N are all true is a QL -interpretation on which C is true also.

For instance, let's show that the following inference, called *universal instantiation (UI)*, is QL -valid:

$$\frac{(\mathbf{x})P}{P[\mathbf{x} \rightarrow \mathbf{a}]}$$

Pick any interpretation, \mathcal{I} , which makes the premise, $(\mathbf{x})P$, true. A wff of the form $(\mathbf{x})P$ is true on an interpretation \mathcal{I} only if every substitution instance of the wff is true. But $P[\mathbf{x} \rightarrow \mathbf{a}]$ is a substitution instance of $(\mathbf{x})P$. So $P[\mathbf{x} \rightarrow \mathbf{a}]$ must be true on the interpretation \mathcal{I} too.

Therefore, every QL -interpretation which makes a wff of the form $(\mathbf{x})P$ true makes a wff of the form $P[\mathbf{x} \rightarrow \mathbf{a}]$ true as well. So *UI* is QL -valid.

An argument of QL is QL -INVALID if and only if it is *not* QL -valid. That is:

A QL -argument $P_1 / P_2 / \dots / P_N // C$ is QL -INVALID if and only if there is some QL -interpretation on which P_1, P_2, \dots, P_N are all true and C is false.

We may introduce the notion of a QL -COUNTEREXAMPLE. This is just an interpretation which makes all of the premises of an argument of QL true, yet makes its conclusion false.

A QL -COUNTEREXAMPLE to a QL -argument $P_1 / P_2 / \dots / P_N // C$ is a QL -interpretation on which P_1, P_2, \dots, P_N are all true and C is false.

With this notion of a QL -counterexample in hand, we may give a simpler definition of QL -validity and QL -invalidity. An argument of QL is QL -valid if and only if it has no QL -counterexample. And it is QL -invalid if and only if it has a QL -counterexample.

An argument of QL is QL -VALID if and only if it has no QL -counterexample.

An argument of QL is QL -INVALID if and only if it has some QL -counterexample.

Therefore, to show that an argument is QL -INVALID, you can just provide a QL -counterexample. That is: you may just provide a QL -interpretation on which the premises of the argument are true, yet the conclusion is false.

For instance, to show that the following argument is QL -invalid:

$$\frac{(\exists x)Ax \cdot (\exists x)Bx}{(\exists x)(Ax \cdot Bx)}$$

It suffices to provide the following (partial) QL -interpretation:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \{ 1, 2 \} \\ Ax & = x \text{ is odd} \\ Bx & = x \text{ is even} \end{cases}$$

Given this interpretation, the premise of the above argument is true, yet its conclusion is false. So, this interpretation provides a QL -counterexample to the validity of the argument $(\exists x)Ax \bullet (\exists x)Bx // (\exists x)(Ax \bullet Bx)$.

13.3 QL -tautologies, QL -self-contradictions, & QL -contingencies

In PL , we classified *wffs* of PL according to whether they were:

1. True on every truth-value assignment;
2. False on every truth-value assignment; or
3. True on some truth-value assignments and false on other truth-value assignments.

If a *wff* of PL was true on every truth-value assignment, then we said that it was a PL -tautology,

A *wff* of PL is a PL -TAUTOLOGY if and only if it is true on every truth-value assignment

If a *wff* of PL was false on every truth-value assignment, then we said that it was a PL -self-contradiction,

A *wff* of PL is a PL -SELF-CONTRADICTION if and only if it is false on every truth-value assignment.

And if a *wff* of PL was true on some truth-value assignments and false on other, then we said that it was a PL -contingency.

A *wff* of PL is PL -CONTINGENT if and only if it is true on some truth-value assignments and false on other truth-value assignments.

In QL , we will categorize *wffs* of QL in a precisely analogous manner, simply swapping out the notion of a *truth-value assignment* for the notion of a QL -interpretation. Thus, a *wff* of QL may be either:

1. True on every QL -interpretation;
2. False on every QL -interpretation; or
3. True on some QL -interpretations and false on other QL -interpretations

In the first case, we will say that the *wff* is a QL -tautology,

A *wff* of QL is a QL -TAUTOLOGY if and only if it is true on every QL -interpretation.

For instance,

$$(\exists x)Fx \vee (x) \sim Fx$$

is a QL -tautology. For the domain \mathcal{D} of every QL interpretation either has something in it which has the property represented by F or it has nothing in it which has the property represented by F . Suppose there is something in the domain which has the property represented by F . Since everything in the domain of the interpretation has a name, this thing must have a name—denote that name, whatever that name is, with ‘a’. Then, the left-hand-side disjunct, $(\exists x)Fx$ will have a true substitution instance, namely, Fa . Therefore, $(\exists x)Fx$ will be true. Therefore, $(\exists x)Fx \vee (x) \sim Fx$ will be true.

Suppose, on the other hand, that nothing in the domain which has the property represented by F . Then, for everything in the domain a , $\sim Fa$ will be true. Thus, every substitution instance of $(x) \sim Fx$ will be true. Thus, $(x) \sim Fx$ will be true. And if $(x) \sim Fx$ is true, then $(\exists x)Fx \vee (x) \sim Fx$ must be true.

So, either way, $(\exists x)Fx \vee (x) \sim Fx$ will be true. So it will be true on every QL -interpretation. So, it is a QL -tautology.

In case (2) above, we will say that the wff of QL is a QL -self-contradiction,

A wff of QL is a QL -SELF-CONTRADICTION if and only if it is false on every QL -interpretation.

For instance,

$$\sim(\exists x)(Fx \vee \sim Fx)$$

is a QL -self-contradiction. For every interpretation, its domain must have something in it. Pick one such thing, and call it ‘a’. a either has the property F or it doesn’t. If it does, then Fa is true—hence, $Fa \vee \sim Fa$ is true. If it doesn’t then $\sim Fa$ is true—hence $Fa \vee \sim Fa$ is true. Either way, $Fa \vee \sim Fa$ is true. Therefore, $(\exists x)(Fx \vee \sim Fx)$ has a true substitution instance, so it is true. So $\sim(\exists x)(Fx \vee \sim Fx)$ is false. So $(\exists x)(Fx \vee \sim Fx)$ is false on every QL -interpretation. Hence, it is a QL -self-contradiction.

Finally, in case (3) above, we will say that the wff of QL is a QL -contingency.

A wff of QL is QL -CONTINGENT if and only if it is true on some QL -interpretations and false on other QL -interpretations.

For instance, the wff of QL

$$(x)Fx$$

is a QL -contingency. For it is true on the (partial) interpretation on the left below and false on the one on the right.

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \{1\} \\ Fx & = x \text{ is odd} \end{cases}$$

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \{1, 2\} \\ Fx & = x \text{ is odd} \end{cases}$$

13.4 QL -Equivalence & QL -Contradiction

In PL , recall, we said that two wffs of PL are PL -EQUIVALENT if and only if they have the same truth-value on every truth-value assignment,

Two wffs are PL -EQUIVALENT if and only if there is no truth-value assignment in which they have different truth values (i.e., if and only if their truth values match on every truth-value assignment).

and we said that two wffs of PL were PL -contradictory if and only if they have different truth-values on every truth-value assignment,

Two wffs are PL -CONTRADICTORY if and only if there is no truth-value assignment in which they have the same truth value (i.e., if and only if they have different truth values in every truth-value assignment).

In QL , we're going to do the same thing, but we'll swap out the notion of a truth-value assignment for the notion of a QL -interpretation. Thus, two wffs of QL are QL -EQUIVALENT if and only if they have the same truth-value on every QL -interpretation.

Two wffs are QL -EQUIVALENT if and only if there is no QL -interpretation on which they have different truth values (i.e., if and only if their truth values match on every QL -interpretation).

For instance, the wffs

$$\sim(x)Fx \quad \text{and} \quad (\exists x) \sim Fx$$

are QL -equivalent. Suppose that $\sim(x)Fx$ is true on a QL -interpretation. Then, $(x)Fx$ must be false on that interpretation. Which means that $(x)Fx$ has some false substitution instance, Fa . Since Fa is false, $\sim Fa$ must be true. Hence, there is something in the domain of the interpretation which makes $\sim Fa$ true. So $(\exists x) \sim Fx$ must have a true substitution instance. So $(\exists x) \sim Fx$ must be true. Putting it all together, if $\sim(x)Fx$ is true on a QL -interpretation, then $(\exists x) \sim Fx$ must be true on that interpretation as well.

Suppose, on the other hand, that $\sim(x)Fx$ is false on a QL -interpretation. Then, $(x)Fx$ must be true on that interpretation. But that means that every substitution instance of $(x)Fx$ is true. Which means that everything in the domain has the property F . So, Fa is true for every a . So $\sim Fa$ is false for every a . So $(\exists x) \sim Fx$ does not have a true substitution instance. So $(\exists x) \sim Fx$ is false. Putting it all together, if $\sim(x)Fx$ is false on a QL -interpretation, then $(\exists x) \sim Fx$ must be false on that interpretation as well.

So $\sim(x)Fx$ and $(\exists x) \sim Fx$ have exactly the same truth-value on every QL -interpretation.

In a similar fashion, we define two wffs of to be QL -CONTRADICTORY if and only if they necessarily have *different* truth-values on every QL -interpretation.

Two wffs are QL -CONTRADICTORY if and only if there is no QL -interpretation on which they have the same truth value (i.e., if and only if they have different truth values on every QL -interpretation).

For instance, the wffs

$$(x)Fx \quad \text{and} \quad (\exists x) \sim Fx$$

are QL -contradictory. For suppose that there is an interpretation which makes $(x)Fx$ true. Then, everything in the domain of that interpretation must have the property represented by F . But if everything in the domain has the property represented by F , then for nothing a will $\sim Fa$ be true. So $(\exists x) \sim Fx$ will not have any true substitution instance. So $(\exists x) \sim Fx$ will be false.

Suppose on the other hand that there is an interpretation which makes $(x)Fx$ false. Then, $(x)Fx$ must have a false substitution instance Fa . But if Fa is false, then $\sim Fa$ must be true. So $(\exists x) \sim Fx$ will have a true substitution instance. So $(\exists x) \sim Fx$ will be true.

So, if $(x)Fx$ is true on a QL -interpretation, then $(\exists x) \sim Fx$ is false on that interpretation. And if $(x)Fx$ is false on a QL -interpretation, then $(\exists x) \sim Fx$ is true on that interpretation. So $(x)Fx$ and $(\exists x) \sim Fx$ have different truth-values on every QL -interpretation. So they are QL -contradictories.

13.5 QL -Consistency & QL -Inconsistency

Recall, in PL , we said that a set of wffs of PL were PL -consistent if and only if there was some truth-value assignment which made them all true,

A set of wffs of PL is PL -CONSISTENT if and only if there is some truth-value assignment on which all of the wffs are true.

and we said that a set of wffs of PL were PL -inconsistent if and only if there was no truth-value assignment which made them all true,

A set of wffs of PL is PL -INCONSISTENT if and only if there is no truth-value assignment on which all of the wffs are true.

In QL , we will do precisely the same thing, but we will substitute out the notion of a truth-value assignment for the notion of a QL -interpretation. Thus, we will say that a set of wffs of QL is QL -consistent if and only if there is some QL -interpretation which makes them all true.

A set of wffs of QL is QL -CONSISTENT if and only if there is some QL -interpretation which makes them all true.

For instance, suppose that we wish to show that the following set of wffs of QL are QL -consistent:

$$\{(x)(Fx \supset Gx), (x)(Gx \supset Hx), \sim(\exists x)(Fx \bullet Hx)\}$$

To show this, it suffices to note that, on the following (partial) QL -interpretation, all of the wffs in this set are true (recall that 1 is not prime):

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \{1\} \\ Fx & = x \text{ is even} \\ Gx & = x \text{ is prime} \\ Hx & = x \text{ is odd} \end{cases}$$

On this interpretation, $(x)(Fx \supset Gx)$ has no false substitution instance, since nothing in the domain is even. Thus, on every substitution instance, the antecedent of the conditional is false; so, on every substitution instance, the conditional is true. Nor does $(x)(Gx \supset Hx)$ have a false substitution instance, since nothing in the domain is prime. So, on every substitution instance, the antecedent of the conditional is false; so, on every substitution instance, the conditional is true. And $(\exists x)(Fx \bullet Hx)$ has no true substitution instance, since nothing in the domain is even. So $(\exists x)(Fx \bullet Hx)$ is false. So $\sim(\exists x)(Fx \bullet Hx)$ is true. So this is a (partial) QL -interpretation which makes all three claims true at once. So these three claims are consistent.

In similar fashion, we will say that a set of wffs of QL is QL -inconsistent if and only if there is no QL -interpretation which makes them all true,

A set of wffs of QL is QL -INCONSISTENT if and only if there is no QL -interpretation on which all of the wffs are true (i.e., if and only if, on every QL -interpretation, at least one of the wffs in the set is false).

For instance, the following set of wffs is QL -inconsistent.

$$\{(\exists x)Fx, (x)(Fx \supset Gx), \sim(\exists x)Gx\}$$

For suppose that $(\exists x)Fx$ is true. Then, there must be some thing in the domain—call it ‘a’—which has the property F . But, if $(x)(Fx \supset Gx)$ is also true, then $Fa \supset Ga$ must be true (since this is a substitution instance of $(x)(Fx \supset Gx)$). So Fa and $Fa \supset Ga$. By *modus ponens*, Ga . But then $(\exists x)Gx$ has a true substitution instance. So $(\exists x)Gx$ must be true. But then, $\sim(\exists x)Gx$ must be false. So, if the first two wffs in the set are true, then the final wff must be false. So there is no QL -interpretation which makes all of the wffs in the set true at once. So $\{(\exists x)Fx, (x)(Fx \supset Gx), \sim(\exists x)Gx\}$ is QL -inconsistent.

Chapter 14

Quantificational Logic: Derivations

14.1 Substitution Instances

Just to refresh your memory: we're using expressions like

$$P[x], Q[x]$$

as variables ranging over the wffs of QL in which the variable x occurs freely. And we're using expressions like

$$P[x \rightarrow t], Q[x \rightarrow t]$$

to refer to the wffs of QL that you get when you replace every free occurrence of x in $P[x]$ and $Q[x]$ with the term t . That is: given a wff $P[x]$, you get the wff $P[x \rightarrow t]$ by going through $P[x]$, and every time x appears free, you swap it out for the term t .

Using this notation,

$$P[x \rightarrow t]$$

refers to a **SUBSTITUTION INSTANCE** of the quantified formulae

$$(\forall x)P[x]$$

and

$$(\exists x)P[x]$$

A point of clarification about substitution instances. Our official definition of **SUBSTITUTION INSTANCE** is given below.

A **SUBSTITUTION INSTANCE** of a quantified wff of the form ' $(\forall x)P$ ' or ' $(\exists x)P$ ' is the wff that you get by removing the quantifier, leaving behind just its immediate subformula, and uniformly replacing every instance of the variable x which was bound by the quantifier with some constant a or some **free** variable y .

It will be important for learning the derivation system for QL that we pay careful attention to the bolded word above—**free**. We must recognize that, if the variable you substitute for the previously bound variable ends up being bound by some other quantifier, then what you end up with is *not* a substitution instance. For instance, consider the wff

$$(y)(\exists x)Pxy$$

To obtain a substitution instance of this wff, we may remove its main operator, the quantifier ‘ (y) ’, and replace every occurrence of y within the scope of (y) with a constant, as in

$$(\exists x)P_x k \quad \leftarrow \text{a substitution instance of } (y)(\exists x)P_x y$$

We may also remove its main operator and replace every occurrence of y within the scope of (y) with a **free** variable, as in

$$(\exists x)P_x z \quad \leftarrow \text{a substitution instance of } (y)(\exists x)P_x y$$

However, we may *not* replace every occurrence of y with a variable which ends up being *bound by some other quantifier in the scope of (y)* . So, for instance, the following is *not* a substitution instance of $(y)(\exists x)P_x z$:

$$(\exists x)P_x x \quad \leftarrow \text{not a substitution instance of } (y)(\exists x)P_x y$$

14.2 QL-Derivations

It’s not nearly as easy to prove that arguments are *QL*-valid using interpretations as it was to show that arguments were *PL*-valid using truth-tables. So rather than dwell on establishing validity, we’re going to march right into the *QL*-derivation system. This derivation system is an extension of the derivation system for *PL*. We just add to it eight new rules of replacement and six new rules of implication. The resulting system has the following property: if there is a legal derivation with assumptions P_1, P_2, \dots, P_N and which has C on its final line, then the argument $P_1 / P_2 / \dots / P_N // C$ is *QL*-valid.

14.2.1 New Rules of Replacement

The new rules of replacement all trade on the fact that, given our semantics, ‘ (\mathbf{x}) ’ has the same meaning as ‘ $\sim (\exists \mathbf{x}) \sim$ ’ and ‘ $(\exists \mathbf{x})$ ’ has the same meaning as ‘ $\sim (\mathbf{x}) \sim$ ’. It says that we may replace any instance of ‘ (\mathbf{x}) ’ with ‘ $\sim (\exists \mathbf{x}) \sim$ ’ (and *vice versa*). Moreover, we may replace any instance of ‘ $\sim (\mathbf{x})$ ’ with ‘ $(\exists \mathbf{x}) \sim$ ’ (and *vice versa*). And we may replace any instance of ‘ $(\exists \mathbf{x})$ ’ with ‘ $\sim (\mathbf{x}) \sim$ ’ (and *vice versa*). Finally, we may replace any instance of ‘ $\sim (\exists \mathbf{x})$ ’ with ‘ $(\mathbf{x}) \sim$ ’ (and *vice versa*). We do so, we should cite the line on which the original wff appeared and write ‘*QN*’.

<u>Quantifier Negation (QN)</u>		
$(\mathbf{x})\mathbf{P}$	$\triangleleft \triangleright$	$\sim (\exists \mathbf{x}) \sim \mathbf{P}$
$\sim (\mathbf{x})\mathbf{P}$	$\triangleleft \triangleright$	$(\exists \mathbf{x}) \sim \mathbf{P}$
$(\exists \mathbf{x})\mathbf{P}$	$\triangleleft \triangleright$	$\sim (\mathbf{x}) \sim \mathbf{P}$
$\sim (\exists \mathbf{x})\mathbf{P}$	$\triangleleft \triangleright$	$(\mathbf{x}) \sim \mathbf{P}$

In short: pushing negations inside of quantifiers (or pulling them outside of quantifiers) flips universal quantifiers to existential quantifiers, and flips existential quantifiers to universal quantifiers.

Because *QN* is a rule of replacement, it may be applied to subformulae of a wff. For instance, the following is a legal derivation.

- 1 $\sim (\exists z)Fz \equiv (x)(\exists y)Lxy$
- 2 $(z) \sim Fz \equiv (x)(\exists y)Lxy$ 1, *QN*
- 3 $(z) \sim Fz \equiv \sim (\exists x) \sim (\exists y)Lxy$ 2, *QN*
- 4 $(z) \sim Fz \equiv \sim (\exists x)(y) \sim Lxy$ 3, *QN*

14.2.2 New Rules of Implication

Universal Instantiation

The first new rule of replacement says that, whenever you have a universally quantified formula, you can write down a substitution instance of it.

<p><u>Universal Instantiation (UI)</u></p> $(\mathbf{x})\mathbf{P}$ $\triangleright \mathbf{P}[\mathbf{x} \rightarrow \mathbf{a}]$ <p>where 'a' is a constant; or:</p> $(\mathbf{x})\mathbf{P}$ $\triangleright \mathbf{P}[\mathbf{x} \rightarrow \mathbf{y}]$ <p>where 'y' is a variable.</p>
--

NOTE: when you use *UI*, you must be sure that you replace *every* occurrence of the bound variable with *the same* constant or *the same* variable. Otherwise, what we write down won't be a substitution instance of the wff we started with.

For instance, the following *QL*-derivation is *not* legal.

$$\begin{array}{llll} 1 & (y)(Fy \supset Gy) & & \\ 2 & Fa \supset Gb & 1, UI & \leftarrow \text{MISTAKE!!!} \end{array}$$

for we replaced with first bound 'y' with 'a', and the second bound 'y' with 'b'. Similarly, the following derivation is not legal:

$$\begin{array}{llll} 1 & (z)(Az \equiv Bz) & & \\ 2 & Aa \equiv Bx & 1, UI & \leftarrow \text{MISTAKE!!!} \end{array}$$

For we replaced the first bound 'z' with 'a', and the second bound 'z' with 'x'.

These QL-derivations, on the other hand, *are* legal.

$$\begin{array}{llll} 1 & (y)(Fy \supset Gy) & & \\ 2 & Fa \supset Ga & 1, UI & \end{array}$$

$$\begin{array}{llll} 1 & (z)(Az \equiv Bz) & & \\ 2 & Ax \equiv Bx & 1, UI & \end{array}$$

Keep in mind: the thing you write down must actually *be* a substitution instance of the quantified wff you started with. The following derivation is not legal:

- 1 $(y)(\exists x)Lyx$
- 2 $(\exists x)Lxx$ 1, *UI* ← MISTAKE!!!

Line 2 does not follow from line 1 because the instantiated variable, x , is not free in line 2. It is bound by the existential quantifier. So $(\exists x)Lxx$ is not a substitution instance of $(y)(\exists x)Lyx$. It's a good thing that our derivation system does not allow this, for $(\exists x)Lxx$ does not follow from $(y)(\exists x)Lyx$. Consider the following (partial) QL -interpretation:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} = \{1, 2, 3, \dots\} \\ Lxy = x \text{ is less than } y \end{cases}$$

Because, for every number, there's some number that it's less than, $(y)(\exists x)Lyx$ is true on this interpretation. However, since there's no number that's less than itself, $(\exists x)Lxx$ is false on this interpretation. Hence, the argument $(y)(\exists x)Lyx \parallel (\exists x)Lxx$ is QL -invalid.

It might help to keep yourself from committing these errors, and also to follow what's going on in the derivation if, after you justify a rule by *UI*, you write in brackets the constant or variable which you instantiated, and the variable for which you instantiated it. So, for instance, you could write out the preceding derivations like so:

- 1 $(y)(Fy \supset Gy)$
- 2 $Fa \supset Ga$ 1, *UI* [$y \rightarrow a$]

- 1 $(z)(Az \equiv Bz)$
- 2 $Ax \equiv Bx$ 1, *UI* [$z \rightarrow x$]

This is not required for a derivation to be legal; but you I encourage you to adopt this convention. I will adopt it from here on out.

Existential Generalization

The second new rule of implication tells us that, if you have a wff of QL according to which some *particular* thing a has a certain property, then you may infer that *something* has that property. That is, if you have a substitution instance of an existentially quantified formula, then you may write down that existentially quantified formula.

Existential Generalization (EG)

$P[x \rightarrow a]$

▷ $(\exists x)P$

where 'a' is a constant; or:

$P[x \rightarrow y]$

▷ $(\exists x)P$

where 'y' is a variable.

For instance, the following are legal QL -derivations.

$$\begin{array}{lll}
1 & (x)(y)Bxy & \\
2 & (y)Bay & 1, UI [x \rightarrow a] \\
3 & (\exists z)(y)Bzy & 2, EG
\end{array}$$

$$\begin{array}{lll}
1 & (x)(Px \equiv Qx) & \\
2 & Pz \equiv Qz & 1, UI [x \rightarrow z] \\
3 & (\exists y)(Py \equiv Qy) & 2, EG
\end{array}$$

$$\begin{array}{lll}
1 & (x)Acx & \\
2 & Acc & 1, UI [x \rightarrow c] \\
3 & (\exists x)Axc & 2, EG
\end{array}$$

This final proof is legal because line 2 is a substitution instance of line 1.

A potential confusion: when you *instantiate* a variable by writing down a substitution instance of a quantified wff of QL , you must replace every instance of the bound variable with the same term. Thus, the derivation below is *not* legal:

$$\begin{array}{lll}
1 & (\exists x)Rxxx & \\
2 & Raxxx & 1, UI [x \rightarrow a] \quad \leftarrow \text{MISTAKE!!!}
\end{array}$$

For line 2 is not a substitution instance of line 1 (all of the bound 'x's must be replaced with the same term in order for it to be a substitution instance).

However, when you *existentially generalize* from a substitution instance of a quantified wff to that quantified wff, you *needn't* replace every instance of the term from which you are generalizing. Thus, the derivation below is legal:

$$\begin{array}{lll}
1 & Raaaa & \\
2 & (\exists x)Rxaax & 1, EG \\
3 & (\exists y)(\exists x)Rxyaa & 2, EG \\
4 & (\exists z)(\exists y)(\exists x)Rxyzza & 3, EG
\end{array}$$

That's because line 1 is a substitution instance of line 2, line 2 is a substitution instance of line 3, and line 3 is a substitution instance of line 4.

Existential Instantiation

The third new rule of implication says that, if you have an existentially quantified wff, $(\exists x)P$, then you may write down a substitution instance of that wff, $P[x \rightarrow a]$ —provided that the constant that you introduce is entirely new (it doesn't appear on any previous line), and provided that you get rid of it before you're done (that is, provided that it doesn't appear on the conclusion line).

Existential Instantiation (EI)

$(\exists x)P$

$\triangleright P[x \rightarrow a]$

where 'a' is a constant.

provided that:

1. a does not appear on any previous line
2. a does not appear on the conclusion line

It is important to keep these provisions in mind. The idea behind this rule is that, if you know that *there is* something which is P, then it's o.k. to give that thing a name. *However*, you don't want to assume anything about this thing *other* than that it's P. So you'd better give it an entirely *new* name; otherwise, you'd be assuming more about the thing than that it is P. Similarly, you'd better get rid of the name before you're done, since, on a *QL*-interpretation, that name has a meaning—it refers to something in the domain. You don't know what that thing is, so leaving it behind at the end of the derivation would allow you to conclude more than you know.

The following derivation is *not* legal:

- | | | | |
|---|---------------------------------------|---------------|--------------|
| 1 | $(\exists y)(Dy \equiv (He \vee Jy))$ | | |
| 2 | $De \equiv (He \vee Je)$ | 1, EI [y → e] | ← MISTAKE!!! |
| 3 | $(\exists x)(Dx \equiv (Hx \vee Jx))$ | 2, EG | |

The constant 'e' appears on line 1, so it cannot be instantiated on line 2 by EI.

This derivation, however, *is* legal:

- | | | | |
|---|---------------------------------------|---------------|--|
| 1 | $(\exists y)(Dy \equiv (He \vee Jy))$ | | |
| 2 | $Da \equiv (He \vee Ja)$ | 1, EI [y → a] | |
| 3 | $(\exists z)(Dz \equiv (He \vee Jz))$ | 2, EG | |

Similarly, the following derivation is not legal:

- | | | | |
|---|------------------------------|---------------|----------------|
| 1 | $(y)(Fy \supset Ky)$ | | } ← MISTAKE!!! |
| 2 | $(\exists x)(Fx \bullet Qx)$ | | |
| 3 | $Fk \bullet Qk$ | 2, EI [x → k] | |
| 4 | Fk | 3, Simp | |
| 5 | $Fk \supset Kk$ | 1, UI [y → k] | |
| 6 | Kk | 4, 5, MP | |

The constant *k* was existentially instantiated on line 3; however, it appears on the final line of the derivation. EI, however, only allows you to existentially instantiate a constant if it disappears by the time the derivation is through.

This derivation, on the other hand, *is* legal:

1	$(y)(Fy \supset Ky)$	
2	$(\exists x)(Fx \bullet Qx)$	
3	$Fk \bullet Qk$	2, <i>EI</i> [$x \rightarrow k$]
4	Fk	3, <i>Simp</i>
5	$Fk \supset Kk$	1, <i>UI</i> [$y \rightarrow k$]
6	Kk	4, 5, <i>MP</i>
7	$(\exists x)Kx$	6, <i>EG</i>

Here, too, it will help to keep yourself from committing errors, and follow what's going on in the derivation if, after you justify a rule by *EI*, you write in brackets the constant or variable which you instantiated. Again, this is entirely optional, but I encourage you to do it. I have adopted this convention above.

A Sample Derivation

1	$(x)(Fx \bullet (\exists y)Gy)$	$/(\exists z)(Fz \bullet Gz)$
2	$Fa \bullet (\exists y)Gy$	1, <i>UI</i> [$x \rightarrow a$]
3	$(\exists y)Gy \bullet Fa$	2, <i>Com</i>
4	$(\exists y)Gy$	3, <i>Simp</i>
5	Gc	4, <i>EI</i> [$y \rightarrow c$]
6	$Fc \bullet (\exists y)Gy$	1, <i>UI</i> [$x \rightarrow c$]
7	Fc	6, <i>Simp</i>
8	$Fc \bullet Gc$	5, 7, <i>Conj</i>
9	$(\exists z)(Fz \bullet Gz)$	8, <i>EG</i>

Universal Generalization

The final new rule of implication says that, if you have a wff of *QL* in which a variable occurs freely, then you may replace it with a(nother) variable and tack on a quantifier out front—*provided that* the freely occurring variable does not occur free in either the assumptions or the first line of any accessible subderivation, and provided that it does not occur freely in any line which is justified by *EI*.

Universal Generalization (UG)

$$P[x \rightarrow y]$$

$$\triangleright (\forall x)P$$

provided that:

1. y does not occur free in the assumptions
2. y does not occur free in the first line of an accessible subderivation.
3. y does not occur free in any accessible line justified by *EI*.
4. y does not occur free in $(\forall x)P$.

Again, it is important to keep these provisions in mind. Let us begin with the final provision. There is an important difference between *existential* generalization and *universal* generalization. With existential generalization, you are allowed to leave behind occurrences of the variable from which you existentially generalize. That is, derivations like the following are allowed:

- | | | |
|---|------------------|------------------------------------|
| 1 | $(z)Rzz$ | |
| 2 | Rxx | 1, <i>UI</i> [$z \rightarrow x$] |
| 3 | $(\exists z)Rxz$ | 2, <i>EG</i> |

However, provision 4 above tells us that this is not allowed with *universal* generalization. The following derivation is *not* legal:

- | | | |
|---|----------|------------------------------------|
| 1 | $(z)Rzz$ | |
| 2 | Rxx | 1, <i>UI</i> [$z \rightarrow x$] |
| 3 | $(z)Rxz$ | 2, <i>UG</i> ← MISTAKE!!! |

This is very good, because $(z)Rxz$ does not follow from $(z)Rzz$. There are *QL*-interpretations on which the first wff is true while the second is false. For instance,

$$\mathcal{I}_p = \begin{cases} \mathcal{D} = \{1, 2\} \\ Rxy = x \text{ is less than or equal to } y \\ x = 2 \end{cases}$$

Both 1 and 2 are less than or equal to themselves, so every substitution instance of $(z)Rzz$ is true on this (partial) *QL*-interpretation. However, 2 is not less than or equal to 1, so $(z)Rxz$ has a false substitution instance on this (partial) *QL*-interpretation.

For another instance in which failure to abide by provision 4 would lead us into trouble, consider the following derivation:

1	Rzz	ACP	
2	$Rzz \vee Rzz$	$1, Taut$	
3	Rzz	$2, Taut$	
4	$Rzz \supset Rzz$	$1-3, CP$	
5	$(y)(Rzy \supset Ryz)$	$4, UG$	\leftarrow MISTAKE!!!
6	$(x)(y)(Rxy \supset Ryx)$	$5, UG$	

Line 5 does not follow from line 4, because occurrences of the variable z were left behind. And this is good. If this derivation were legal, then we would falsely conclude that it is a QL -tautology that every two-place relation of QL is symmetric. But that is not a QL -tautology, as the QL -interpretation above shows (1 is less than or equal to 2, but 2 is not less than or equal to 1).

Similarly, provision 1 tells us that the following derivation is not legal:

1	$(x)(Fx \supset Gy)$		
2	$(y)Fy$		
3	$Fc \supset Gy$	$1, UI [x \rightarrow c]$	
4	Fc	$2, UI [y \rightarrow c]$	
5	Gy	$3, 4, MP$	
6	$(x)Gx$	$5, UG$	\leftarrow MISTAKE!!!

The variable y appears free in one of the assumptions of the derivation. Therefore, we may not universally generalize from that variable. This is a good thing, too, for $(x)Gx$ does not follow from $(x)(Fx \supset Gy)$ and $(y)Fy$. There are QL -interpretations on which the premises are true yet the conclusion is false. For instance, the following (partial) QL -interpretation provides a QL -counterexample to the QL -validity of this argument:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} = \{ 1, 2 \} \\ Fx = x \text{ is positive} \\ Gx = x \text{ is even} \\ y = 2 \end{cases}$$

Had we stopped at line 5, on the other hand, our derivation *would* be legal.

1	$(x)(Fx \supset Gy)$	
2	$(y)Fy$	
3	$Fc \supset Gy$	$1, UI [x \rightarrow c]$
4	Fc	$2, UI [y \rightarrow c]$
5	Gy	$3, 4, MP$

For an example in which provision 2 is violated, consider the following derivation:

1	Fx $(y)Fy$	ACP $1, UG$	← MISTAKE!!!
3	$Fx \supset (y)Fy$	$1-2, CP$	
4	$(z)(Fz \supset (y)Fy)$	$3, UG$	

Line 2 does not follow from line 1, since the variable x appears free in the assumption of an accessible subderivation (the one starting at line 1). (Good thing, too, since ‘ $(z)(Fz \supset (y)Fy)$ ’ is false on any interpretation in which one thing is F and another is not F —so it is not a QL -tautology.)

For an example in which provision 3 is violated, consider the following derivation:

1	$(x)(\exists y)Axy$		
2	$(\exists y)Azy$	$1, UI [x \rightarrow z]$	
3	Azc	$2, EI [y \rightarrow c]$	
4	$(x)Axc$	$3, UG$	← MISTAKE!!!
5	$(\exists y)(x)Axy$	$4, EG$	

Line 4 does not follow from line 3, since the variable z appears free on a line of the derivation which is justified by ‘ EP ’—namely, line 3. It’s a good thing, too, since $(\exists y)(x)Axy$ doesn’t follow from $(x)(\exists y)Axy$ —there are QL -interpretations on which the first is true but the second false. For instance, consider the following (partial) QL -interpretation:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} = \{1, 2, 3, 4, \dots\} \\ Axy = x \text{ is less than } y \end{cases}$$

It is also important to note that UG only allows you to universally generalize from *variables*. It does not allow you to universally generalize from *constants*. Thus, the following derivation is not legal:

1	$(x)(Yx \bullet Zx)$		
2	$Ya \bullet Za$	$1, UI [x \rightarrow a]$	
3	Ya	$2, Simp$	
4	$(x)Yx$	$3, UG$	← MISTAKE!!!

This derivation, however, is legal.

1	$(x)(Yx \bullet Zx)$		
2	$Yy \bullet Zy$	$1, UI [x \rightarrow y]$	
3	Yy	$2, Simp$	
4	$(x)Yx$	$3, UG$	

Sample Derivations

1	$\sim(x)(Ax \bullet Bx)$	$/(\exists y)(Ay \supset \sim By)$
2	$(\exists x) \sim(Ax \bullet Bx)$	1, QN
3	$\sim(Ac \bullet Bc)$	2, EI [x → c]
4	$\sim Ac \vee \sim Bc$	3, DM
5	$Ac \supset \sim Bc$	4, Impl
6	$(\exists y)(Ay \supset \sim By)$	5, EG

1	$(z)(Fz \bullet Gz)$	
2	$(x)Fx \supset (\exists y)Qy$	
3	$(x)Hx \supset (y) \sim Qy$	$/(\exists x) \sim Hx$
4	$Fx \bullet Gx$	1, UI [z → x]
5	Fx	4, Simp
6	$(x)Fx$	5, UG
7	$(\exists y)Qy$	2, 6, MP
8	$\sim(y) \sim Qy$	7, QN
9	$\sim(x)Hx$	3, 8, MT
10	$(\exists x) \sim Hx$	9, QN

14.2.3 Final Thoughts

It is important, throughout, to make sure that what you are *taking* to be a substitution instance of a wff of *QL* *actually* is a substitution instance of that wff. For instance, consider the following illegal derivation:

1	$(y)(\exists x)Pxy$	
2	$(\exists x)Pxz$	1, UI [y → z]
3	$(x)(\exists x)Pxx$	2, UG ← MISTAKE!!!

Line 3 does not follow from line 2 by *UG* for the simple reason that $(\exists x)Pxz$ **is not a substitution instance of** $(x)(\exists x)Pxx$. For both *x* variables in $(x)(\exists x)Pxx$ are bound by the existential quantifier. The only substitution instance of $(x)(\exists x)Pxx$ is $(\exists x)Pxx$, because (x) binds no variables whatsoever.

Consider also the following illegal derivation:

1	$(x)(\exists y)Lxy$	
2	$(\exists y)Lzy$	1, UI [x → z]
3	Lza	2, EI [y → a]
4	$(\exists z)Lzz$	3, EG ← MISTAKE!!!

Line 4 does not follow from line 3. For line Laz is **not a substitution instance** of $(\exists z)Lzz$. A substitution instance of $(\exists z)Lzz$ would *uniformly* replace the occurrences of the bound variable z with *the same term of QL throughout*.

On the other hand, the following derivation is legal:

1	$(x)(\exists y)Lxy$		
2	$(\exists y)Lzy$	1, UI [$x \rightarrow z$]	
3	Lza	2, EI [$y \rightarrow a$]	
4	$(\exists y)Lzy$	3, EG	

Here, line 4 does follow from line 3, because Lza is a substitution instance of $(\exists y)Lzy$.

One final note on the four new rules of implication. **These are rules of implication—as such, they cannot be applied to subformulae.** For instance, all of the following derivations are illegal:

1	$(x)Fx \supset Ga$		
2	$Fa \supset Ga$	1, UI [$x \rightarrow a$]	← MISTAKE!!!

1	$(x)Fx \supset (x)Gx$		
2	$Fy \supset Gy$	1, UI [$x \rightarrow y$]	← MISTAKE!!!

1	$Pj \bullet Qr$		
2	$(\exists y)Py \bullet Qr$	1, EG	← MISTAKE!!!

1	$\sim(x)Fx$		
2	$\sim Fa$	1, UI [$x \rightarrow a$]	← MISTAKE!!!

14.3 QL-Derivability and the Logical Notions of QL

Let me begin with some notation. If and only if the QL-argument $P_1 / P_2 / \dots / P_N // C$ is QL-valid, I will write:

$$P_1, P_2, \dots, P_N \models_{QL} C$$

This expression just means ‘the QL-argument whose premises are P_1, P_2, \dots , and P_N and whose conclusion is C is QL-valid’. (Or, equivalently, ‘every QL-interpretation which makes all of P_1, P_2, \dots , and P_N true makes C true as well’.)

And similarly, if and only if it is possible to construct a legal QL-derivation whose assumptions are P_1, P_2, \dots, P_N and whose final line is C , I will write

$$P_1, P_2, \dots, P_N \vdash_{QL} C$$

This expression just means ‘there is a possible legal QL-derivation whose assumptions are P_1, P_2, \dots, P_N , and whose final line is C ’. Or, for short ‘ C is QL-derivable from P_1, P_2, \dots, P_N ’.

14.3.1 QL -Validity

Everything that was true about the derivation system in PL is true about the derivation system in QL , too. For instance,

Fact 1: $P_1 / P_2 / \dots / P_N // C$ is QL -valid if and only if C is QL -derivable from P_1, P_2, \dots , and P_N .

$$P_1, P_2, \dots, P_N \models_{QL} C \quad \text{if and only if} \quad P_1, P_2, \dots, P_N \vdash_{QL} C$$

14.3.2 QL -Tautologies and QL -Self-Contradictions

We can additionally use our derivation system to show that a wff of QL is a QL -tautology, if and only if it is a QL -tautology; and we can use it to show that a wff of QL is a QL -self-contradiction, if and only if it is a QL -self-contradiction—just as we could with PL .

We defined a QL -tautology to be a wff of QL that was true in every QL interpretation. However, it turns out that a wff of QL , P , is a QL -tautology if and only if there is a legal QL -derivation without any assumptions whose final line is P . In that case, let's say that P is ' QL -derivable' from *no* assumptions.

Fact 2: A wff of QL , P , is a QL -tautology if and only if

$$\vdash_{QL} P$$

For instance, suppose that we wish to show that $(\exists x)Fx \vee (x) \sim Fx$ is a QL -tautology.

We may do so by showing that

$$\vdash_{QL} (\exists x)Fx \vee (x) \sim Fx$$

We may show this by providing a QL derivation like the following:

1	$\sim(\exists x)Fx$	ACP
2	$(x) \sim Fx$	$1, QN$
3	$\sim(\exists x)Fx \supset (x) \sim Fx$	$1-2, CP$
4	$\sim\sim(\exists x)Fx \vee (x) \sim Fx$	$3, Impl$
5	$(\exists x)Fx \vee (x) \sim Fx$	$4, DN$

Together with **Fact 1**, by the way, **Fact 2** tells us that P is a QL -tautology if and only if

$$\models_{QL} P$$

Similarly, it turns out that a wff of QL , P , is a QL -self-contradiction if and only if there is a legal QL -derivation whose only assumption is P and whose final line is an explicit contradiction of the form $Q \bullet \sim Q$.

Fact 3: A wff of QL , P , is a QL -self-contradiction if and only if

$$P \vdash_{QL} Q \bullet \sim Q$$

for some Q .

For instance, suppose that we wish to show that $\sim(\exists x)(Fx \vee \sim Fx)$ is a QL -self-contradiction. Then, we may do so by showing that

$$\sim(\exists x)(Fx \vee \sim Fx) \vdash_{QL} \mathbf{Q} \bullet \sim \mathbf{Q}$$

for some \mathbf{Q} . We may do this by providing a QL -derivation like the following.

$$\begin{array}{lll} 1 & \sim(\exists x)(Fx \vee \sim Fx) & \\ 2 & (x) \sim(Fx \vee \sim Fx) & 1, QN \\ 3 & \sim(Fa \vee \sim Fa) & 2, UI [x \rightarrow a] \\ 4 & \sim Fa \bullet \sim \sim Fa & 3, DM \end{array}$$

By the way, together with **Fact 1**, **Fact 3** tells us that \mathbf{P} is a QL -self-contradiction if and only if

$$\mathbf{P} \models_{QL} \mathbf{Q} \bullet \sim \mathbf{Q}$$

for some \mathbf{Q} .

14.3.3 QL -Equivalence and QL -Contradiction

We may additionally use QL -derivations to establish that two wffs of QL are QL -equivalent by appealing to the following fact:

Fact 4: Two wffs of QL , \mathbf{P} and \mathbf{Q} are QL -equivalent if and only if

$$\vdash_{QL} \mathbf{P} \equiv \mathbf{Q}$$

For instance, suppose that we wish to show that the wffs

$$\sim(x)Fx \quad \text{and} \quad (\exists x) \sim Fx$$

are QL -equivalent. To do so, it suffices to show that

$$\vdash_{QL} \sim(x)Fx \equiv (\exists x) \sim Fx$$

That is: it suffices to provide a QL -derivation like the following:

$$\begin{array}{lll} 1 & \left| \begin{array}{l} \sim(x)Fx \\ (\exists x) \sim Fx \end{array} \right. & ACP \\ 2 & & 1, QN \\ 3 & \sim(x)Fx \supset (\exists x) \sim Fx & 1-2, CP \\ 4 & \left| \begin{array}{l} (\exists x) \sim Fx \\ \sim(x)Fx \end{array} \right. & ACP \\ 5 & & 4, QN \\ 6 & (\exists x) \sim Fx \supset \sim(x)Fx & 4-5, CP \\ 7 & (\sim(x)Fx \supset (\exists x) \sim Fx) \bullet ((\exists x) \sim Fx \supset \sim(x)Fx) & 3, 6, Conj \\ 8 & \sim(x)Fx \equiv (\exists x) \sim Fx & 7, Equiv \end{array}$$

By the way, together with **Fact 1**, **Fact 4** tells us that \mathbf{P} and \mathbf{Q} are QL -equivalent if and only if

$$\models_{QL} \mathbf{P} \equiv \mathbf{Q}$$

In a similar fashion, we can appeal to the following fact to use our derivation system to show that two wffs of QL are QL -contradictories.

Fact 5: Two wffs of QL , P and Q are QL -contradictories if and only if

$$\vdash_{QL} P \equiv \sim Q$$

Fact 5 tells us that, if $P \equiv \sim Q$ is QL -derivable from no assumptions, then P and $\sim Q$ are QL -contradictories: that is, any QL -interpretation which makes P is true is a QL -interpretation which makes Q is false; and any QL -interpretation which makes P is false makes Q is true.

For instance, suppose that we wish to show that the wffs

$$(x)Fx \quad \text{and} \quad (\exists x) \sim Fx$$

are QL -contradictory. To do so, it suffices to show that

$$\vdash_{QL} (x)Fx \equiv \sim(\exists x) \sim Fx$$

That is: it suffices to provide a QL -derivation like the following:

1	$(x)Fx$	ACP
2	$\sim(\exists x) \sim Fx$	1, QN
3	$(x)Fx \supset \sim(\exists x) \sim Fx$	1-2, CP
4	$\sim(\exists x) \sim Fx$	ACP
5	$(x)Fx$	4, QN
6	$\sim(\exists x) \sim Fx \supset (x)Fx$	4-5, CP
7	$((x)Fx \supset \sim(\exists x) \sim Fx) \bullet (\sim(\exists x) \sim Fx \supset (x)Fx)$	3, 6 $Conj$
8	$(x)Fx \equiv \sim(\exists x) \sim Fx$	7, $Equiv$

By the way, Fact 5, together with Fact 1, tells us that P and Q are QL -contradictories if and only if

$$\vDash_{QL} P \equiv \sim Q$$

14.3.4 QL -Inconsistency

We can similarly use QL -derivations to show that a set of wffs of QL is QL -inconsistent, by appealing to the following fact.

Fact 6: A set of wffs of QL , $\{P_1, P_2, \dots, P_N\}$ is QL -inconsistent if and only if

$$P_2, \dots, P_N \vdash_{QL} \sim P_1$$

That is: if, by beginning a QL -derivation with all but one of the members of a set of wffs of QL , we can construct a legal derivation whose final line is the negation of the remaining member, then the original set of wffs of QL is QL -inconsistent.

Fact 6 tells us that the following QL -derivation establishes that the set

$$\{(x)(Px \supset (Qx \vee Rx)), (x)(Px \supset \sim Qx), (x)(Px \supset \sim Rx), (\exists x)Px\}$$

is QL -inconsistent.

1	$(x)(Px \supset (Qx \vee Rx))$	
2	$(x)(Px \supset \sim Qx)$	
3	$(x)(Px \supset \sim Rx)$	
4	Pz	<i>AIP</i>
5	$Pz \supset (Qz \vee Rz)$	1, <i>UI</i> [$x \rightarrow z$]
6	$Qz \vee Rz$	4, 5, <i>MP</i>
7	$Pz \supset \sim Qz$	2, <i>UI</i> [$x \rightarrow z$]
8	$\sim Qz$	4, 7, <i>MP</i>
9	Rz	6, 8, <i>DS</i>
10	$Pz \supset \sim Rz$	3, <i>UI</i> [$x \rightarrow z$]
11	$\sim Rz$	4, 10, <i>MP</i>
12	$Rz \cdot \sim Rz$	9, 11, <i>Conj</i>
13	$\sim Pz$	4-12, <i>IP</i>
14	$(x) \sim Px$	13, <i>UG</i>
15	$\sim (\exists x)Px$	14, <i>QN</i>

This derivation demonstrates that

$$(x)(Px \supset (Qx \vee Rx)), (x)(Px \supset \sim Qx), (x)(Px \supset \sim Rx) \vdash_{QL} \sim (\exists x)Px$$

which, together with **Fact 6**, tells us that the set

$$\{(x)(Px \supset (Qx \vee Rx)), (x)(Px \supset \sim Qx), (x)(Px \supset \sim Rx), (\exists x)Px\}$$

is *QL*-inconsistent.

By the way, **Fact 6**, together with **Fact 1**, tells us that a set of wffs of *QL*, $\{P_1, P_2, \dots, P_N\}$ is *QL*-inconsistent if and only if

$$P_2, \dots, P_N \models_{QL} \sim P_1$$

Chapter 15

Quantificational Logic with Identity

We're going to learn about a slight extension of the language QL which allows us to talk about 1) whether two things are identical; 2) whether something is unique; and 3) how many things there are. It will also allow us to translate sentences of English like 'Only Bob and Eric went to the party', 'The King of France is bald', and 'Nobody likes Janet except for Rudy'. All of these translations require the addition of a relation of *identity* to our language. With this addition, we'll call our language ' QLI '.

15.1 The Language QLI

15.1.1 Syntax for QLI

We're now going to make one change to our grammar for QL . We're going to introduce a special two-place predicate for *identity*, $=$. Thus, we will add the following clause to our rules for well formed formulae:

\Rightarrow) If t_1 is a term of QL and t_2 is a term of QL , then $t_1 = t_2$ is a wff of QL .

We add this to the rules for well-formed formulae that we already have to get the following recursive definition of wff:

\mathcal{F}) If ' \mathcal{F}^n ' is an n -place predicate and ' t_1, t_2, \dots, t_n ' are n terms, then ' $\mathcal{F}^n t_1 t_2 \dots t_n$ ' is a wff.

\Rightarrow) If t_1 is a term of QL and t_2 is a term of QL , then $t_1 = t_2$ is a wff of QL .

\sim) If ' P ' is a wff, then ' $\sim P$ ' is a wff.

\bullet) If ' P ' and ' Q ' are wffs, then ' $(P \bullet Q)$ ' is a wff.

\vee) If ' P ' and ' Q ' are wffs, then ' $(P \vee Q)$ ' is a wff.

\supset) If ' P ' and ' Q ' are wffs, then ' $(P \supset Q)$ ' is a wff.

\equiv) If ' P ' and ' Q ' are wffs, then ' $(P \equiv Q)$ ' is a wff.

x) If ' P ' is a wff and x is a variable, then ' $(x)P$ ' is a wff.

\exists) If ' P ' is a wff and x is a variable, then ' $(\exists x)P$ ' is a wff.

– Nothing else is a wff.

Then, we can show that, e.g., ' $\sim(\exists x) \sim x = x$ ' is a wff of QL , by carrying out a proof like the following:

- a) ' $x = x$ ' is a wff [from (=)]
- b) So, ' $\sim x = x$ ' is a wff [from (a) and (\sim)]
- c) So, ' $(\exists x) \sim x = x$ ' is a wff [from (b) and (\exists)]
- d) So, ' $\sim(\exists x) \sim x = x$ ' is a wff [from (c) and (\sim)]

Thus, ' $\sim(\exists x) \sim x = x$ ' has the following syntax tree:

$$\begin{array}{c}
 \sim(\exists x) \sim x = x \\
 | \\
 (\exists x) \sim x = x \\
 | \\
 \sim x = x \\
 | \\
 x = x
 \end{array}$$

As a matter of convention, we will write ' $\sim t_1 = t_2$ ' as ' $t_1 \neq t_2$ '. Thus, ' $\sim(\exists x) \sim x = x$ ' may be written as:

$$\sim(\exists x)x \neq x$$

15.1.2 Semantics for QLI

We'll account for the meaning of the expressions of QLI just as we did with QL : with the aid of the notion of a QLI -interpretation. And our definition of a QLI -interpretation will be almost identical to our definition of a QL -interpretation. Thus:

A QLI -INTERPRETATION, \mathcal{I} , provides

1. A specification of which things fall in the *domain*, \mathcal{D} , of the interpretation.^a
2. A unique constant of QLI to name every thing in the domain.
3. For every term (constant or variable) of QLI , a specification of which thing in the domain \mathcal{D} it represents.
4. For every predicate of QLI except for $=$, a specification of the property or relation it represents.

^a Note: the domain must be non-empty, and it must be countable.

We don't give an interpretation of the meaning of the 2-place predicate ' $=$ ' because we're going to want to hold its interpretation fixed across QLI -interpretations. It will always mean the same thing: identity.

We may also give a *partial* QLI -interpretation (just as we did for QL). A *partial* QLI -interpretation is as specified below.

Given a wff, set of wffs, or argument of *QLI*, a PARTIAL *QLI*-INTERPRETATION, \mathcal{I}_p provides:

1. A specification of which things fall in the *domain*, \mathcal{D} , of the partial interpretation.^a
2. For the constants and free variables appearing in the wff, set of wffs, or argument of *QLI*, a specification of which thing in the domain \mathcal{D} they represent.
3. For the predicates other than = appearing in the wff, set of wffs, or argument of *QLI*, a specification of the property or relation they represent.

^a Note: the domain must be non-empty, and it must be countable.

The meaning of '=' is exactly what you would expect it to be. ' $a = b$ ' means that the thing denoted by ' a ' is the same thing as the thing denoted by ' b '. Thus, if ' a ' denotes Samuel Clemes and ' b ' denotes Mark Twain, then ' $a = b$ ' says that Samuel Clemens is Mark Twain.

Thus, we'll add the following semantic clause to the semantics that we gave for *QL*:

9. A wff of the form ' $t_1 = t_2$ ' is true on the interpretation \mathcal{I} if the thing denoted by ' t_1 ' on the interpretation \mathcal{I} is identical to the thing denoted by ' t_2 ' on the interpretation \mathcal{I} . Otherwise, it is false on the interpretation \mathcal{I} .

Everything else remains the same.

15.2 *QLI* Derivations

Our derivation system for *QLI* will carry over all of the rules from the *QL* derivation system, plus three new ones, all of which have the same name—'*Identity (Id)*'.

Identity (Id)

▷ $t = t$

for any term t

$t_1 = t_2$

▷ $t_2 = t_1$

for any terms t_1, t_2

$P[t_1]$

$t_1 = t_2$

▷ $P[t_1 \rightarrow t_2]$

for any terms t_1, t_2

Identity allows us to do three things. Firstly, we may, *whenever we wish*, write down an identity claim on which the identity sign is flanked by the same term of *QLI* on both sides. When we do so, we should write '*Id*' on the justification line—though we needn't cite any other line of the derivation when we do so.

Note that this means that, with *Identity*, we can prove tautologies without ever having to start a subderivation. For instance, the following one-line derivation establishes that $a = a$ is a *QLI*-tautology:

$$1 \quad a = a \quad Id$$

Similarly, the following derivation establishes that $(\exists x)x = x$ is a *QLI*-tautology:

Similarly, the following derivation establishes that $(x)x = x$ is a *QLI*-tautology:

$$\begin{array}{l} 1 \quad z = z \quad Id \\ 2 \quad (x)x = x \quad 1, UG \end{array}$$

We may utilize *UG* here because, even though z appears free on the first line of the derivation, it does not appear free in the derivation's *assumptions* (because the derivation has no assumptions).

Because $(x)x = x$ is a *QLI*-tautology, this tells us that identity is a *reflexive* relation.

REFLEXIVITY
A binary (2-place) relation R is *reflexive* if and only if,
 $(x)Rxx$

Secondly, *Identity* allows us to commute an identity statement. So, if we have ' $a = b$ ' written down on an accessible line of our derivation, then we may write down ' $b = a$ '. When we do so, we should write the line on which ' $a = b$ ' appeared and write '*Id*' in the justification line. For instance, the following is a legal derivation:

$$\begin{array}{l} 1 \quad | \quad x = y \quad ACP \\ 2 \quad | \quad y = x \quad 1, Id \\ 3 \quad x = y \supset y = x \quad 1-2, CP \\ 4 \quad (y)(x = y \supset y = x) \quad 3, UG \\ 5 \quad (x)(y)(x = y \supset y = x) \quad 4, UG \end{array}$$

We may use *UG* on lines 4 and 5 because, even though x and y both appear free in the assumption of the subderivation running from lines 1–2, that subderivation is not *accessible* at lines 4 and 5.

Thus, we may conclude that ' $(x)(y)(x = y \supset y = x)$ ' is a tautology of *QLI*. This tells us, by the way, that identity, $=$, is a *symmetric* relation.

SYMMETRY
A binary (2-place) relation R is *symmetric* if and only if,
 $(x)(y)(Rxy \supset Ryx)$

Note: the second rule of *Identity* is a rule of replacement. So it may be applied to subformulae. For instance, the following derivation is legal:

$$\begin{array}{l} 1 \quad (x)(Fx \supset x = a) \\ 2 \quad (x)(Fx \supset a = x) \quad 1, Id \end{array}$$

Thirdly, *Identity* tells us that, if we have a wff of *QLI*, $P[t_1]$ in which a term t_1 appears, and we have a wff of *QLI* of the form $t_1 = t_2$, then we may replace some or all of the occurrences of t_1 in $P[t_1]$ with the term t_2 —so long as t_2 *doesn't end up getting bound by a quantifier when we do so*.

This provision is important. So, for instance, the following derivation is not legal:

- 1 $(\exists y)Fxy$
- 2 $x = y$
- 3 $(\exists y)Fyy$ 1, 2, *Id* \leftarrow MISTAKE!!!

However, the following derivation is legal:

- 1 $(\exists y)Fxy$
- 2 $x = z$
- 3 $(\exists y)Fzy$ 1, 2, *Id*

Note: the order of the terms flanking '=' matters. *Identity* tells us that, if we have a wff in which the term on the *left hand side* of the identity sign appears, then we may replace it with the term on the *right hand side*. It *does not* tell us that, if we have a wff in which the term on the *right hand side* of the identity sign appears, then we may replace it with the term on the *left hand side*. Thus, the following derivation is *not* legal:

- 1 $(x)(Rax \supset Fx)$
- 2 $b = a$
- 3 $(x)(Rbx \supset Fx)$ 1, 2, *Id* \leftarrow MISTAKE!!!

For ' a ' appears on the right hand side of $b = a$, so we may not apply the third Identity rule to swap out ' b ' for ' a '. On the other hand, the following derivation is legal:

- 1 $(x)(Rax \supset Fx)$
- 2 $b = a$
- 3 $a = b$ 2, *Id*
- 4 $(x)(Rbx \supset Fx)$ 1, 3, *Id*

The following *QLI* derivation is legal:

1	$x = y \bullet y = z$	<i>ACP</i>
2	$y = z \bullet x = y$	1, <i>Com</i>
3	$x = y$	1, <i>Simp</i>
4	$y = z$	2, <i>Simp</i>
5	$x = z$	3, 4, <i>Id</i>
6	$(x = y \bullet y = z) \supset x = z$	1-5, <i>CP</i>
7	$(z)((x = y \bullet y = z) \supset x = z)$	6, <i>UG</i>
8	$(y)(z)((x = y \bullet y = z) \supset x = z)$	7, <i>UG</i>
9	$(x)(y)(z)((x = y \bullet y = z) \supset x = z)$	8, <i>UG</i>

We may use *UG* on lines 7, 8, and 9 because, even though x , y , and z all appear free in the assumption of the subderivation running from lines 1-5, that subderivation is not *accessible* at lines 7, 8, and 9.

Thus, we may conclude that ‘ $(x)(y)(z)((x = y \bullet y = z) \supset x = z)$ ’ is a tautology of *QLI*. This tells us, by the way, that identity, =, is a *transitive* relation.

TRANSITIVITY
 A binary (2-place) relation R is *transitive* if and only if,

$$(x)(y)(z)((Rxy \bullet Ryz) \supset Rxz)$$

Though there are three different *Identity* rules, you will always know which is being invoked by the number of lines cited. If an application of *Identity* cites no lines, then the first rule is being invoked. If it cites one line, then the second rule is being invoked. And if it cites two lines, then the third rule is being invoked.

Sample Derivations

1	$(x)(x = c \supset Nx)$	<i>INc</i>
2	$c = c \supset Nc$	1, <i>UI</i> [$x \rightarrow c$]
3	$c = c$	<i>Id</i>
4	Nc	2, 3, <i>MP</i>

1	$Haa \supset Waa$	
2	Hab	
3	$a = b$	<i>Wab</i>
4	$Hab \supset Wab$	1, 3, <i>Id</i>
5	Wab	2, 4, <i>MP</i>

1	$(x)x = a$	
2	$(\exists x)Rx$	$/Ra$
3	Rq	2, $EI [x \rightarrow q]$
4	$q = a$	1, $UI [x \rightarrow q]$
5	Ra	3, 4, Id

1	Ke	
2	$\sim Kn$	$/e \neq n$
3	$e = n$	AIP
4	Kn	1, 3, Id
5	$Kn \cdot \sim Kn$	2, 4, $Conj$
6	$e \neq n$	3-5, IP

1	$(x)(Px \supset x = a)$	
2	$(x)(x = c \supset Qx)$	
3	$a = c$	$/ (x)(Px \supset Qx)$
4	Pz	ACP
5	$Pz \supset z = a$	1, $UI [x \rightarrow z]$
6	$z = a$	4, 5, MP
7	$z = c$	3, 6, Id
8	$z = c \supset Qz$	2, $UI [x \rightarrow z]$
9	Qz	7, 8, MP
10	$Pz \supset Qz$	4-9, CP
11	$(x)(Px \supset Qx)$	10, UG

15.3 Translations from English to QLI

15.3.1 Number Claims

At Least One

Suppose that we wish to say (only) that there is at least one thing in the domain of our interpretation. We may do so as follows:

$$(\exists x)x = x$$

' $(\exists x)x = x$ ' will be true if it has a true substitution instance, $\mathbf{a} = \mathbf{a}$. But, for every constant of the language, \mathbf{a} , whatever \mathbf{a} refers to will be the same thing as the thing that \mathbf{a} refers to, so long as there's at least one thing in the domain for \mathbf{a} to

refer to. So $(\exists x)x = x$ is true if and only if there's at least one thing in the domain. (Since we require our domains to be non-empty, there will be such a thing in every *QLI*-interpretation—that's why $(\exists x)x = x$ is a *QLI*-tautology. If we didn't require our domains to be non-empty, then it wouldn't be a *QLI*-tautology.)

At Least Two

Suppose that we wish to say something that is true in all and only the interpretations in which there are two distinct things in the domain of the interpretation. We may do this as follows:

$$(\exists x)(\exists y)x \neq y$$

This is true if and only if $(\exists x)(\exists y)x \neq y$ has a true substitution instance $(\exists y)\mathbf{a} \neq y$. And $(\exists y)\mathbf{a} \neq y$ is true if and only if it has a true substitution instance $\mathbf{a} \neq \mathbf{b}$. So $(\exists x)(\exists y)x \neq y$ is true if and only if there are two constants \mathbf{a} and \mathbf{b} which name different things. So $(\exists x)(\exists y)x \neq y$ is true if and only if there are at least two things in the model. $(\exists x)(\exists y)x \neq y$ is *not* a *QLI*-tautology, since we don't require that our domains have more than one thing in them.

Here, we don't need to additionally specify that $y \neq x$, since this follows from the symmetry of $=$. From ' $(\exists x)(\exists y)x \neq y$ ', we could derive that $(\exists x)(\exists y)(x \neq y \cdot y \neq x)$, as follows (to make it clearer what's going on, I've replaced ' $x \neq y$ ' with ' $\sim x = y$ ')

1	$(\exists x)(\exists y) \sim x = y$	
2	$(\exists y) \sim a = y$	1, <i>EI</i> [$x \rightarrow a$]
3	$\sim a = b$	2, <i>EI</i> [$x \rightarrow b$]
4	$\sim b = a$	3, <i>Id</i>
5	$\sim a = b \cdot \sim b = a$	4, <i>Conj</i>
6	$(\exists y)(\sim a = y \cdot \sim y = a)$	5, <i>EG</i>
7	$(\exists x)(\exists y)(\sim x = y \cdot \sim y = x)$	6, <i>EG</i>

At Least Three

Similarly, if we wish to say something that is true in all and only the *QLI*-interpretations in which there are three distinct things in the domain, we may say the following:

$$(\exists x)(\exists y)(\exists z)((x \neq y \cdot y \neq z) \cdot x \neq z)$$

Here, it is not enough to say merely that $x \neq y$ and that $y \neq z$. For it could still be that $x = z$. So we must rule this out by specifying that $x \neq z$. The wff above says that the domain contains three things, all of which are distinct from one another. Of course, there could be other things besides these three. So what the wff above says is just that there are *at least* three things in the domain.

At Least Four

We could go on. Suppose that we wish to say something that's true in all and only the *QLI*-interpretations on which there are at least *four* things in the domain. Then, we could say the following:

$$(\exists x)(\exists y)(\exists z)(\exists x_1)((((x \neq y \cdot x \neq z) \cdot x \neq x_1) \cdot y \neq z) \cdot y \neq x_1) \cdot z \neq x_1)$$

A CHALLENGE

See if you can come up with a wff of *QLI* that's true if and only if there are *infinitely many* things in the domain. (You'll have to use more than identity. *Hint*: to think it through, use an interpretation whose domain is the counting numbers, use the predicate Gxy for 'x is greater than y ($x > y$)', and try to think of a collection of claims which will tell you that there are an infinite number of numbers. *A further hint*: translate the claims "nothing is greater than itself", "every number has some number that's greater than it", and then think about how, with the foregoing claims laid down, you could rule out 'loops' of greatness, like, e.g., Gab , Gbc , and Gca .)

At Least Two P s

Now, suppose that we wish to say, not just that there's at least two *things* in the domain, but additionally, that there's at least two things *that are P* in the domain. We can say that by saying, first, that there's at least two things, and next, that those two things are P :

$$(\exists x)(\exists y)((x \neq y \cdot Px) \cdot Py)$$

At Least Three P s

Similarly, suppose that we wish to say that there are at least three things *that are P* in the domain. We can say that by saying, first, that there's at least three things, and next, that those three things are P :

$$(\exists x)(\exists y)(\exists z)((x \neq y \cdot y \neq z \cdot x \neq z) \cdot Px) \cdot Py) \cdot Pz)$$

No More Than One

The previous translations put a *lower* bound on the number of things in the domain. Suppose that, instead, we wish to be an *upper* bound on the number of things in the domain. Suppose that we wish to say that there is *no more than one* thing in the domain. If we wish to say this, then we could just say that there's *something* which everything in the domain is identical to. If everything in the domain is the same as that one thing, then there can only be one thing in the domain. Thus,

$$(\exists x)(\forall y)y = x$$

translates "there is at least one thing in the domain." (Note: since we require the domain to contain at least one thing, this is equivalent to saying that there is *exactly* one thing in the domain.)

No More Than Two

Suppose, on the other hand, that we wish to say that there are no more than two things in the domain. We may say that by saying that there are two things x and y such that, for any thing in the domain z , z is either identical to x or z is identical to y .

$$(\exists x)(\exists y)(\forall z)(z = x \vee z = y)$$

This could also be true if there is but one thing in the domain. So it doesn't say that there are *exactly* two things in the domain. Rather, it says that there are *no more than* two things in the domain.

No More Than Three

We could go on. For instance, the following wff says that there are no more than three things in the domain.

$$(\exists x)(\exists y)(\exists z)(x_1)((x_1 = x \vee x_1 = y) \vee x_1 = z)$$

In English, this says that there are three things, x , y , and z , such that every thing in the domain is either identical to x , or it's identical to y , or it's identical to z . Again, this could be true if there's only one or two things in the domain (in that case, either $x = y$ or $y = z$ or $x = z$). So it doesn't say that there are *exactly* three things in the domain. But it does say that there are no more than three.

Exactly One

Suppose that we wish to put *both* an upper bound *and* a lower bound on the number of things in the domain. In the case of one, saying that there is exactly one thing in the domain is equivalent to saying that there is no more than one thing in the domain (since we required that our domains have at least one thing in them). And we have already seen that the way to say that there is no more than one thing in the domain is:

$$(\exists x)(y)y = x$$

However, we could also just conjoin the claim that there is at least one thing with the claim that there is no more than one thing in the domain, as follows:

$$(\exists x)x = x \bullet (\exists x)(y)y = x$$

These two wffs are *QLI*-equivalent, as the following *QLI* derivation establishes.

1	$(\exists x)(y)y = x$	<i>ACP</i>
2	$a = a$	<i>Id</i>
3	$(\exists x)x = x$	2, <i>UG</i>
4	$(\exists x)x = x \bullet (\exists x)(y)y = x$	1, 3, <i>Conj</i>
5	$(\exists x)(y)y = x \supset ((\exists x)x = x \bullet (\exists x)(y)y = x)$	1-4, <i>CP</i>
6	$(\exists x)x = x \bullet (\exists x)(y)y = x$	<i>ACP</i>
7	$(\exists x)(y)y = x \bullet (\exists x)x = x$	6, <i>Com</i>
8	$(\exists x)(y)y = x$	7, <i>Simp</i>
9	$((\exists x)x = x \bullet (\exists x)(y)y = x) \supset (\exists x)(y)y = x$	6-8, <i>CP</i>
10	$((\exists x)(y)y = x \supset ((\exists x)x = x \bullet (\exists x)(y)y = x))$ $\bullet ((\exists x)x = x \bullet (\exists x)(y)y = x) \supset (\exists x)(y)y = x$	5, 9, <i>Conj</i>
11	$(\exists x)(y)y = x \equiv ((\exists x)x = x \bullet (\exists x)(y)y = x)$	10, <i>Equiv</i>

Exactly Two

Suppose that we wish to say that there are *exactly two* things in the domain. Then, we may just conjoin the claims that there are at least two things with the claim that there are no more than two things, as follows:

$$(\exists x)(\exists y)x \neq y \bullet (\exists x)(\exists y)(z)(z = x \vee z = y)$$

This works, but it's a bit more complicated than it needs to be. We may alternatively just say that there are two things which are non-identical, and that anything else in the domain is identical to one of them:

$$(\exists x)(\exists y)(x \neq y \bullet (z)(z = x \vee z = y))$$

These two claims are equivalent. We could provide a *QLI*-derivation to establish this, but the shortest one I was able to produce was 38 lines long, so I won't reproduce it here.

Exactly Three

We could go further. Here's a way of saying that there are exactly three things in the domain.

$$(\exists x)(\exists y)(\exists z)((x \neq y \bullet y \neq z) \bullet z \neq x) \bullet (x_1)((x_1 = x \vee x_1 = y) \vee x_1 = z))$$

This wff says: 1) there are three things; 2) those things are distinct; and 3) everything in the domain is identical to one of them. And this tells us that there are exactly three things in the domain.

15.4 The Only

Suppose that we wish to translate claims like

James is the only person in the class taller than 7 feet.

If we have the following *QL*-interpretation,

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \text{people in the class} \\ j & = \text{James} \\ T\mathbf{x} & = \mathbf{x} \text{ is taller than 7 feet} \end{cases}$$

then we may translate this sentence of English by saying two things. First, we must say that James is taller than 7 feet. This part of the translation is easy:

$$Tj$$

Then, we must say that he is the *only* one taller than 7 feet. We can accomplish this within *QLI* by saying that all people who are taller than 7 feet are identical to James.

$$(x)(Tx \supset x = j)$$

Thus, the following sentence will translate “James is the only person in the class taller than 7 feet”:

$$Tj \bullet (x)(Tx \supset x = j)$$

In general, we may translate claims of the form “*a* is the only *P*” as follows:

$$Pa \bullet (x)(Px \supset x = a)$$

Suppose that we wish to translate “Only Bob and Eric went to the party”, given the following partial *QLI*-interpretation:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \text{contextually salient people} \\ b & = \text{Bob} \\ e & = \text{Eric} \\ P\mathbf{x} & = \mathbf{x} \text{ went to the party} \end{cases}$$

We may do this as follows:

$$(Pb \bullet Pe) \bullet (y)(Py \supset (y = b \vee y = e))$$

15.5 Definite Descriptions

Suppose that we wish to translate a sentence like

The King of France is bald.

In order to attempt to translate this claim, we should think about what it's saying. It appears to be saying:

1. There is a King of France (the *existence* claim);
2. He is the only King of France (the *uniqueness* claim); and
3. He is bald (the *predication* claim).

Our translation should include these three elements of the original claim. But we already know how to say these three things in *QLI*. Suppose that we have the following (partial) *QLI*-interpretation,

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \text{the set of people} \\ Kx & = x \text{ is King of France} \\ Bx & = x \text{ is bald} \end{cases}$$

Then, I submit, "The King of France is bald" may be translated by the wff

$$(\exists x)((Kx \bullet (y)(Ky \supset y = z)) \bullet Bx)$$

In this wff, the claim that there is a King of France (the *existence* claim) is made by:

$$(\exists x)((Kx \dots$$

The *uniqueness* claim is made by:

$$\dots (y)(Ky \supset y = x) \dots$$

And the *predication* claim (that he is bald), is made by:

$$\dots Bx \dots$$

In general, if we wish to translate a definite description of the form

The P is Q.

Then we may do so with a wff of the following form:

$$(\exists x)((Px \bullet (y)(Py \supset y = x)) \bullet Qx)$$

For instance, suppose that we wish to translate

The owner is tired.

given the following (partial) *QLI*-interpretation:

$$\mathcal{I}_p = \begin{cases} \mathcal{D} & = \text{the set of people under discussion} \\ Ox & = x \text{ is an owner} \\ Tx & = x \text{ is tired} \end{cases}$$

Then, we may do so by saying:

1. There is an owner: $(\exists x)Ox \dots$;
2. He is the only owner: $\dots (y)(Oy \supset y = x) \dots$; and
3. He is tired: $\dots Tx$.

Thus, "The owner is tired" is translated by:

$$(\exists x)((Ox \bullet (y)(Oy \supset y = x)) \bullet Tx)$$